Techniques for Constructing Small Regular Graphs of Given Girth and Related Topics

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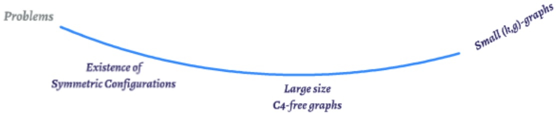
Problems
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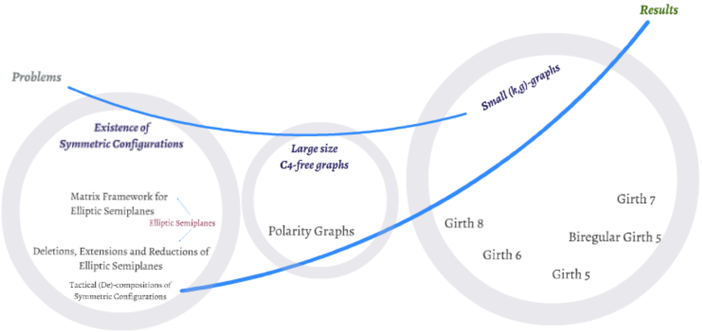
Symmetric Configurations
Large size
C4-free graphs
Small (k,g)-graphs



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Girth 8
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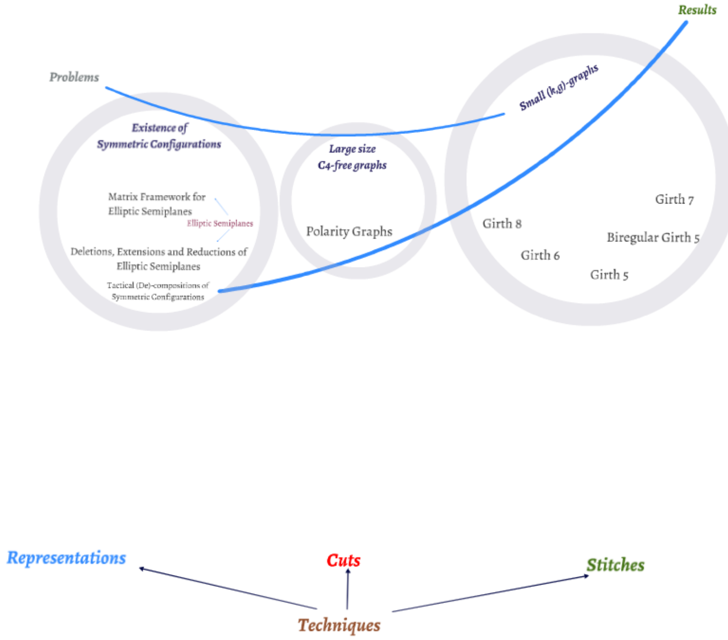
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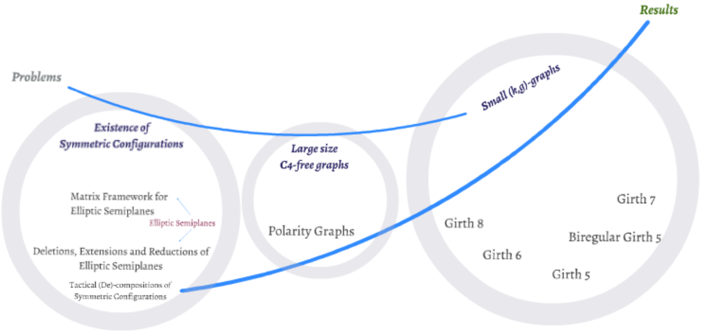
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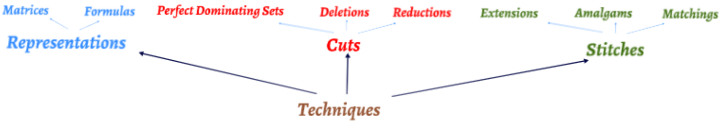
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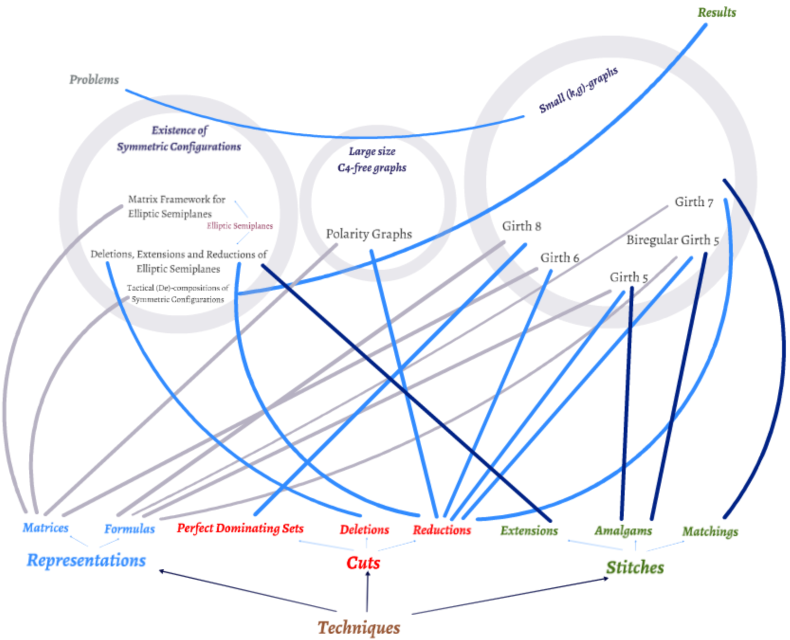
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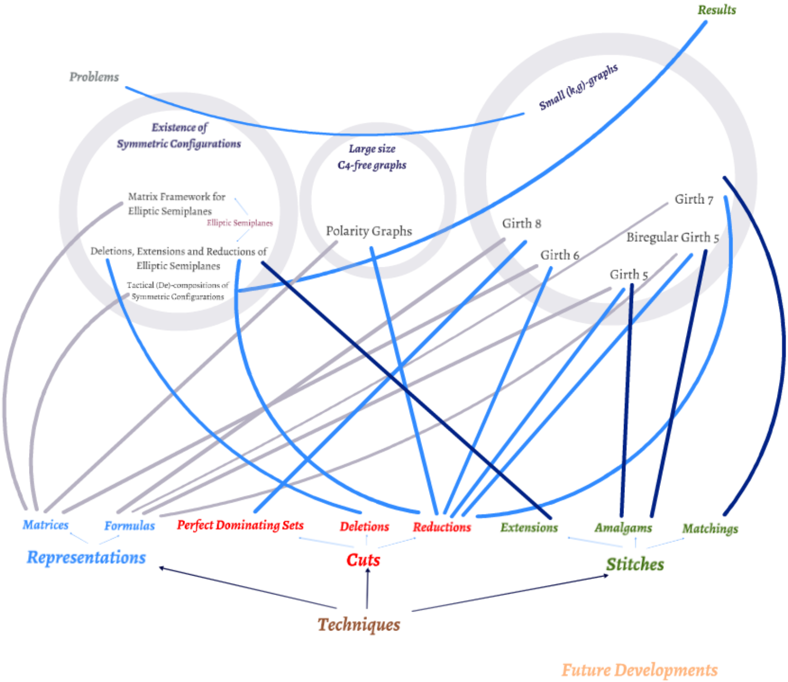

