A family of dense mixed graphs of diameter 2

Gabriela Araujo-Pardo
Camino M. Balbuena
Mirka Miller
Mária Ždímalová

Spain, Mexico, Australia, Czech Republic, Slovakia

Introduction

Introduction

Motivation: design of interconnection networks

Introduction

Motivation: design of interconnection networks

Degree-diameter problem for graphs, digraphs or mixed graphs:
determination of the largest number $n(d, k)$ of vertices in a graph (digraph, mixed graph) of a given maximum degree d (out-degree, mixed degree) and diameter k

Introduction

Motivation: design of interconnection networks

Degree-diameter problem for graphs, digraphs or mixed graphs: determination of the largest number $n(d, k)$ of vertices in a graph (digraph, mixed graph) of a given maximum degree d (out-degree, mixed degree) and diameter k

Restricted classes of graphs and digraphs: vertex-transitive, Cayley, bipartite graphs (digraphs), graphs (digraphs) embaddable in a fixed surface...

Introduction

Motivation: design of interconnection networks

Degree-diameter problem for graphs, digraphs or mixed graphs: determination of the largest number $n(d, k)$ of vertices in a graph (digraph, mixed graph) of a given maximum degree d (out-degree, mixed degree) and diameter k

Restricted classes of graphs and digraphs: vertex-transitive, Cayley, bipartite graphs (digraphs), graphs (digraphs) embaddable in a fixed surface...

The idea of mixed or partially directed graphs is a generalization of both undirected and directed graphs.

Directed case

Directed case

For directed graphs (mixed Moore graphs admitting only (directed) arcs) with maximum out-degree d and diameter k are graphs of order

$M_{d,k} = 1 + d + d^2 + \ldots + d^k$ (directed Moore bound).

Directed case

For directed graphs (mixed Moore graphs admitting only (directed) arcs) with maximum out-degree d and diameter k are graphs of order

$M_{d,k} = 1 + d + d^2 + ... + d^k$ (directed Moore bound).

Moore digraphs do not exist for $d > 1$ and $k > 1$. Directed cycles Z_{k+1} and complete digraphs on $d + 1$ vertices are the only Moore digraphs.

Undirected case

Undirected case

Undirected graphs (mixed Moore graphs admitting only (undirected) edges) with maximum degree d and diameter k are graphs of order:

$M^*_{d,k} = 1 + d + d(d-1) + ... + d(d-1)^{k-1}$ (undirected Moore bound).

Undirected case

Undirected graphs (mixed Moore graphs admitting only (undirected) edges) with maximum degree d and diameter k are graphs of order:

$M^*_{d,k} = 1 + d + d(d-1) + \ldots + d(d-1)^{k-1}$ (undirected Moore bound).

They are no Moore graphs of degree $d \geq 3$ and diameter $k \geq 3$.

- For $k = 1$ and $d \geq 1$, complete graphs K_{d+1} are the only Moore graphs.
- For $k \geq 3$ and $d + 2$, the cycles C_{2k+1} are the only Moore graphs.
- For $k = 2$, apart from C_5 $(d = 2)$, Moore graphs exist only when $d = 3$ (Petersen graph), $d = 7$ (the Hoffman-Singleton graph), and possibly $d = 57$.

Mixed case

Mixed case

The idea of mixed or "partially directed" graphs: a generalization of both undirected and directed graphs.

Mixed case

The idea of mixed or "partially directed" graphs: a generalization of both undirected and directed graphs.

Motivation: World Wide Web Networks, Small World Networks...

Mixed case

The idea of mixed or "partially directed" graphs: a generalization of both undirected and directed graphs.

Motivation: World Wide Web Networks, Small World Networks...

Models of networks: a mixture of both undirected and bidirected connection. The mixed graphs are used in this case as a good modelling tool.

Directed and undirected Moore graphs are special cases of mixed Moore graphs where the graphs admit only arcs or only edges. Nguyen and Miller (2008) showed that both undirected and directed Moore graphs are special case of mixed Moore graphs.

A mixed graph with parameters z, r and diameter k: (z, r, k)-mixed graph of directed degree z, undirected degree r, and diameter k.

A mixed graph with parameters z, r and diameter k: (z, r, k)-mixed graph of directed degree z, undirected degree r, and diameter k.

