Graphs similar to strongly regular graphs

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Joint work with Martin Mačaj

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Degree/diameter problem

Definition

The degree/diameter problem is the problem of finding the largest possible graph with given diameter d and given maximum degree k.

Moore bound

- natural upper bound on number of vertices of graph with diameter d and maximum degree k

$$M(k,d) = \begin{cases} 1 + k\frac{(k-1)^d - 1}{k-2}, & \text{if } k > 2, \\ 2d+1, & \text{if } k = 2, \end{cases}$$

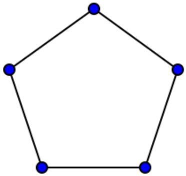
- graphs with diameter 2 : $M(k,2) = k^2 + 1.$

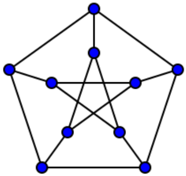
Moore graphs

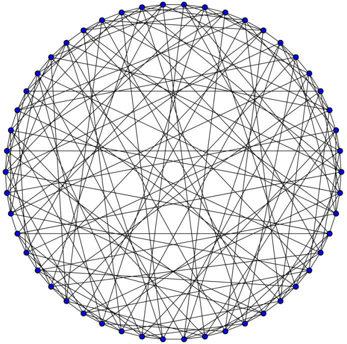
- attain Moore bound $\Rightarrow$ answer to degree/diameter problem

Moore graphs

- attain Moore bound $\Rightarrow$ answer to degree/diameter problem
- (Hoffman, Singleton 1960)
  - if $d = 2$ Moore graphs exist for $k = 2, 3, 7$ and possibly 57
  - if $d = 3$ unique Moore graph for $k = 2$ (heptagon)
- (Damerell 1973, Bannai and Ito 1973) no Moore graphs for $d \geq 3$ and $k \geq 3$







Graphs where $|V(G)| =$ Moore bound -1

- small number of nontrivial Moore graphs $\Rightarrow$ investigation of graphs where $|V(G)| =$ Moore bound -1

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- small number of nontrivial Moore graphs $\Rightarrow$ investigation of graphs where $|V(G)| =$ Moore bound -1
- (Erdös, Fajtlowicz, Hoffman 1980) if $d = 2$ unique graph for $k = 2$ (C_4)
- (Kurosawa and Tsujii 1981, Bannai and Ito 1981)
    - if $k = 2$ only such graphs are C_{2d}
    - no graphs for $k \geq 3$

Methods (Hoffman, Singleton 1960)

Moore graphs with diameter 2

- matrix equation for adjacency matrix A

$$A^2 + A - (k-1)I = J$$

  where

  - I - identity matrix
  - J - all-ones matrix

- analysis of eigenvalues and eigenvectors of A

Methods (Hoffman, Singleton 1960)

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$$\Downarrow$$

- $k = 2, 3, 7$ and possibly 57

Methods (Erdös, Fajtlowicz, Hoffman 1980)

Graphs where $|V(G)| =$ Moore bound -1

- matrix equation for adjacency matrix A

  $$A^2 + A - (k-1)I = J + K,$$

  where K is matrix of 1-factor, which we get as direct sum of matrices 2×2

  $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- analysis of eigenvalues of A

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$$\Downarrow$$

- C_4 or $k = 12$
- analysis of eigenvalues of A^3

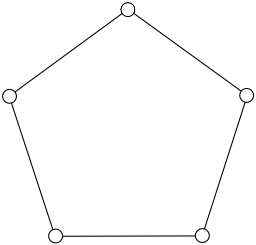
$$\Downarrow$$

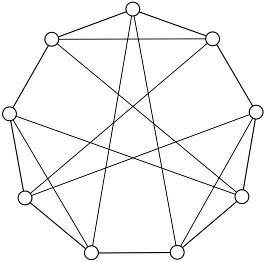
- only C_4

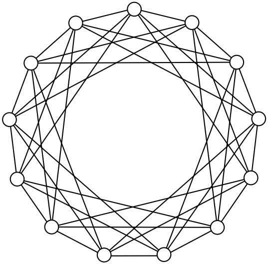
Definition

Graph G is strongly regular with parameters (n, k, a, c) if:

- it has n vertices
- it is k-regular graph
- every two adjacent vertices have a common neighbours
- every two non-adjacent vertices have c common neighbours







Strongly regular graphs

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- Moore graphs with diameter 2 are strongly regular graphs $(n, k, 0, 1)$

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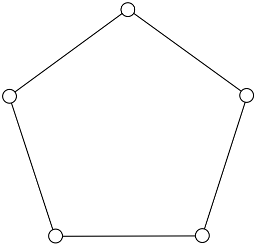