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- C_4–free graphs of large size (Polarity graphs)

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- Existence of symmetric configurations.

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- Many authors are trying to construct cages, or at least smaller (k, g)–graphs than previously known ones.
- We denote by $rec(k, g)$ the order of smallest known (k, g)–graphs.

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The polarity graph $G(\mathbb{P}, \pi)$ of a finite projective plane $\mathbb{P}$ and a polarity π of $\mathbb{P}$ is a graph whose vertex set is the set of points of $\mathbb{P}$ and whose edge set is $\{pp' : p \in \pi(p')\}$.

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- The polarity is said to be orthogonal if this bound is sharp.

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Question

What is the value of $ex(n; C_4)$ for a given n?

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- The *deficiency* of a symmetric configuration $\mathcal{C}$ is $d := v - k^2 + k - 1$. The deficiency is zero if and only if $\mathcal{C}$ is a finite projective plane.

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Which are the types v_k for which a symmetric configuration exists?

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- Existence of new (construction of) symmetric configurations.

Techniques

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- Representations
  - Matrices: Cyclic Schemes and Blow Up
  - Formulas

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- Representations
- Cuts
  - Perfect Dominating Sets,
  - Deletions,
  - Reductions.

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- Representations
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- Stitches
  - Extensions,
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Techniques

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- Applied over:
  - $(q+1,6)$–cages $=$ incidence graphs of Finite Projective Planes,
  - $(q+1,8)$–cages $=$ incidence graphs of Generalized Quadrangles,
  - Elliptic Semiplanes.

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An Elliptic Semiplane *is an n_k configuration satisfying the axiom of parallels: given a non–incident point-line pair (p, l),*

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- Through this axiom, the Elliptic Semiplane is partitioned into parallel classes, all of the same size, say m, and it has $n(n+1)+m$ points and lines.
- Dembowski provided a classification of elliptic semiplanes in types called $\mathcal{O}$, $\mathcal{C}$, $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{B}$.

Classification of Elliptic Semiplanes: Demboski (1968)

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  This subset $\mathcal{B}(p, l)$ is called a flag.
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- *($\mathcal{D}$)* $\mathcal{S} = \mathcal{P} - \mathcal{B}$ and $m = n + 1 - \sqrt{n+1}$
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A Baer subplane $\mathcal{B}$ is a proper subplane of a projective plane such that:

- For every point not in $\mathcal{B}$ there exists a unique line in $\mathcal{B}$ containing it
- For every line not in $\mathcal{B}$ there exists a unique point of $\mathcal{B}$ in it

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- *($\mathcal{B}$)* $m < n + 1 - \sqrt{n+1}$ [Baker 1977]: 45 points, order 6, $m = 3$

We will mainly use elliptic semiplanes of types $\mathcal{C}$, $\mathcal{L}$ and $\mathcal{D}$

Configuration Existence Spectrum

Elliptic Semiplanes provide many symmetric configurations

$k\backslash d$	0	1	2	3	4	5	6	7	8	9
3	7	8	9	10	11	12	13	14	15	16
4	13	14	15	16	17	18	19	20	21	22
5	21	22	23	24	25	26	27	28	29	30
6	31	32	33	34	35	36	37	38	39	40
7	43	44	45			48	49	50	51	52
8	57	58					63	64	65	66
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Projective planes

