A proof of the non existence of a mixed Moore graph of order 486

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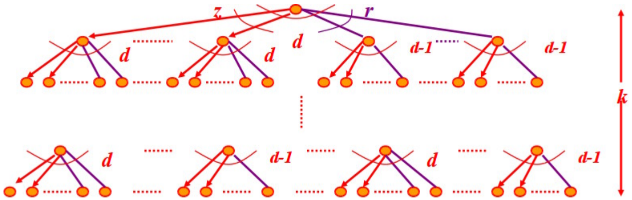

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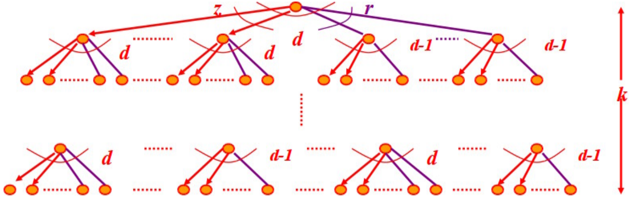

$$n = 1 + d + zd + r(d-1) + \cdots + zd^{k-1} + r(d-1)^{k-1}, \text{ where } d = r + z$$

[Nguyen, Miller, 2008]

Moore graphs and Directed Moore graphs

- $r = 0$ (no edges) $\rightarrow$ *Directed Moore graphs* [Plesník and Znám '74, Bridges and Toueg '80] only exists for:
  - $k = 1$ (complete digraph K_{z+1})
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- $z = 0$ (no arcs) $\rightarrow$ *Moore graphs* [Banai and Ito '73, Hoffman and Singleton '60, Damerell '73] only exists for:
  - $k = 1$ and $r \geq 1$ (Complete graph K_{r+1});
  - $k \geq 3$ and $r = 2$ (Cycle graph C_{2k+1});
  - $k = 2$ and $r = 2$ (Cycle graph C_5);
  - $k = 2$ and $r = 3$ (Petersen graph);
  - $k = 2$ and $r = 7$ (Hoffman-Singleton graph);
  - $k = 2$ and $r = 57$ (?)

Proper Mixed Moore graphs

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Theorem (Nguyen,Gimbert,Miller, 2007)

Proper Mixed Moore graphs of diameter $k \geq 3$ do not exist.

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Theorem (Nguyen,Gimbert,Miller, 2007)

Proper Mixed Moore graphs of diameter $k \geq 3$ do not exist.

Bosák necessary condition for the existence of mixed Moore graphs of diameter $k = 2$:

$$\textit{Exists odd } c \in \mathbb{Z} \textit{ such that } r = \frac{1}{4}(c^2 + 3) \textit{ and } c | (4z - 3)(4z + 5)$$

Matrix equation of Mixed Moore graphs of diameter two

Let A be the adjacency matrix of a mixed Moore graph of diameter two, then there is a unique trail of length ≤ 2 between any pair of vertices, that is,

$$I + A + A^2 = J + rI$$

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The characteristic polynomial of A is given by $\Phi_A(x) = (x-d)(x-\alpha)^a(x-\beta)^b$ where $a+b = n-1$, $\alpha = \frac{-1+\sqrt{4r-3}}{2}$ and $\beta = \frac{-1-\sqrt{4r-3}}{2}$.

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$$\mathrm{Tr}(A^m) = n \cdot \#\{\text{ closed walks of length } m \text{ of any vertex }\}$$

Since a and b must be non-negative integers, The two ways calculation of $\mathrm{Tr}(A^m)$ can be used to determine necessary conditions on the existence of mixed Moore graphs.

