The degree-diameter problem for circulant graphs of degree 8 and 9

Rob Lewis

July 2014

Some definitions

A circulant graph $X = \mathrm{Cay}(\mathbb{Z}_n, C)$ is the Cayley graph of a cyclic group $\mathbb{Z}_n$ where $C \subset \mathbb{Z}_n \setminus \{0\}$ is the connection set: (i, j) is an edge of X iff $j - i \in C$.

X is vertex-transitive and has degree $|C|$. X is undirected iff C is inverse-closed.

Notation:

- d degree
- k diameter
- n number of vertices / order of the graph
- f dimension $= \begin{cases} d/2 & d \text{ even} \\ (d-1)/2 & d \text{ odd} \end{cases}$ / # of independent generators

$CC(d,k)$ maximum size for an undirected Cayley graph of a cyclic group

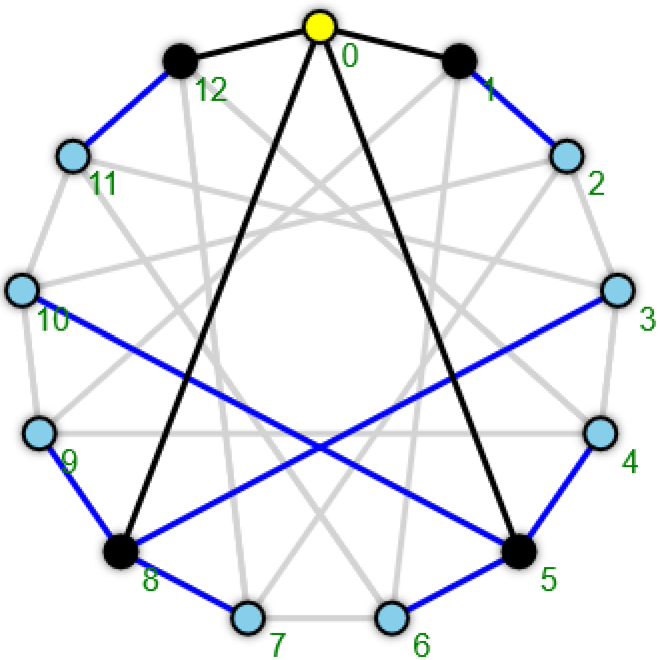
$AC(d,k)$ maximum size for an undirected Cayley graph of an Abelian group

$M(d,k)$ the Moore bound for any undirected graph

$M_{AC}(d,k)$ an upper bound for $AC(d,k)$, sharper than $M(d,k)$

$M_{CC}(d,k)$ a conjectured upper bound for $CC(d,k)$, sharper than $M_{AC}(d,k)$

Example:
dimension $f = 2$, degree $d = 4$,
diameter $k = 2$, order $n = 13$

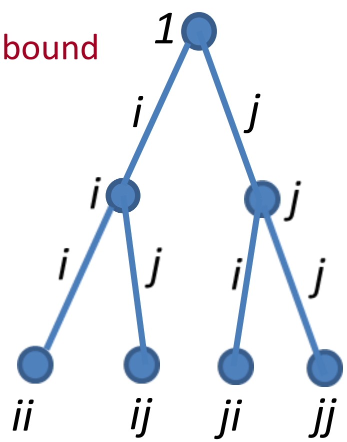


Connection set $C = \{1, 5, 8, 12\} = \{\pm 1, \pm 5\}$
Generator set $\{1, 5\}$
Distance from Vertex 0: black = 1, blue = 2

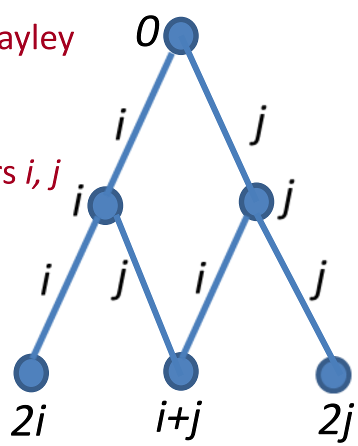
So $CC(4, 2) \geq 13$
Moore bound $M(4, 2) = 17$
Is $CC(4, 2) = 13$? Yes

Sharper upper bound for Abelian Cayley graphs of degree $d \geq 3$, $M_{AC}(d,k)$