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Projective planes   Non existence

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6	31	32	33	34	35	36	37	38	39	40
7	43	44	45			48	49	50	51	52
8	57	58					63	64	65	66
9	73	74				78		80	81	
10	91	92						98		
11	111									120
12	133	134	135							

Projective planes

Non existence

Sporadic configurations

Configuration Existence Spectrum

Elliptic Semiplanes provide many symmetric configurations

$k \backslash d$	0	1	2	3	4	5	6	7	8	9
3	7	8	9	10	11	12	13	14	15	16
4	13	14	15	16	17	18	19	20	21	22
5	21	22	23	24	25	26	27	28	29	30
6	31	32	33	34	35	36	37	38	39	40
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Flag Diagonal: Elliptic Semiplanes of type $\mathcal{C}$

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3	7	8	9	10	11	12	13	14	15	16
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Flag Diagonal: Elliptic Semiplanes of type $\mathcal{C}$

Antiflag Diagonal: Elliptic Semiplanes of type $\mathcal{L}$

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Golomb configurations

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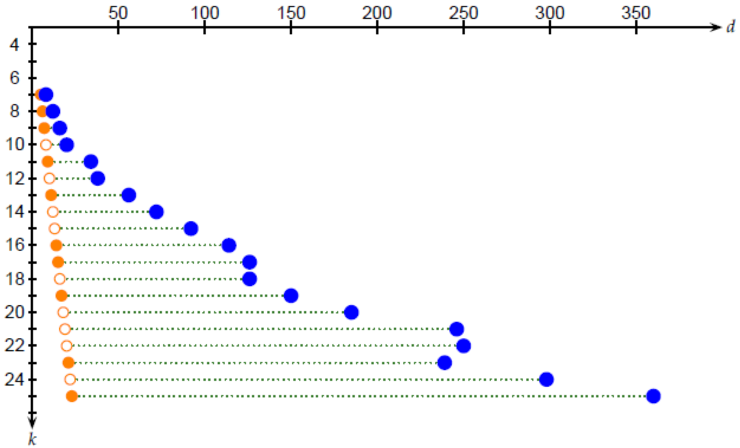
Golomb configurations Gropp's Theorem 1990

Configuration Existence Spectrum: in other scale



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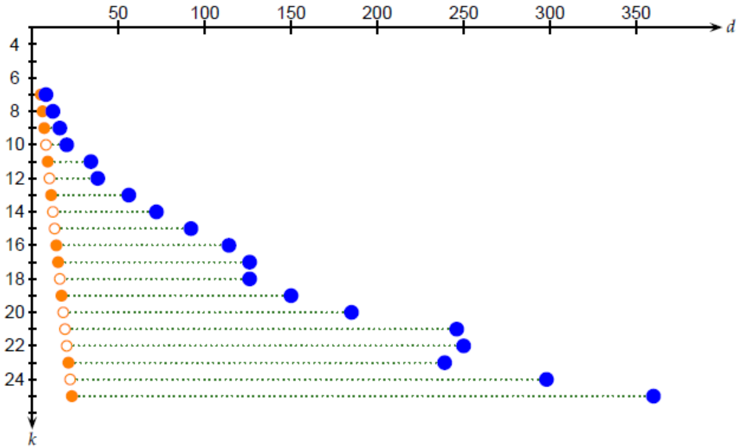

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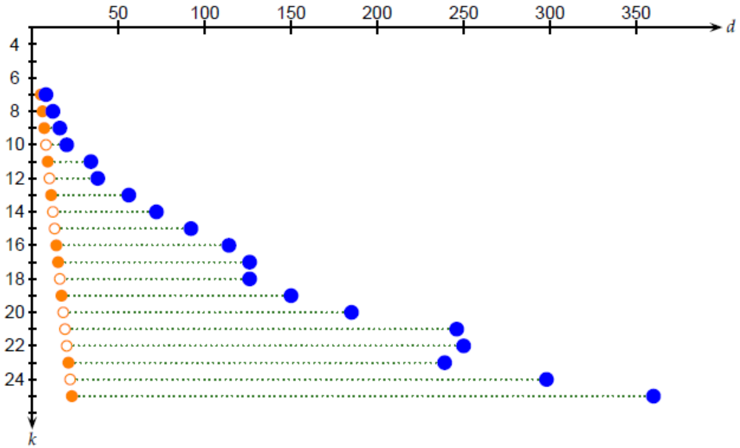

Easier to see that there is a huge area between the antiflag diagonal and the Golomb configurations, where the existence of configurations is undecided. Below the antiflag diagonal, apparently non-existence prevails.