$M_{z,r,k}$ - the upper bound on the order of a (z, r, k)-mixed graph (mixed Moore bound).

A mixed graph with parameters z, r and diameter k: (z, r, k)-mixed graph of directed degree z, undirected degree r, and diameter k.

$M_{z,r,k}$ - the upper bound on the order of a (z, r, k)-mixed graph (mixed Moore bound).

For $k = 2$:

$$M_{z,r,2} = 1 + r + z + r(r - 1 + z) + z(r + z) = (r + z)^2 + z + 1. \quad (1)$$

Note that

$M_{z,r,k} = M^*_{d,k}$ when $z = 0$ and $M_{z,r,k} = M_{d,k}$ when $r = 0$ $(d = z + r)$. (mixed Moore bound)

Mixed Moore graph of diameter two:

Mixed Moore graph of diameter two:

a mixed graph of diameter two has order equal to $M_{z,r,2} = (r+z)^2 + z + 1$

Mixed Moore graph of diameter two:

a mixed graph of diameter two has order equal to $M_{z,r,2} = (r+z)^2 + z + 1$

Then the two parameters z and r must satisfy a tight arithmetic condition obtained by eigenvalue analysis [Bosák, 1979].

Mixed Moore graph of diameter two:

a mixed graph of diameter two has order equal to $M_{z,r,2} = (r+z)^2 + z + 1$

Then the two parameters z and r must satisfy a tight arithmetic condition obtained by eigenvalue analysis [Bosák, 1979].

Accordingly, apart from the trivial cases $r = 0$ and $z = 1$ (graph Z_3), $r = 2$ and $z = 0$ (graph C_5), there must exists a positive oddinteger c such that

Mixed Moore graph of diameter two:

a mixed graph of diameter two has order equal to $M_{z,r,2} = (r+z)^2 + z + 1$

Then the two parameters z and r must satisfy a tight arithmetic condition obtained by eigenvalue analysis [Bosák, 1979].

Accordingly, apart from the trivial cases $r = 0$ and $z = 1$ (graph Z_3), $r = 2$ and $z = 0$ (graph C_5), there must exists a positive oddinteger c such that

$$c|(4z-3)(4z+5) \text{ and } r = \frac{1}{4}(c^2+3). \tag{2}$$

Mixed Moore graph of diameter two:

a mixed graph of diameter two has order equal to $M_{z,r,2} = (r+z)^2 + z + 1$

Then the two parameters z and r must satisfy a tight arithmetic condition obtained by eigenvalue analysis [Bosák, 1979].

Accordingly, apart from the trivial cases $r = 0$ and $z = 1$ (graph Z_3), $r = 2$ and $z = 0$ (graph C_5), there must exists a positive oddinteger c such that

$$c|(4z-3)(4z+5) \text{ and } r = \frac{1}{4}(c^2+3). \tag{2}$$

We highlight that this condition is applicable only to the case of diameter $k = 2$.

Mixed Moore graph of diameter two:

a mixed graph of diameter two has order equal to $M_{z,r,2} = (r+z)^2 + z + 1$

Then the two parameters z and r must satisfy a tight arithmetic condition obtained by eigenvalue analysis [Bosák, 1979].

Accordingly, apart from the trivial cases $r = 0$ and $z = 1$ (graph Z_3), $r = 2$ and $z = 0$ (graph C_5), there must exists a positive oddinteger c such that

$$c|(4z-3)(4z+5) \text{ and } r = \frac{1}{4}(c^2+3). \tag{2}$$

We highlight that this condition is applicable only to the case of diameter $k = 2$.

It was proved [Nguyen, Miller, Gimbert, 2007] that mixed Moore graphs of diameter greater than 2 do not exist.

The aim

The aim

We construct mixed graphs of diameter 2, which come close to the Moore bound, the so called $(z, r, 2)$ - mixed graphs, sometimes referred as *dense* $(z, r, 2)-$ graphs, thereby producing good lower bounds for the degree diameter problem for those parameter values.

Particularly

The aim

We construct mixed graphs of diameter 2, which come close to the Moore bound, the so called $(z, r, 2)$ - mixed graphs, sometimes referred as *dense* $(z, r, 2)-$ graphs, thereby producing good lower bounds for the degree diameter problem for those parameter values.