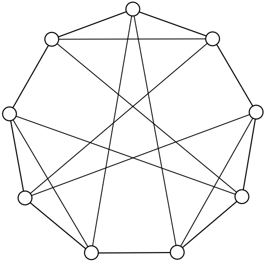

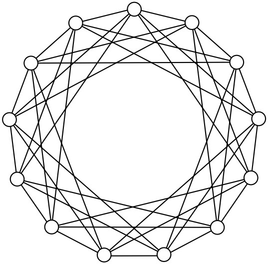

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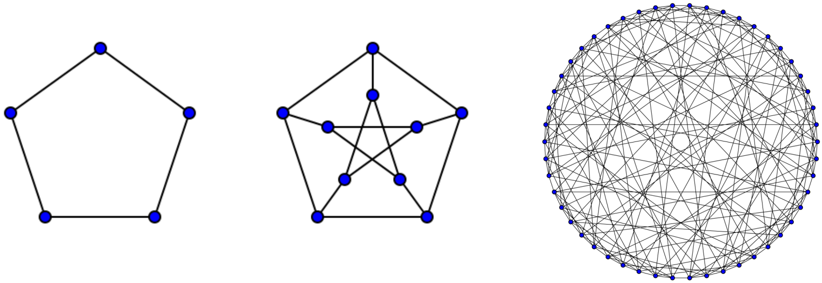

Strongly regular graphs

- Adjacency matrix A of graph satisfies equation:

$$A^2 + (c - a)A + (c - k)I = cJ,$$

  where
    - I - identity matrix
    - J - all-ones matrix
    - A - adjacency matrix of graph
- methods of Hoffman and Singleton

$$\Downarrow$$

Integral criterion (multiplicities of eigenvalues have to be integral)

Generalization of Moore graphs

Moore graphs, i.e. strongly regular graphs with $(n, k, 0, 1)$

$$A^2 + A - (k-1)I = J$$

$$\Downarrow$$

strongly regular graphs (n, k, a, c)

$$A^2 + (c-a)A + (c-k)I = cJ$$

Graphs similar to strongly regular graphs

Erdős, Fajtlowicz and Hoffman

$$A^2 + A - (k-1)I = J + K,$$

where K is matrix of 1-factor

$$\Downarrow$$

Generalization towards strongly regular graphs (we are trying to find graphs satisfying equation):

$$A^2 + (c-a)A + (c-k)I = cJ + K$$

Basic properties

- interesting combinatorial interpretation of our graphs with parameters (n, k, a, c)
  - k-regular graph on n vertices
  - for each vertex v there exists unique vertex, denoted as s_v, such that
    - if v is adjacent with s_v then v and s_v have $a+1$ common neighbours
    - if v is not adjacent with s_v then v and s_v have $c+1$ common neighbours
    - all other vertices, which are neighbours or non-neighbours of v have with vertex v a or c common neighbours respectively
- closed under complement (if graph G is similar to SRG then complement $\bar{G}$ is also similar to SRG)
- parity of ka globally determines whether v is adjacent with s_v or not

Examples

- perfect matchings (corresponding with matrix K)
- complements of perfect matchings ($K_n - K$)

$$\Downarrow$$

imprimitive graphs (all other graphs are primitive)

Eigenvalues of A

$$A^2 + (c-a)A + (c-k)I = cJ + K$$

from this equation and the spectrum of 1-factor K (it has eigenvalues $\{-1, 1\}$)

$$\Downarrow$$

five eigenvalues of A

- k
- λ_1, λ_2 corresponding to 1, which is eigenvalue of K
- θ_1, θ_2 corresponding to -1, which is eigenvalue of K

Necessary conditions for parameters (n, k, a, c)

five equations:

- one from the eigenvalues corresponding to all-ones vector
- two from the spectrum of 1-factor
- one from the trace of A
- one from the trace of A^3

$$
\begin{aligned}
k^2 + (c-a)k + c - k - cn - 1 &= 0\\
m_1 + m_2 - \frac{n}{2} + 1 &= 0\\
n_1 + n_2 - \frac{n}{2} &= 0\\
k + \frac{a-c}{2}(n-1) + \frac{u_1}{2}(m_1 - m_2) + \frac{u_2}{2}(n_1 - n_2) &= 0\\
k^3 + m_1\lambda_1^3 + m_2\lambda_2^3 + n_1\theta_1^3 + n_2\theta_2^3 - akn - \mathrm{st}(\mathrm{KA}) &= 0
\end{aligned}
$$

where m_1, m_2, n_1, n_2 are multiplicities of eigenvalues of A

Simplification of necessary conditions

$$\begin{aligned} 0 &= k^2 + (c-a)k + c - k - cn - 1 \\ x_1 u_1 &= \mathrm{tr}(KA) - 2k + (c-a)\left(\frac{n}{2} - 1\right) \\ x_2 u_2 &= -\mathrm{tr}(KA) + (c-a)\frac{n}{2} \end{aligned}$$

where

- $x_1 = m_1 - m_2$ is difference of multiplicities of eigenvalues λ_1, λ_2
- $x_2 = n_1 - n_2$ is difference of multiplicities of eigenvalues θ_1, θ_2
- u_1 and u_2 depend only on n, k, a, c
- $\mathrm{tr}(KA) = 0$ (up to complement)