Techniques: Representations

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  - Result: ◇ An explicit formula for obtaining $(q+1,8)$–cages and others small regular graphs of girth 8
    [Abreu, Araujo, Balbuena, DL - ArXiv 2011; 2014]

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Cyclic Schemes and Blow Up

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$$M := \begin{pmatrix} 1,4 & 0 \\ 0 & 2,3 \end{pmatrix} \text{ Example of } \mathbb{Z}_5\text{–scheme}$$

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$$\begin{pmatrix} 1,4 & 0 \\ 0 & 2,3 \end{pmatrix} \longrightarrow \left(\begin{array}{ccccc|ccccc} 0&1&0&0&1&1&0&0&0&0 \\ 1&0&1&0&0&0&1&0&0&0 \\ 0&1&0&1&0&0&0&1&0&0 \\ 0&0&1&0&1&0&0&0&1&0 \\ 1&0&0&1&0&0&0&0&0&1 \\ \hline 1&0&0&0&0&0&0&1&1&0 \\ 0&1&0&0&0&0&0&0&1&1 \\ 0&0&1&0&0&1&0&0&0&1 \\ 0&0&0&1&0&1&1&0&0&0 \\ 0&0&0&0&1&0&1&1&0&0 \end{array}\right)$$

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- Cyclic Schemes $\Longleftrightarrow$ Voltage Graphs (topological graph theory 1968 – Gross 1978).

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- Cyclic Schemes $\Longleftrightarrow$ Voltage Graphs (topological graph theory 1968 – Gross 1978).
- Blow up of cyclic schemes $\Longleftrightarrow$ Lift of Voltage Graphs
(cf. [Exoo, Jajcay. Dynamic Cage Survey, (2013)]).

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Position Matrices.

The $(0,1)$–matrices used in the blow up of a cyclic scheme, usually arise as *position matrices* of a symbol in a scheme.

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Let A be a scheme whose entries are symbols.

A				$P_x(A)$			
a	b	x	y	0	0	1	0
b	a	y	x	0	0	0	1
x	y	a	b	1	0	0	0
y	x	b	a	0	1	0	0

The *position matrix $P_x(A)$ of the symbol x* in A is a $(0,1)$–matrix with the same dimension as A which satisfies:

$$(P_x(A))_{i,j} = 1 \text{ if and only if } x \in A_{i,j}$$

(cf. [C.Balbuena. Incidence matrices of projective planes and of some regular bipartite graphs of girth 6 with few vertices, Siam Journal of Discrete Maths. 22(4) (2008), 1351-1363.])

Results: Tactical (De-)compositions of Symmetric Configurations [Funk, DL, Napolitano - Discrete Math. - 2009]

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Theorem (Funk, DL, Napolitano - Discrete Math. - 2009)

There exists an infinite class of symmetric configurations of type $(2p^2)_{p+s}$ where p is any prime and $s \leqslant t$ is a positive integer such that $t - 1$ is the greatest prime power with $t^2 - t + 1 \leqslant p$.

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- New configuration: 98_{10} $\rightarrow$ lies below the antiflag diagonal.

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[Abreu, Funk, DL, Napolitano - Ars Combin. - 2008]

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- Blow ups of cyclic schemes $\Longrightarrow$ construction of three families of small $(q-\lambda, 6)$–graphs:

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- For $\lambda = 0$ the cyclic schemes arise from extended additive and multiplication tables of finite fields.

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- For $\lambda = 0$ the cyclic schemes arise from extended additive and multiplication tables of finite fields.

- For $\lambda \geqslant 1$ reductions of such schemes are performed, and will be presented later.

Results: Adjacency matrices of polarity graphs and of other C_4-free graphs of large size [Abreu, Balbuena, DL - Des.Cod.Cryptogr. - 2010]

Let $q = p^h$ be a prime power, denote by $\widehat{G_q}$ the polarity graph of a certain orthogonal polarity in $PG(2, q)$ and by G_q its associated simple graph.

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- Schemes arising from latin squares $\Longrightarrow$ representation for polarity graphs G_q
- Blow up + position matrices of symbols in the latin squares $\Longrightarrow$ adjacency matrix for G_q
- Lower bounds on $ex(n; C_4)$ require reductions of such schemes and will be presented later.

Representations: Formulas

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In particular we use:

- $y = mx + b$ for incidences in the elliptic semiplanes of type $\mathcal{C}$

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- Elliptic Semiplanes
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In particular we use:

- $y = mx + b$ for incidences in the elliptic semiplanes of type $\mathcal{C}$
- The coordinatization of a Moore $(q+1,8)$–cage.

[Abreu, Araujo, Balbuena, DL; ArXiv 2011; 2014]

Results: An explicit formula for obtaining $(q+1,8)$–cages and others small regular graphs of girth 8 [Abreu, Araujo, Balbuena, DL; ArXiv 2011; 2014]

Theorem (Abreu, Araujo, Balbuena, DL; ArXiv 2011; 2014)

Let $\mathbb{F}_q$ be a finite field with $q \geqslant 2$ a prime power and ρ a symbol not belonging to $\mathbb{F}_q$. Let $\Gamma_q = \Gamma_q[V_0, V_1]$ be a bipartite graph with vertex sets $V_i = \mathbb{F}_q^3 \cup \{(\rho, b, c)_i, (\rho, \rho, c)_i : b, c \in \mathbb{F}_q\} \cup \{(\rho, \rho, \rho)_i\}$, $i = 0, 1$, and edge set defined as follows:

For all $a \in \mathbb{F}_q \cup \{\rho\}$ and for all $b, c \in \mathbb{F}_q$:

$$N_{\Gamma_q}((a,b,c)_1) = \begin{cases} \{(w,\ aw+b,\ a^2w+2ab+c)_0 : w \in \mathbb{F}_q\} \cup \{(\rho, a, c)_0\} & \text{if } a \in \mathbb{F}_q; \\ \{(c,b,w)_0 : w \in \mathbb{F}_q\} \cup \{(\rho,\rho,c)_0\} & \text{if } a = \rho. \end{cases}$$

$$N_{\Gamma_q}((\rho,\rho,c)_1) = \{(\rho, c, w)_0 : w \in \mathbb{F}_q\} \cup \{(\rho,\rho,\rho)_0\}$$

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Then Γ_q is a $(q+1,8)$–cage.