Proposition

$\mathrm{Tr}(A) = 0 \Leftrightarrow \exists$ *odd* $c \in \mathbb{Z}$ *such that* $r = \frac{1}{4}(c^2+3)$ *and* $c|(4z-3)(4z-5)$

Cases of order $n \leq 110$

Exists odd $c \in \mathbb{Z}$ such that $r = \frac{1}{4}(c^2+3)$ and $c|(4z-3)(4z+5)$

Moore bound	r	z	d	Existence	Uniqueness
6	1	1	2	$\mathrm{Ka}(2,2)$	Yes
12	1	2	3	$\mathrm{Ka}(3,2)$	Yes
18	3	1	4	Bosák	Yes
20	1	3	4	$\mathrm{Ka}(4,2)$	Yes
30	1	4	5	$\mathrm{Ka}(5,2)$	Yes
40	3	3	6	?	?
42	1	5	6	$\mathrm{Ka}(6,2)$	Yes
54	3	4	7	?	?
56	1	6	7	$\mathrm{Ka}(7,2)$	Yes
72	1	7	8	$\mathrm{Ka}(8,2)$	Yes
84	7	2	9	?	?
88	3	6	9	?	?
90	1	8	9	$\mathrm{Ka}(9,2)$	Yes
108	3	7	10	Jørgensen	No
110	1	9	10	$\mathrm{Ka}(10,2)$	Yes

mixed Moore graphs of directed degree one

When $z = 1$ there are only three possibilities for r, according to Bosàk eq.

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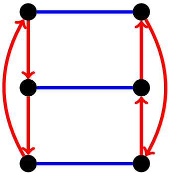

$Ka(2,2)$

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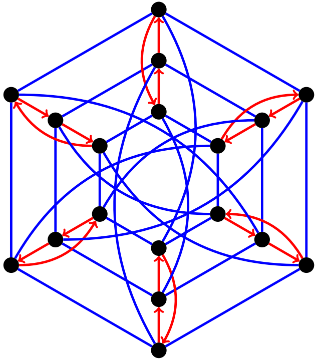

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3. $r = 21$. A mixed Moore graph G_{21} of order $n = 486$,

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Properties of a mixed Moore graph G_r of directed degree one.

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- *Every vertex of G_r belongs to a unique directed cycle of length 3.*
- *There is a partition $\Upsilon = \{V_1, V_2, \ldots, V_{\frac{n}{3}}\}$ of the vertex set V of G_r, $|V| = n$, such that the subgraph of G_r induced by V_i is a directed cycle of length 3, for all $1 \leq i \leq \frac{n}{3}$.*

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Proposition

Let V and V' be two different sets of the partition Υ of G_r. Then, the subgraph of G_r induced by $V \cup V'$ is one of the following (mixed) graphs:

- *The union of two directed cycles of length 3.*
- *The Kautz digraph $Ka(2,2)$.*

The proof of the nonexistence of G_{21}

We define the (undirected) *reduced graph* of G_r, denoted by G_r^*, as follows:

- The vertex set of G_r^* is the partition set $\Upsilon = \{V_1, V_2, \ldots, V_{\frac{n}{3}}\}$,
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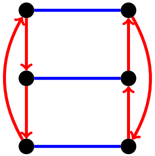

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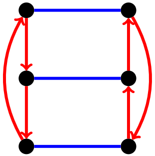

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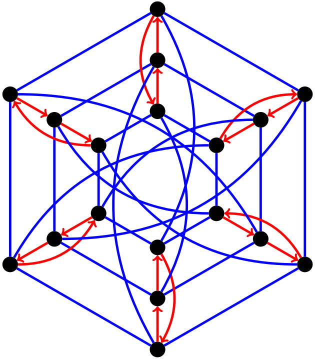

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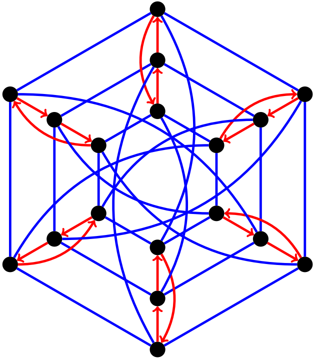

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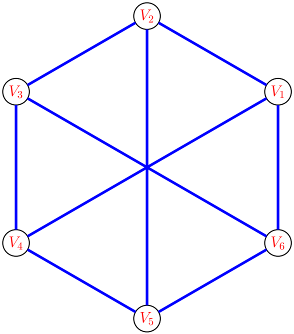

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Properties of G_r^*:

Observation

G_r^* is a regular (undirected) graph of degree r containing $\frac{n}{3}$ vertices, where $n = r^2 + 2r + 3$. Moreover, $\mathrm{diam}(G_r^*) = 2$ *for $r > 1$ and* $\mathrm{diam}(G_1^*) = 1$.