Analysis of multiplicities of eigenvalues

4 cases:

1. $x_1 = 0,\ x_2 = 0$
2. $x_1 \neq 0,\ x_2 \neq 0$
3. $x_1 = 0,\ x_2 \neq 0$
4. $x_1 \neq 0,\ x_2 = 0$

where

- $x_1 = 0 \Leftrightarrow m_1 = m_2$
- $x_2 = 0 \Leftrightarrow n_1 = n_2$

Analysis of multiplicities of eigenvalues

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1. $x_1 = 0,\ x_2 = 0 \Rightarrow$ no graphs
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2. $x_1 \neq 0,\ x_2 \neq 0 \Rightarrow$ imprimitive graphs
3. $x_1 = 0,\ x_2 \neq 0 \Rightarrow$ no graphs
4. $x_1 \neq 0,\ x_2 = 0 \Rightarrow$ infinite class of parameters

where

- $x_1 = 0 \Leftrightarrow m_1 = m_2$
- $x_2 = 0 \Leftrightarrow n_1 = n_2$

Case $x_1 \neq 0$, $x_2 = 0$

Transformation of necessary conditions to integral parameters (z, w, y) such that

$$\begin{aligned} n &= z^2(y+2) \\ k &= z^2 + wz \\ a &= wz + 1 \\ c &= wz + 1, \end{aligned}$$

where

$$\begin{array}{cl} z & \text{is even} \\ w & \text{is odd} \\ y & \text{is even} \end{array}$$

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Transformation of necessary conditions to integral parameters (z, w, y) such that

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Modified necessary conditions:

$$y(1 + wz) = w^2 + z^2 - 3$$

Basic solutions

$$y(1 + wz) = w^2 + z^2 - 3$$

- triple (z, w, y) is a solution iff triple $(yz - w, z, y)$ is
- method of descent

$$\Downarrow$$

set of solutions $\{(w^3 - 3w, w, w^2 - 3) | w \geq 2\}$

z	w	y	$a = c$	k	n
18	3	6	55	378	2592
52	4	13	209	2912	40560
110	5	22	551	12650	290400
198	6	33	1189	40392	1372140

Advanced solutions

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complete set of solutions (up to complements)

z	w	y	$a = c$	k	n
18	3	6	55	378	2592
105	18	6	1891	12915	88200
612	105	6	64261	438804	2996352
3567	612	6	2183005	14906493	101787912

Conclusion

- systemic application of trace of the third power of adjacency matrix A
- combinatorial consequences of parity of term ka
- methods of number theory

$$\Downarrow$$

- complete classification of feasible parameters
- existence of primitive graphs remains an open problem

Thank you for your attention

Combinatorial properties

Combinatorial properties of such graph with parameters (n, k, a, c) is

- it has n vertices
- k-regular
- for each vertex v there exists unique vertex, denoted as s_v, such that
    - if v is incident with s_v then v and s_v have $a + 1$ common neighbours
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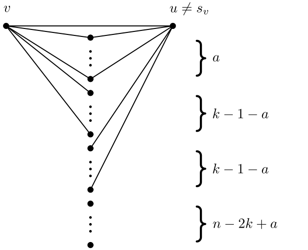
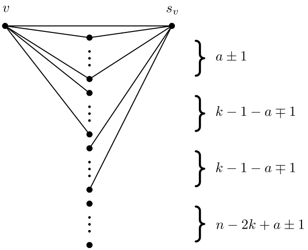

v
$u \neq s_v$
a
$k-1-a$
$k-1-a$
$n-2k+a$

A


v
s_v
$a \pm 1$
$k-1-a \mp 1$
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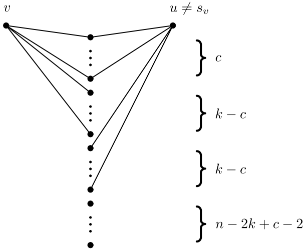
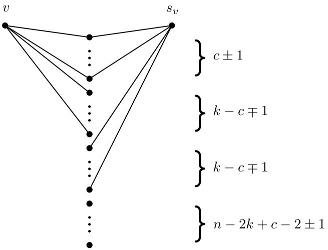
$n-2k+a \pm 1$

B


v
$u \neq s_v$
c
$k-c$
$k-c$
$n-2k+c-2$

C


v
s_v
$c \pm 1$
$k-c \mp 1$
$k-c \mp 1$
$n-2k+c-2 \pm 1$

D