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  - Their removal leaves a regular graph of smaller regularity

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  - Their removal leaves a regular graph of smaller regularity
  - Results:

    ◇ A construction of small $(q-1)$–regular graphs of girth 8

    [Abreu, Araujo, Balbuena, DL - 2014]

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  - 1–factor deletions of incidence graphs of elliptic semiplanes of type $\mathcal{C}$, $\mathcal{L}$ and $\mathcal{D}$ $\Rightarrow$ new symmetric configurations
  - Results:
    $\diamond$ Deletions, extensions, and reductions of elliptic semiplanes

    [Abreu, Funk, DL, Napolitano - Innov. Incidence Geom. - 2010]

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      [Abreu,Balbuena,DL - Des.Cod.Cryptogr.2010]
    - Used together with the amalgam technique to obtain:
      ◇ Families of small regular graphs of girth 5
      [Abreu, Araujo, Balbuena, DL - Discrete Math. - 2012]
      ◇ Biregular cages of girth five
      [Abreu, Araujo, Balbuena, DL, López - Electron. J. Combin. - 2013]

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    - Results:
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      [Abreu,Balbuena,DL - Des.Cod.Cryptogr.2010]
    - Used together with the matchings technique to obtain:
      ◇ Small regular graphs of girth 7
      [Abreu, Araujo, Balbuena, DL, Salas- 2014]

Techniques: Cuts

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  - Type 1: Removal of vertices or subgraphs
  - Type 2: Removal of blocks of vertices or parallel classes in elliptic semiplanes.
    - Frequently used by other authors as well [Brown; Gács,Héger; Araujo,Balbuena et. al.]

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Techniques: Cuts

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    - Results:
      ◇ Small regular graphs of girth 5 and 6 for regularities that are not prime powers plus one.
      ◇ New symmetric configurations from elliptic semiplanes of type $\mathcal{C}$, $\mathcal{L}$ and $\mathcal{D}$.

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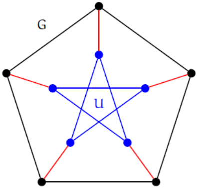
- A perfect dominating set of a graph G is a subset U of its vertices such that each vertex not in U, has exactly one neighbour in U.



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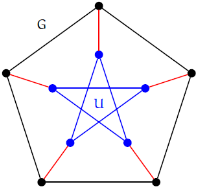

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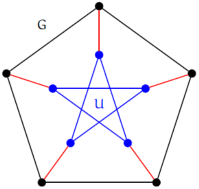

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- Perfect Dominating Sets in $(q+1, 8)$-cages and $(q, 8)$–graphs $\Longrightarrow$ known $(q, 8)$-graphs and new $(q-1, 8)$–graphs.

Results: A construction of small $(q-1)$–regular graphs of girth 8

[Abreu, Araujo, Balbuena, DL - 2014]

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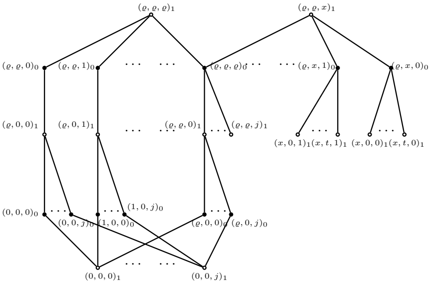
Theorem (Abreu, Araujo–Pardo, Balbuena, DL - 2014)

Let $q \geqslant 2$ be a prime power and let Γ_q be a $(q+1, 8)$-cage with the previous coordinatization. Then D



is a perfect dominating set of size $2(q^2+3q+1)$ and gives rise to a $(q,8)$–graph $G_q^x := \Gamma_q - D$ of order $2q(q^2-2)$.

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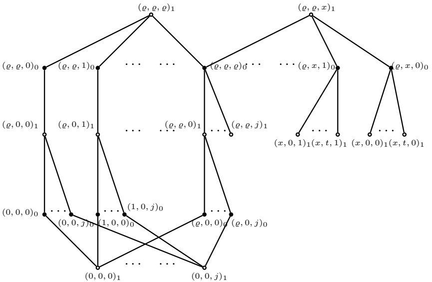

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In 2008, Gács and Héger, found this same values using *t*–good structures

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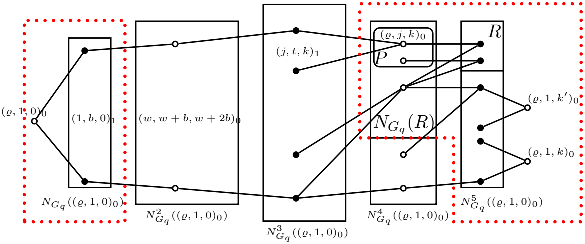

is a perfect dominating set of size $4q^2 - 6q$ and gives rise to a $(q-1, 8)$–graph $G_q^x - S$ of order $2q(q-1)^2$.