Moore bound



Abelian Cayley graph
A pair of generators *i, j*



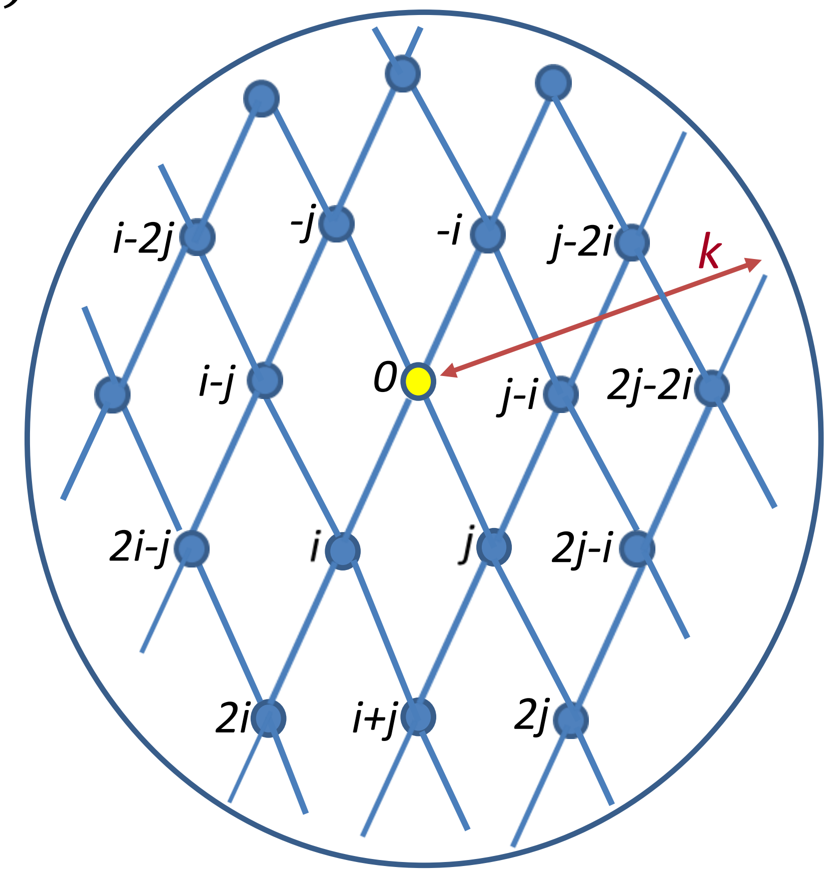
Forms an *f*-dimensional Lee sphere of radius *k* (Manhattan metric).

Example: *f*=2

The vertices in the lattice will repeat.

Order of sphere

$$S(f,k) = \sum_{i=0}^{f} 2^i \binom{f}{i}\binom{k}{i}$$



An upper bound for the order of Abelian Cayley graphs is

$$M_{AC}(d,k) = \begin{cases} S(f,k) & d \text{ even}, f = d/2 \\ S(f,k) + S(f,k-1) & d \text{ odd}, f = (d-1)/2 \end{cases}$$

Upper bound, $M_{AC}(d,k)$

$$M_{AC}(d,k)=\begin{cases}\dfrac{2^f}{f!}k^f+\dfrac{2^{f-1}}{(f-1)!}k^{f-1}+O\left(k^{f-2}\right) & d \text{ even}, f=d/2\\ \dfrac{2^{f+1}}{f!}k^f+O\left(k^{f-2}\right) & d \text{ odd}, f=(d-1)/2\end{cases}$$

Now consider the case $k\equiv 0 \pmod{f}$.

Define $a=\frac{4}{f}k$ and let the vector $\left(c_f\ c_{f-1}\ \ldots\ c_1\ c_0\right)$ represent the polynomial

$$c_f a^f+c_{f-1}a^{f-1}+\cdots c_1 a+c_0\ .$$

Then

$$M_{AC}(d,k)=\begin{cases}\dfrac{f^f}{2^{f-1}f!}\left(\tfrac{1}{2}\ 1\ \ldots\right) & d \text{ even}, f=d/2\\ \dfrac{f^f}{2^{f-1}f!}\left(1\ \ 0\ \ \ldots\right) & d \text{ odd}, f=(d-1)/2\end{cases}$$

Extremal and largest known circulant graphs of degree 2 to 9 for arbitrary diameter k