Particularly

We give a construction of dense mixed graphs of diameter 2 undirected degree q, directed degree $\frac{q-1}{2}$ and order $2q^2$, when q is an odd prime power. Since the Moore bound for a mixed Moore graph with these parameters is equal to $\frac{9q^2-4q+3}{4}$, the defect of these mixed graphs is $(\frac{q-2}{2})^2 - \frac{1}{4}$.

History

History

Bosák (1979)
Nguyen, Miller and Gimbert (2007)
Nguyen, Miller (2008)

History

Bosák (1979)
Nguyen, Miller and Gimbert (2007)
Nguyen, Miller (2008)

Known new results:

History

Bosák (1979)
Nguyen, Miller and Gimbert (2007)
Nguyen, Miller (2008)

Known new results:

Leif Jorgensen
Mikhail Klin and Štefan Gyürki
Nacho Lopez
Jana Šiagiová

Preliminaries

Preliminaries

Biaffine planes, bipartite graphs B_q.

A *partial plane* can be defined as two finite sets $\mathcal{P}$ and $\mathcal{L}$, called points and lines, where $\mathcal{L}$ consists of subsets of $\mathcal{P}$ such that: any line is incident with at least two points, and two points are incident with at most one line.

Preliminaries

Biaffine planes, bipartite graphs B_q.

A *partial plane* can be defined as two finite sets $\mathcal{P}$ and $\mathcal{L}$, called points and lines, where $\mathcal{L}$ consists of subsets of $\mathcal{P}$ such that: any line is incident with at least two points, and two points are incident with at most one line.

Let F_q is a finite field of order q.

Preliminaries

Biaffine planes, bipartite graphs B_q.

A *partial plane* can be defined as two finite sets $\mathcal{P}$ and $\mathcal{L}$, called points and lines, where $\mathcal{L}$ consists of subsets of $\mathcal{P}$ such that: any line is incident with at least two points, and two points are incident with at most one line.

Let F_q is a finite field of order q.

A *projective plane* is a partial plane in which two points are incident with exactly one line and two lines have exactly two points in common. It is not difficult to derive that every line of a projective plane has the same number of points, say $q+1$, and every point is incident with $q+1$ lines; q is called the *order* of the projective plane.

The *incidence graph* of a partial plane is a bipartite graph in which the elements of one part $\mathcal{L}$ are called *lines* and the elements of the other part $\mathcal{P}$ are called *points*. The terminology for incidence graphs is geometric. A point and a line are said to be *incident* if they are adjacent. The incidence graph of a partial plane is clearly a bipartite graph with even girth $g \geq 6$. And the incidence graph of a projective plane is a $(q+1)$-regular graph of girth exactly 6.

A *biaffine plane* is obtained from a projective plane deleting the infinite line and all its points and the infinite point and all its lines. From an algebraic point of view we can define a biaffine plane in the following way.

Definition

Let $\mathbb{F}_q$ be the finite field of order q.

(i) Let $\mathcal{L} = \{1\} \times \mathbb{F}_q \times \mathbb{F}_q$ and $\mathcal{P} = \{0\} \times \mathbb{F}_q \times \mathbb{F}_q$. Then the following set of q^2 lines define a biaffine plane:

$$(m, b)_1 = \{(x, mx + b)_0 : x \in \mathbb{F}_q\} \text{ for all } m, b \in \mathbb{F}_q.$$

(ii) The incidence graph of the biaffine plane is a bipartite graph $B_q = (\mathcal{P}, \mathcal{L})$ which is q-regular, has order $2q^2$, diameter 4, and girth 6 if $q \geq 3$; and girth 8 if $q = 2$.

The diameter of B_q is 4 and the vertices mutually at distance four are the vertices of the sets $V_x = \{(x, y)_0 : y \in \mathbb{F}_q\}$ and $V_m = \{(m, b)_1 : b \in \mathbb{F}_q\}$, for all $x, m \in \mathbb{F}_q$.

The diameter of B_q is 4 and the vertices mutually at distance four are the vertices of the sets $V_x = \{(x, y)_0 : y \in \mathbb{F}_q\}$ and $V_m = \{(m, b)_1 : b \in \mathbb{F}_q\}$, for all $x, m \in \mathbb{F}_q$.