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Let V and V' be two non-adjacent vertices of G_r^*. Then, V and V' have exactly three common neighbours.

The proof of the nonexistence of G_{21}

Theorem

G_r^* exists if and only if $r = 1$ or $r = 3$.

Sketch of the proof:
Let $A = (a_{ij})$ be the adj. matrix of G_r^*, $r > 1$, and $A^2 = (b_{ij})$. Then,

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1. $b_{ii} = r$ for all $1 \leq i \leq \frac{n}{3}$;

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Hence, every row of A^2 contains r 0's, $\frac{n}{3} - 1 - r$ 3's, and just one r. This means $r + 3(\frac{n}{3} - 1 - r)$ is an eigenvalue of A^2 corresponding to j. Taking into account that r is an eigenvalue of A corresponding to j, then

$$r^2 = r + 3(\frac{n}{3} - 1 - r)$$

must hold. The solution to this equation is precisely $r = 3$.

Open Problems and research lines

(Open Problem)

Discover new mixed Moore graphs (or prove that they do not exist).

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Research lines:

- **Algorithmic search:** We develop an algorithm in Python (using NetworkX library) in order to find mixed Moore graphs $M_{r,z}$. To this end, we give:
  - Specific functions for mixed graphs.
  - The generating tree $T_{r,z}$ of any mixed Moore graph.
  - Specific arcs/edges to $T_{r,z}$ to find a generating subgraph $G_{r,z}$ of $M_{r,z}$.
  - An heuristic to complete the regularity of $G_{r,z}$. Then check if $G_{r,z}$ is a mixed Moore graph.

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Research lines:

- **Particular cases:** Try different pairs (r, z) satisfying Bosák equation.
- **Related problems:** We have studied the underlying undirected graph UG_r of a mixed Moore graph with $z = 1$ and we find out that

Proposition

UG_r is a distance regular graph of diameter 4 with intersection array:

$$\begin{pmatrix} r & r-1 & 2 & 1 \\ 1 & 1 & r-1 & r \end{pmatrix}$$

In particular, the spectra of UG_{21} is $-6^{56}, -\sqrt{21}^{162}, 3^{105}, \sqrt{21}^{162}, 21^{1}$.

Does this distance regular graph exist?

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Research lines:

- **Cayley digraphs:** Construction techniques given by groups.
    - Almost every known mixed Moore graphs is a Cayley digraph.
    - We use *Sage* in order to work with the *SmallGroups* library (*GAP*) together with *NetworkX* library (*Python*). For an small group of order n, we generate all the 'feasible' generating sets that gives a mixed graph with undirected degree r and directed degree z. Then we compute the diameter of these digraphs. We find out that

Proposition

A mixed Moore graph of order 40 *must be a non-Cayley digraph.*
A mixed Moore graph of order 54 *must be a non-Cayley digraph.*

Open Problems and research lines

Research lines:

- **Matrix construction:** Try to generate adjacency matrices A of mixed Moore graphs
  - We know A partially.
  - There are strong conditions for A to be the adjacency matrix of a mixed Moore graph.
  - Try to construct A for small values of n.

Open Problems

- Give more necessary conditions for the existence of mixed Moore graphs.
- Discover new mixed Moore graphs (or prove that they do not exist for specific cases).
- Any mixed Moore digraph is a directed strongly regular graph with $\lambda = 1$ and $\mu = 0$, where

$$A^2 = rI + \lambda A + \mu(J - A - I)$$

  Can we say something new using this point of view?
- Are there non-Cayley proper mixed Moore digraphs for $r > 1$?

Devoted to our friend *Joan Gimbert*. He would loved to see this proof.