A family of dense mixed graphs of diameter 2

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Spain, Mexico, Australia, Czech Republic, Slovakia
Introduction

Motivation:
Design of interconnection networks

Degree-diameter problem for graphs, digraphs or mixed graphs:
Determination of the largest number $n(d, k)$ of vertices in a graph (digraph, mixed graph) of a given maximum degree $d$ (out-degree, mixed degree) and diameter $k$.

Restricted classes of graphs and digraphs:
Vertex-transitive, Cayley, bipartite graphs (digraphs), graphs (digraphs) embeddable in a fixed surface...

The idea of mixed or partially directed graphs is a generalization of both undirected and directed graphs.

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**The idea of mixed** or partially directed graphs is a generalization of both undirected and directed graphs.
Directed case

For directed graphs (mixed Moore graphs admitting only (directed) arcs) with maximum out-degree $d$ and diameter $k$ are graphs of order $M_{d,k} = 1 + d + d^2 + ... + d^k$ (directed Moore bound). Moore digraphs do not exist for $d > 1$ and $k > 1$. Directed cycles $Z_{k+1}$ and complete digraphs on $d+1$ vertices are the only Moore digraphs.
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Undirected graphs (mixed Moore graphs admitting only (undirected) edges) with maximum degree $d$ and diameter $k$ are graphs of order:

$$M^*d, k = 1 + d + d(d - 1) + \ldots + d(d - 1)^{k-1} \text{ (undirected Moore bound)}.$$

They are no Moore graphs of degree $d \geq 3$ and diameter $k \geq 3$.

For $k = 1$ and $d \geq 1$, complete graphs $K_{d+1}$ are the only Moore graphs.

For $k \geq 3$ and $d + 2$, the cycles $C_{2k+1}$ are the only Moore graphs.

For $k = 2$, apart from $C_5 (d = 2)$, Moore graphs exist only when $d = 3$ (Petersen graph), $d = 7$ (the Hoffman-Singleton graph), and possibly $d = 57$.
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The idea of mixed or "partially directed" graphs: a generalization of both undirected and directed graphs.

Motivation: World Wide Web Networks, Small World Networks...

Models of networks: a mixture of both undirected and bidirected connection. The mixed graphs are used in this case as a good modelling tool.

Directed and undirected Moore graphs are special cases of mixed Moore graphs where the graphs admit only arcs or only edges. Nguyen and Miller (2008) showed that both undirected and directed Moore graphs are special case of mixed Moore graphs.

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A mixed graph with parameters $z$, $r$ and diameter $k$: $(z, r, k)$-mixed graph of directed degree $z$, undirected degree $r$, and diameter $k$. 

$M_{z, r, k}$ - the upper bound on the order of a $(z, r, k)$-mixed graph (mixed Moore bound).

For $k = 2$:

$$M_{z, r, 2} = 1 + r + z + r(r - 1 + z) + z(r + z) = (r + z)^2 + z + 1. \quad (1)$$

Note that $M_{z, r, k} = M_{d, k}$ when $z = 0$ and $M_{z, r, k} = M_{d, k}$ when $r = 0$ (mixed Moore bound).
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Accordingly, apart from the trivial cases \( r = 0 \) and \( z = 1 \) (graph \( Z_3 \)), \( r = 2 \) and \( z = 0 \) (graph \( C_5 \)), there must exist a positive odd integer \( c \) such that

\[
\begin{align*}
&c | (4z - 3)(4z + 5) \quad \text{and} \\
&r = \frac{4c^2 + 3}{4}
\end{align*}
\]

We highlight that this condition is applicable only to the case of diameter \( k = 2 \).

It was proved [Nguyen, Miller, Gimbert, 2007] that mixed Moore graphs of diameter greater than 2 do not exist.
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$$c \mid (4z - 3)(4z + 5) \quad \text{and} \quad r = \frac{1}{4}(c^2 + 3).$$

(2)
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The aim

We construct mixed graphs of diameter 2, which come close to the Moore bound, the so-called \((z, r, 2)\)-mixed graphs, sometimes referred as dense \((z, r, 2)\)-graphs, thereby producing good lower bounds for the degree-diameter problem for those parameter values.

Particularly, we give a construction of dense mixed graphs of diameter 2 undirected degree \(q\), directed degree \(q - 1\) and order \(2q^2\), when \(q\) is an odd prime power. Since the Moore bound for a mixed Moore graph with these parameters is equal to \(9q^2 - 4q + 3\), the defect of these mixed graphs is \((q - 2)^2 - 1\).
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History

Known new results:
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Preliminaries

Biaffine planes, bipartite graphs $B_q$. A partial plane can be defined as two finite sets $P$ and $L$, called points and lines, where $L$ consists of subsets of $P$ such that: any line is incident with at least two points, and two points are incident with at most one line.

Let $F_q$ is a finite field of order $q$. A projective plane is a partial plane in which two points are incident with exactly one line and two lines have exactly two points in common. It is not difficult to derive that every line of a projective plane has the same number of points, say $q + 1$, and every point is incident with $q + 1$ lines; $q$ is called the order of the projective plane.