Dimension f	Degree d	Order, n	Proven extremal	Source
1	2	$2k+1$	All k	
1	3	$4k$	All k	
2	4	$2k^2+2k+1$	All k	
2	5	$4k^2$	All k	
3	6	$(32k^3+48k^2+54k+27)/27 \quad k \equiv 0 \pmod 3$ $(32k^3+48k^2+78k+31)/27 \quad k \equiv 1 \pmod 3$ $(32k^3+48k^2+54k+11)/27 \quad k \equiv 2 \pmod 3$	$k \leq 18$	Dougherty & Faber, 2004
3	7	$(64k^3+108k)/27 \quad k \equiv 0 \pmod 3$ $(64k^3+60k-16)/27 \quad k \equiv 1 \pmod 3$ $(64k^3+60k+16)/27 \quad k \equiv 2 \pmod 3$	$k \leq 10$	Dougherty & Faber, 2004
4	8	$(k^4+2k^3+6k^2+4k)/2 \quad k \equiv 0 \pmod 2$ $(k^4+2k^3+6k^2+6k+1)/2 \quad k \equiv 1 \pmod 2$	$3 \leq k \leq 7$	Lewis, 2013
4	9	$k^4+3k^2+2k \quad k \equiv 0 \pmod 2$ $k^4+3k^2 \quad k \equiv 1 \pmod 2$	$5 \leq k \leq 6$	Lewis, 2013

All these graphs have odd girth equal to $2k+1$, maximal

For largest known graphs of degree $d = 8$ there is just one solution up to isomorphism

	$k \equiv 0 \pmod 2$	$k \equiv 1 \pmod 2$
Order of graph	$(k^4 + 2k^3 + 6k^2 + 4k)/2$	$(k^4 + 2k^3 + 6k^2 + 6k + 1)/2$
Gen set $g1$	1	1
$g2$	$(k^3 + 2k^2 + 6k + 2)/2$	$(k^3 + k^2 + 5k + 3)/2$
$g3$	$(k^4 + 4k^2 - 8k)/4$	$(k^4 + 2k^2 - 8k - 11)/4$
$g4$	$(k^4 + 4k^2 - 6k)/4$	$(k^4 + 2k^2 - 4k - 7)/4$

The connection set is $\{\pm 1, \pm g2, \pm g3, \pm g4\}$

- Proven to exist for all k by constructing a four-dimensional lattice tiling (same method as Dougherty & Faber)
- Largest known solution for $k \geq 3$
- Proven extremal by computer search up to $M_{AC}(8, k)$ for $3 \leq k \leq 7$

For largest known graphs of degree $d = 9$ there is one solution for even diameter k, and two for odd

The prime solution valid for all k

	$k \equiv 0 \pmod 2$	$k \equiv 1 \pmod 4$	$k \equiv 3 \pmod 4$
Order of graph	$k^4 + 3k^2 + 2k$	$k^4 + 3k^2$	$k^4 + 3k^2$
Gen set $g1$	1	1	1
$g2$	$k + 1$	k	k
$g3$	$(k^4 - k^3 + 2k^2 - 2)/2$	$(k^4 + k^3 + k^2 + 3k - 2)/4$	$(k^4 - k^3 + k^2 - 3k - 2)/4$
$g4$	$(k^4 - k^3 + 4k^2 - 2)/2$	$(k^4 + k^3 + 5k^2 + 3k + 2)/4$	$(k^4 - k^3 + 5k^2 - 3k + 2)/4$

The connection set is $\{\pm 1, \pm g2, \pm g3, \pm g4, n/2\}$

This graph is not yet proven to exist for all k
Largest known solution for $k \geq 5$
Proven extremal by computer search up to $M_{AC}(9, k)$ for $5 \leq k \leq 6$

The second degree 9 solution for odd k

Diameter, k	Generator set 1	Generator set 2
$k \equiv 1 \pmod{14}$	1 $(k^4+k^3+5k^2)/7$ $(k^4+k^3+5k^2+7k+7)/7$ $(3k^4+3k^3+8k^2+7k)/7$	1 $(k^4-3k^3+2k^2-7k)/7$ $(2k^4+k^3+4k^2)/7$ $(2k^4+k^3+4k^2+7k-7)/7$
$k \equiv 3 \pmod{14}$	1 $(k^4-k^3+k^2-7k-7)/7$ $(k^4-k^3+k^2)/7$ $(3k^4-3k^3+10k^2-7k)/7$	1 $(2k^4-3k^3+5k^2-7k)/7$ $(3k^4-k^3+11k^2-7k+7)/7$ $(3k^4-k^3+11k^2)/7$
$k \equiv 5 \pmod{14}$	1 $(k^4-3k^3+4k^2-7)/7$ $(2k^4+k^3+8k^2)/7$ $(2k^4+k^3+8k^2+7k+7)/7$	*
$k \equiv 7 \pmod{14}$	1 $(2k^4-3k^3+7k^2-7k)/7$ $(3k^4-k^3+7k^2-7k-7)/7$ $(3k^4-k^3+7k^2)/7$	1 $(2k^4+3k^3+7k^2+7k)/7$ $(3k^4+k^3+7k^2)/7$ $(3k^4+k^3+7k^2+7k-7)/7$
$k \equiv 9 \pmod{14}$	*	1 $(k^4+3k^3+4k^2+7k)/7$ $(2k^4-k^3+8k^2-7k+7)/7$ $(2k^4-k^3+8k^2)/7$
$k \equiv 11 \pmod{14}$	1 $(2k^4+3k^3+5k^2+7k)/7$ $(3k^4+k^3+11k^2)/7$ $(3k^4+k^3+11k^2+7k+7)/7$	1 $(k^4+k^3+k^2)/7$ $(k^4+k^3+k^2+7k-7)/7$ $(3k^4+3k^3+10k^2+7k)/7$
$k \equiv 13 \pmod{14}$	1 $(k^4+3k^3+2k^2+7)/7$ $(2k^4-k^3+4k^2-7k-7)/7$ $(2k^4-k^3+4k^2)/7$	1 $(k^4-k^3+5k^2-7k+7)/7$ $(k^4-k^3+5k^2)/7$ $(3k^4-3k^3+8k^2-7k)/7$

