Mixed Cayley graphs of diameter two of order asymptotically approaching the Moore bound

Jana Šiagiová

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The mixed Moore bound for diameter two

In a (regular) mixed \((\Delta, d)\)-graph, every vertex is incident with \(\Delta \geq 1\) undirected edges and there are \(d \geq 1\) darts from and to each vertex.

The order of such a graph of diameter \(2\) is \(\leq (\Delta + d)^2 + d + 1\), generalizing the undirected and directed Moore bounds for diameter \(2\).

Bosák [1979] If a mixed Moore \((\Delta, d)\)-graph of diameter \(2\) exists, then there is a divisor \(t\) of \((4d - 3)(4d + 5)\) such that \(\Delta = (t^2 + 3)/4\).

Our interest is in large Cayley mixed \((\Delta, d)\)-graphs of diameter \(2\).

Motivation:

• Directed Cayley graphs of diameter \(2\) and defect \(1\) exist for \(\infty\) degrees;
• Undirected diameter-\(2\) Moore bound can be approached by Cayley graphs;
• Computer generation of record-large examples uses Cayley graphs.
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Directed graphs with $d \geq 2$

$M_2(0, d) = d^2 + d + 1$; impossible if $d \geq 2$ (e.g. Bridges, Toueg 1980).

Kautz digraphs $L(\vec{K}_n)$:

$d = n - 1$, diam $\geq 2$, order $M_2(0, d) - 1 = d^2 + d$.

By a deep result of Gimbert [2001], they are unique if $d \geq 3$;

Definition: A Cayley digraph $\vec{C}(H, D)$ for a group $H$ and a generating set $D \neq \{id\}$, has vertex set $H$, dart set $(h, hx)$ for $h \in H$, $x \in D$;

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The Kautz digraph $L(\vec{K}_n)$ is a Cayley digraph iff $n$ is a prime power;

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Undirected graphs with $\Delta \geq 3$

In doubt but, by Higman [1960's] not v-trans (hence non-Cayley).

Current best result on large Cayley graphs of diam 2:

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For every $n \geq 1$ there exists a group $H_n$ of order $|H_n| = 2^{2^n - 1}$ and a symmetric generating set $U_n$ of size $\Delta_n = 2^{2^n} + 2^n + 2 - 6$ in $H_n$ such that the (undirected) Cayley graph $C(H_n, U_n)$ has diameter 2.

Since $|H_n| > \Delta_n - 8\Delta_n / 2^n$, we have $|H_n|/M_2(\Delta_n, 0) \to 1$.

In this sense our Cayley graphs asymptotically approach the Moore bound.

Here, $H_n = AGL(1, F) \sim = F + \cdot F^*$ for a Galois field $F$ of order $2^{2^n}$.

The 'majority' of $U_n$ is formed by the set $\{(x, x^2) ; x \in F^*\}$. 
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Let $H$ be a group and let $X, Y$ be disjoint unit-free subsets of $H$ with $X = X - 1$.

The mixed Cayley graph $C(H; X, Y)$ has vertex set $H$: for every vertex $h \in H$ there is an undirected edge joining $h$ with $hx$ for every $x \in X$ and a directed edge from $h$ to $hy$ for every $y \in Y$.

It is $(\Delta, d)$-regular for $\Delta = |X|$, $d = |Y|$; undirected if $Y = \emptyset$, directed if $X = \emptyset$.

Example: Take $F = GF(p^e)$ and the group $H = AGL(1, F) = F^+ \rtimes F^*$, $Y = \{ (ax + b, x); x \in F^* \setminus (-1)^p \}$, $X = \{ (a(-1)^p + b, (-1)^p) \}$, $a + b \neq 0$.

Then, $C(H; X, Y)$ is the Cayley mixed Moore graph obtained from the Kautz digraph $L(\vec{K}_n)$, $n = p^e$, by suppressing digons.

Note: This replacement works for all $n$, but we focus on the Cayley case.
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Example: Take $F = \text{GF}(p^e)$ and the group $H = AGL(1, F) = F^+ \rtimes F^*$, $Y = \{(ax + b, x); x \in F^* \setminus (-1)p\}$, $X = \{(a(−1)p + b, (−1)p)\}$, $a + b \neq 0$. Then, $C(H; X, Y)$ is the Cayley mixed Moore graph obtained from the Kautz digraph $L(\vec{K}_n)$, $n = p^e$, by suppressing digons. Note: This replacement works for all $n$, but we focus on the Cayley case.
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Mixed Cayley graphs of diameter two of order asymptotically approaching the Moore bound
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Results for mixed Cayley graphs of diameter two

Theorem 1
For every \( c \) such that \( 0 \leq c \leq \infty \) there exists an infinite sequence of mixed \((\Delta_n, d_n)\)-regular Cayley graphs \( G_n \) of diameter 2 such that
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\frac{|G_n|}{M^2(\Delta_n, d_n)} \to 1 \quad \text{and} \quad \frac{\Delta_n}{d_n} \to c \quad \text{as} \quad n \to \infty.
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Proof (by 'cheating'): Replace \( \approx \frac{1}{1 + c} \) undirected edges in the current best construction of Cayley graphs of degree \( \Delta_n \) and diameter 2 by digons!

Instead, one may replace 'a few' darts by edges:

If \( G_n = \vec{C}(H_n, D_n) \) are Cayley digraphs of diameter 2 and degree \( k_n \) with \( |G_n|/k_n^2 \to 1 \) as \( n \to \infty \), take \( U_n \subset D_n \) such that \( |U_n| = o(k_n) \).

Letting \( X_n = U_n \cup U_{n-1} \), \( Y_n = D_n \setminus U_n \), with \( |X_n| = \Delta_n \) and \( |Y_n| = d_n \), and considering the mixed Cayley graphs \( G'_n = C(H_n, X_n, Y_n) \) of diam 2 we still have
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Proper strengthenings of Theorem 1 should be concerned with simple and irredundant mixed Cayley graphs (with no digons and in which removal of any generator increases the diameter). This appears to be much harder.

Theorem 2

There is an infinite sequence of simple and irredundant mixed \((\Delta_n, d_n)\)-regular Cayley graphs \(G_n\) of diameter 2 such that \(4d_n/\Delta_n^2 \to 1\) and \(|G_n|/M_2(\Delta_n, d_n) \to 1\) as \(n \to \infty\).

Proof: Finite fields, affine groups and very particular generating sets. A full extension of Theorem 2 to simple and irredundant mixed Cayley graphs remains open – lack of suitable generating sets for Cayley graphs and digraphs approaching the Moore bound for diameter 2.
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