Results: Improvements on the order of $(q-1,8)$–graphs

- Previously, the smallest known $(q-1,8)$-graphs, for q a prime power, were those of order $2q(q^2-q-1)$

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k	Bound in [B09]	New bound		k	Bound in [B09]	New bound
15	7648	7200		52	292030	286624
22	23230	22264		58	403678	396952
36	98494	95904		63	515968	508032
40	134398	131200		66	592414	583704
46	203134	198904		70	705598	695800

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Let $\mathcal{K}$ be a configuration of type n_k, $G(\mathcal{K})$ its *incidence graph*.

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Reiterate deletions ν times, to obtain new configurations $\mathcal{K}^{(\nu F)}$.

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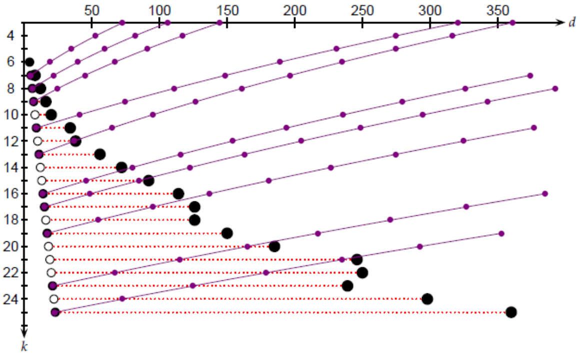

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- Füredi (1983;1996): proved that (most important)

$$ex(q^2 + q + 1; C_4) = \frac{1}{2} q(q+1)^2$$

where q is either a power of 2 or a prime power exceeding 13.

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Adjacency matrices of polarity graphs and of other C_4-free graphs of large size. [Abreu, Balbuena, DL - Des.Cod.Cryptogr. - 2010]

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- If $h = 2t$, the last subgraph corresponds to a Baer subplane $PG(2, \sqrt{q})$ of $PG(2, q)$.

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(i) *prime power* q, $m \in \{1, \ldots, 7\}$ *and* $n = q^2 + q + 1 - m$;

(ii) *prime* p, $q = p^h$ *with* $q \geqslant 4$, $t|h$ *and* $n = q^2 + q - p^{2t} - p^t$;

(iii) *prime* p, $q = p^{2t}$ *with* $q \geqslant 4$ *and* $n = q^2 - \sqrt{q}$;

(iv) *prime power* q *and* $n = q^2 - q - 2$.

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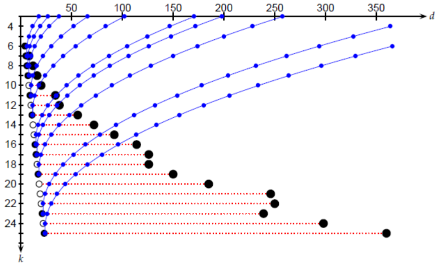

Results: Construction of small (k, g)–graphs:

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- Blow ups of cyclic schemes + Reductions $\Longrightarrow$ construction of small $(q-\lambda,6)$–graphs:

$$G_+(q,\lambda),\ G_*(q,\lambda) \text{ and } G'(q,\lambda)$$

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- Theorem $\Rightarrow$ the following upper bound:

$$n(k,6) \leqslant \begin{cases} 2(q^2 - \lambda q + \lambda - 1) & \text{if} \quad \lambda \leqslant 1 \\ 2(q^2 - \lambda q) & \text{if} \quad \lambda \geqslant 2 \end{cases}$$

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- We conjectured that $G'(q^2,\lambda)$ can be extended to any square prime power q^2 using cyclic schemes.

Constructions

- Let q^2 be a square prime power.

Theorem (Abreu,Funk,DL,Napolitano; AJC - 2006)

The graphs of the class $G'(q^2,\lambda)$ are $(q^2+1-\lambda)$–regular bipartite of girth 6, for $q=2,3,4$, of order $2(q^4+q^2+1-\lambda(q^2+q+1))$.

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- Geometrically proved by Gàcs and Héger (2008) using t–good structures.

[Gàcs and Héger. On geometric constructions of (k,g)–graphs. Contr.Discrete Math., 3 (2008), 63-80.]

Results: Upper bounds for $n(k,6)$

The table of $rec(k,6)$, $k \leqslant 20$ (Dynamic Cage Survey [G.Exoo, R.Jajcay, E-JC 2013]):

Degree k	Lower Bound	Upper Bound	Due to
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8	114	114	Projective Plane
9	146	146	Projective Plane
10	182	182	Projective Plane
11	224	240	Wong
12	266	266	Projective Plane
13	314	336	**Abreu-Funk-DL-Napolitano** $G_+(13,0)$
14	366	366	Projective Plane
15	422	462	**Abreu-Funk-DL-Napolitano** $G'(16,2)$
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- For some values of the regularity $k > 20$, there are better bounds by Araujo–Pardo, Balbuena et. al.