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The incidence graph of a partial plane is a bipartite graph in which the elements of one part \( L \) are called lines and the elements of the other part \( P \) are called points. The terminology for incidence graphs is geometric. A point and a line are said to be incident if they are adjacent. The incidence graph of a partial plane is clearly a bipartite graph with even girth \( g \geq 6 \).

And the incidence graph of a projective plane is a \((q + 1)\)-regular graph of girth exactly 6.

A biaffine plane is obtained from a projective plane deleting the infinite line and all its points and the infinite point and all its lines. From an algebraic point of view we can define a biaffine plane in the following way.

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The *incidence graph* of a partial plane is a bipartite graph in which the elements of one part $\mathcal{L}$ are called *lines* and the elements of the other part $\mathcal{P}$ are called *points*. The terminology for incidence graphs is geometric. A point and a line are said to be *incident* if they are adjacent. The incidence graph of a partial plane is clearly a bipartite graph with even girth $g \geq 6$. And the incidence graph of a projective plane is a $(q + 1)$-regular graph of girth exactly 6.

A *biaffine plane* is obtained from a projective plane deleting the infinite line and all its points and the infinite point and all its lines. From an algebraic point of view we can define a biaffine plane in the following way.
Definition

Let $\mathbb{F}_q$ be the finite field of order $q$.

(i) Let $\mathcal{L} = \{1\} \times \mathbb{F}_q \times \mathbb{F}_q$ and $\mathcal{P} = \{0\} \times \mathbb{F}_q \times \mathbb{F}_q$. Then the following set of $q^2$ lines define a biaffine plane:

$$(m, b)_1 = \{(x, mx + b)_0 : x \in \mathbb{F}_q\} \text{ for all } m, b \in \mathbb{F}_q.$$ 

(ii) The incidence graph of the biaffine plane is a bipartite graph $B_q = (\mathcal{P}, \mathcal{L})$ which is $q$-regular, has order $2q^2$, diameter 4, and girth 6 if $q \geq 3$; and girth 8 if $q = 2$. 
The diameter of $B_q$ is 4 and the vertices mutually at distance four are the vertices of the sets $V_x = \{(x, y)_0 : y \in \mathbb{F}_q\}$ and $V_m = \{(m, b)_1 : b \in \mathbb{F}_q\}$, for all $x, m \in \mathbb{F}_q$. 
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**Construction:**

Let $q \geq 3$ be an odd prime power. Define a mixed graph $G_q$ such that

$V(G_q) = V(B_q), E(G_q) = E(B_q)$ and $A(G_q) = \{((m, b)_1, (m, b + i)_1) : i \in S\} \cup \{((x, y)_0, (x, y + j)_0) : j \in -S\}$,

where $S \subset \mathbb{F}_q - 0$, $|S| = (q - 1)/2$ and for all $u, v \in S$, $u + v \neq 0$. 

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Theorem

The mixed graph $G_q$ defined in the Construction is a mixed graph of diameter 2 with parameters $r = q$, $z = (q - 1)/2$ and $2q^2$ vertices.
A dense \((z, r, 2)\)-mixed graph having the maximum possible number of vertices is called \textit{optimal}. We denote by \(n(z, r, 2)\) the number of vertices in an optimal mixed graph of parameters \(z\) and \(r\) and diameter 2.
A dense \((z, r, 2)\)-mixed graph having the maximum possible number of vertices is called optimal. We denote by \(n(z, r, 2)\) the number of vertices in an optimal mixed graph of parameters \(z\) and \(r\) and diameter 2.

In particular for \(q = 5\) we construct a mixed graph of order 50, undirected degree 5 and directed degree 2. Bosák proved, in 1979, that there is not exits a mixed graph with these parameters that attains the Moore Bound and it is easy to prove that also does not exists a graph with these parameters, and order 51; then we exhibit an "optimal" graph.
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Theorem

For \( q = 3 \), \( G_3 \) is a mixed Moore graph; and for \( q = 5 \), \( G_5 \) is an optimal mixed graph.
Theorem

For $q = 3$, $G_3$ is a mixed Moore graph; and for $q = 5$, $G_5$ is an optimal mixed graph.

Idea of the proof: For $q = 3$ it turns out that $G_3$ has 18 vertices and parameters $r = 3$ and $z = 1$. Since Bosak’s graph is unique [Nguyen, Miller, Gimbert, 2008], we obtain that $G_3$ given in the Construction is isomorphic to Bosak’s graph.
For $q = 5$ it turns out that $G_5$ has 50 vertices and parameters $r = 5$ and $z = 2$. By (1) the upper bound on the number of vertices for this particular case is 52. Let us show that a mixed Moore graph on 52 vertices and parameters $r = 5$ and $z = 2$ cannot exist. Otherwise, by (2) an odd integer $c$ dividing $(4z - 3)(4z + 5) = 65$ exists such that $r = 5 = \frac{1}{4}(c^2 + 3)$. But then $c = \sqrt{17}$ which implies that $c$ is not an integer. Therefore the upper bound on the number of vertices must be at most 51. However, 51 is not possible because otherwise deleting the directions of the arcs we would obtain a regular graph of degree $r + 2z = 9$ and 51 vertices. Thus we conclude that the upper bound is 50, yielding that $G_5$ is an “optimal” mixed graph.
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Thank you for your attention.