* No solutions for $k \equiv 9 \pmod{14}$ for generator set 1 or for $k \equiv 5 \pmod{14}$ for generator set 2

Extremal and largest known circulant graphs compared with $M_{AC}(d,k)$

Consider again the case $k \equiv 0 \pmod{f}$ and the definition: $a = \frac{4}{f}k$.

Then for example, for $f = 1, d = 2$: $n = 2k + 1 = 2\frac{f}{4}a + 1 = \frac{1}{2}a + 1 = \begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix}$

Dimension f	Even degree d	Order n	Odd degree d	Order n	$\frac{f^f}{2^{f-1}f!}$
1	2	$\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix}$	3	$\begin{pmatrix} 1 & 0 \end{pmatrix}$	1
2	4	$\begin{pmatrix} \frac{1}{2} & 1 & 1 \end{pmatrix}$	5	$\begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$	1
3	6	$\begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} & 1 \end{pmatrix}$	7	$\begin{pmatrix} 1 & 0 & 3 & 0 \end{pmatrix}$	$9/8$
4	8	$\begin{pmatrix} \frac{1}{2} & 1 & 3 & 2 & 0 \end{pmatrix}$	9	$\begin{pmatrix} 1 & 0 & 3 & 2 & 0 \end{pmatrix}$	$4/3$
$M_{AC}(d,k)$		$\begin{pmatrix} \frac{1}{2} & 1 & \dots & \end{pmatrix} \times \frac{f^f}{2^{f-1}f!}$		$\begin{pmatrix} 1 & 0 & \dots & \end{pmatrix} \times \frac{f^f}{2^{f-1}f!}$	$625/384$ $(f = 5)$
Lower bound	Chen & Jia, 1993	$\begin{pmatrix} \frac{1}{2} & \dots & \end{pmatrix}$			$\sim 10^{12}$ $(f = 100)$
$M_{CC}(d,k)$	Conjecture	$\begin{pmatrix} \frac{1}{2} & 1 & \dots & \end{pmatrix}$		$\begin{pmatrix} 1 & 0 & \dots & \end{pmatrix}$	

Conclusion: Some conjectures

Initial conjectures

1) The degree 6/7 graphs by Dougherty & Faber and my degree 8/9 graphs are extremal for all greater diameters.

2) It should be possible to establish a much sharper upper bound than $M_{AC}(d,k)$ for circulant graphs, $M_{CC}(d,k) = \left(\frac{1}{2} \quad 1 \quad \dots \qquad \right)$ for even degree, $(1 \quad 0 \quad \dots \qquad)$ for odd.

3) There exists a family of extremal circulant graphs for any degree d with order $n = \left(\frac{1}{2} \quad 1 \quad \dots \qquad \right)$ for even degree, $(1 \quad 0 \quad \dots \qquad)$ for odd.

4) These graphs all have odd girth $2k + 1$.

Revised conjectures (work in progress)

3a) For dimension $f \geq 5$ and even degree, there are no families with order $n = \left(\frac{1}{2} \quad 1 \quad \dots \qquad \right)$. The best achievable is $\left(\frac{1}{2} \quad \frac{1}{2} \dots \qquad \right)$.

4a) For dimension $f \geq 5$, extremal circulant graphs all have odd girth less than $2k + 1$. The highest achievable is $2k - 1$.