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Let $\mathcal{C}_q$ be a Desarguesian elliptic semiplane of type $\mathcal{C}$. Then, for each $\alpha \in \{0, \ldots, q-3\}$, $\beta \in \{0, \ldots, q-\alpha\}$, and $\gamma \in \{0, \ldots, q-\alpha-3\}$, there exists a configuration $\mathcal{C}_q^{(\alpha R)(\beta M)(\gamma F)}$ of type $(q^2-\alpha q+\beta)_{q-\alpha-\gamma}$.

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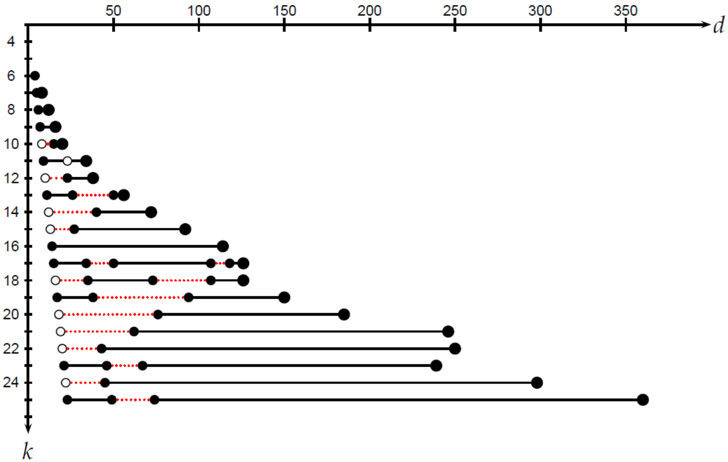
- The 1–factor deletion can always be performed on an elliptic semiplanes of types $\mathcal{D}$.
- The reduction can only be performed when we have a cyclic scheme representing its adjacency matrix.

New Elliptic Semiplane Spectrum



The region between the antiflag diagonal and the Golomb configurations is now mostly filled with configurations arising from our deletions, extensions and reductions. The red gaps are due to the absence of certain finite projective planes.

Stitches: Amalgams.

Let Γ_1 and Γ_2 be two graphs of the same order and with the same label on their vertices

An amalgam of Γ_1 into Γ_2 is a graph obtained by adding all the edges of Γ_1 to Γ_2

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- If instead of *merging* edges, one amalgams graphs after the removal, better results for increased regularity are sometimes obtained.

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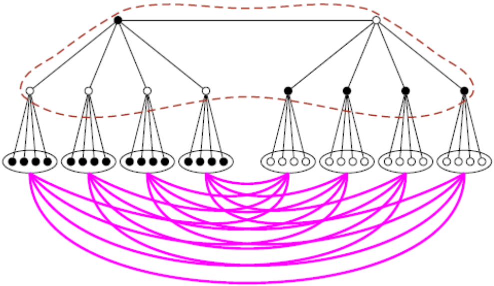

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- These matchings can be expressed by an algebraic rule.

$(x, y)_0 \in V_0$ is adjacent to $(m, b)_1 \in V_1$ if and only if $y = mx + b$

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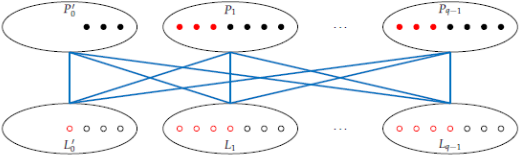

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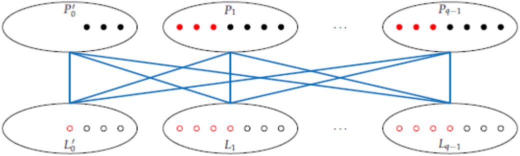

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Lemma (Abreu,Araujo,Balbuena,DL; Discrete Math. - 2012)

$B_q(S,T)$ is $(q-1,q)$-regular of order $2q^2 - |S| - |T|$.

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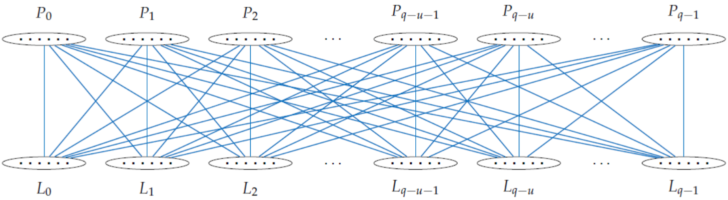

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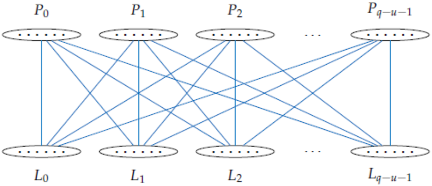

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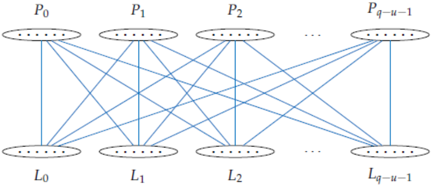

Lemma (Abreu, Araujo, Balbuena, DL - Discrete Math. - 2012)

$B_q(u)$ is $(q-u)$-regular of order $2(q^2 - qu)$

Construction: Combining both reductions on B_q

The resulting graph is denoted by

$$B_q(S, T, u).$$

This graph is $(q-u-1, q-u)$-regular.

Construction: Amalgam on $B_q(S, T, u)$ = The graph $B_q^*(S, T, u)$

Amalgam graphs H_1, H_2, G_1 and G_2 into blocks of $B_q(S, T, u)$ to obtain desired degrees.