Construction:

The diameter of B_q is 4 and the vertices mutually at distance four are the vertices of the sets $V_x = \{(x, y)_0 : y \in \mathbb{F}_q\}$ and $V_m = \{(m, b)_1 : b \in \mathbb{F}_q\}$, for all $x, m \in \mathbb{F}_q$.

Construction:

Let $q \geq 3$ be an odd prime power. Define a mixed graph G_q such that
$V(G_q) = V(B_q)$, $E(G_q) = E(B_q)$ and
$A(G_q) = \{((m, b)_1, (m, b + i)_1) : i \in S\} \cup \{((x, y)_0, (x, y + j)_0) : j \in -S\}$,
where $S \subset \mathbb{F}_q - 0$, $|S| = (q - 1)/2$ and for all $u, v \in S$, $u + v \neq 0$.

Theorem

The mixed graph G_q defined in the Construction is a mixed graph of diameter 2 with parameters $r = q$, $z = (q-1)/2$ and $2q^2$ vertices.

A dense $(z, r, 2)$-mixed graph having the maximum possible number of vertices is called *optimal*. We denote by $n(z, r, 2)$ the number of vertices in an optimal mixed graph of parameters z and r and diameter 2.

A dense $(z, r, 2)$-mixed graph having the maximum possible number of vertices is called *optimal*. We denote by $n(z, r, 2)$ the number of vertices in an optimal mixed graph of parameters z and r and diameter 2.

In particular for $q = 5$ we construct a mixed graph of order 50, undirected degree 5 and directed degree 2. Bosák proved, in 1979, that there is not exits a mixed graph with these parameters that attains the Moore Bound and it is easy to prove that also does not exists a graph with these parameters, and order 51; then we exhibit an "optimal" graph.

A dense $(z, r, 2)$-mixed graph having the maximum possible number of vertices is called *optimal*. We denote by $n(z, r, 2)$ the number of vertices in an optimal mixed graph of parameters z and r and diameter 2.

In particular for $q = 5$ we construct a mixed graph of order 50, undirected degree 5 and directed degree 2. Bosák proved, in 1979, that there is not exits a mixed graph with these parameters that attains the Moore Bound and it is easy to prove that also does not exists a graph with these parameters, and order 51; then we exhibit an "optimal" graph.

Theorem

For $q = 3$, G_3 is a mixed Moore graph; and for $q = 5$, G_5 is an optimal mixed graph.

Theorem

For $q = 3$, G_3 is a mixed Moore graph; and for $q = 5$, G_5 is an optimal mixed graph.

Idea of the proof: For $q = 3$ it turns out that G_3 has 18 vertices and parameters $r = 3$ and $z = 1$. Since Bosak's graph is unique [Nguyen, Miller, Gimbert, 2008], we obtain that G_3 given in the Construction is isomorphic to Bosak's graph.

For $q = 5$ it turns out that G_5 has 50 vertices and parameters $r = 5$ and $z = 2$. By (1) the upper bound on the number of vertices for this particular case is 52. Let us show that a mixed Moore graph on 52 vertices and parameters $r = 5$ and $z = 2$ cannot exist. Otherwise, by (2) an odd integer c dividing $(4z - 3)(4z + 5) = 65$ exists such that $r = 5 = \frac{1}{4}(c^2 + 3)$.

For $q = 5$ it turns out that G_5 has 50 vertices and parameters $r = 5$ and $z = 2$. By (1) the upper bound on the number of vertices for this particular case is 52. Let us show that a mixed Moore graph on 52 vertices and parameters $r = 5$ and $z = 2$ cannot exist. Otherwise, by (2) an odd integer c dividing $(4z - 3)(4z + 5) = 65$ exists such that $r = 5 = \frac{1}{4}(c^2 + 3)$.But then $c = \sqrt{17}$ which implies that c is not an integer. Therefore the upper bound on the number of vertices must be at most 51. However, 51 is not possible because otherwise deleting the directions of the arcs we would obtain a regular graph of degree $r + 2z = 9$ and 51 vertices. Thus we conclude that the upper bound is 50, yielding that G_5 is an "optimal" mixed graph.

Thank you for your attention.