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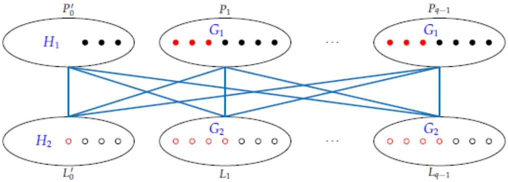

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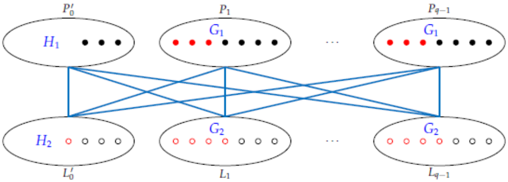

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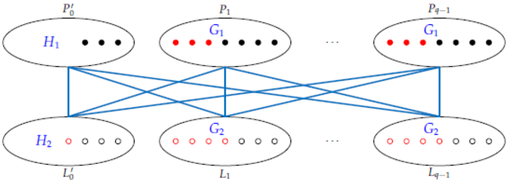

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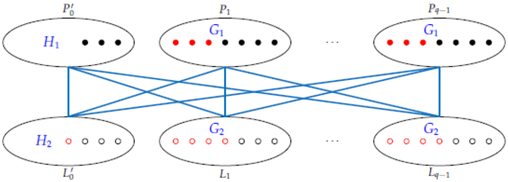

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Construction: The resulting graph $B_q^*(S, T, u)$ is $(q - u + k)$–regular

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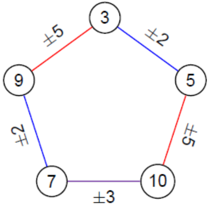

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Theorem (Abreu,Araujo,Balbuena,DL; Discrete Math. - 2012)

Let $T \subseteq S \subseteq GF(q)$. Let H_1, H_2, G_1 and G_2 be defined as above and suppose that the weights $P_\omega \cap L_\omega = \emptyset$. Then the amalgam $B_q^*(S, T)$ is a $(q + k)$–regular graph of girth at least 5 and order $2(q^2) - |S| - |T|$, for $u = 0$.

Similarly for $B_q^*(S, T, u)$, for $u \geqslant 1$

Example: $(13, 5)$–graph of order 236

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Consider B_{11}

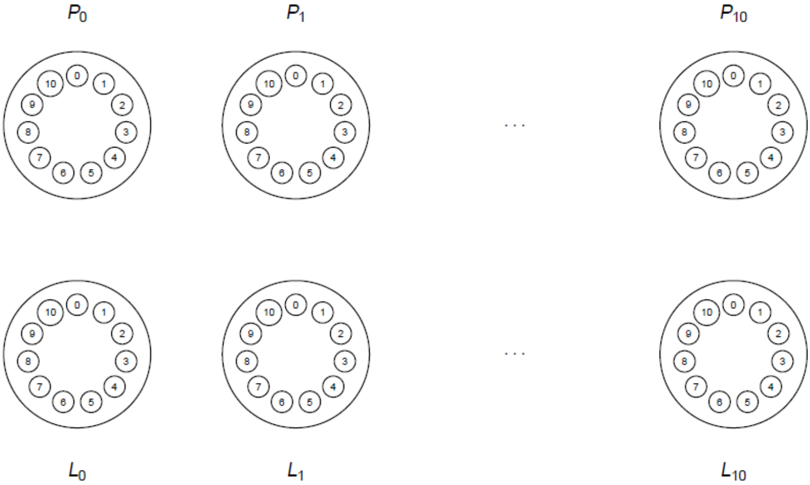
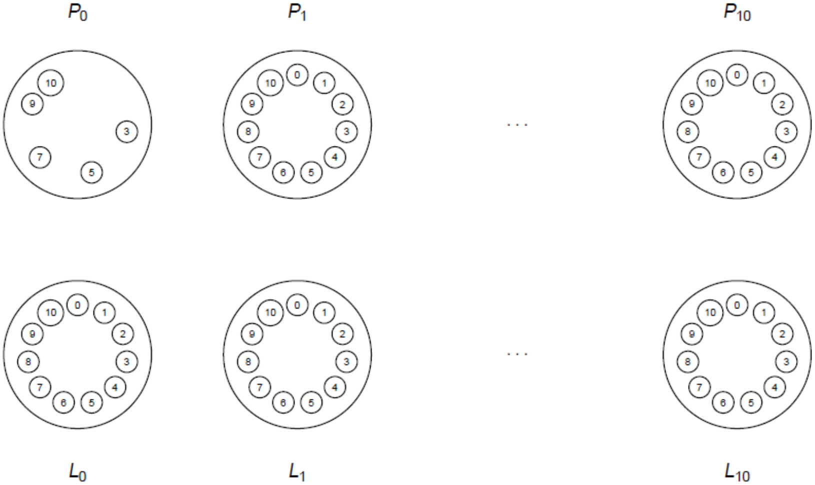


Figure illustrates $B_{11} - E(B_{11})$

Example: $(13, 5)$–graph of order 236

Remove $S = \{0, 1, 2, 4, 6, 8\}$ and $T = \emptyset$ to obtain $B_{11}(S, T)$



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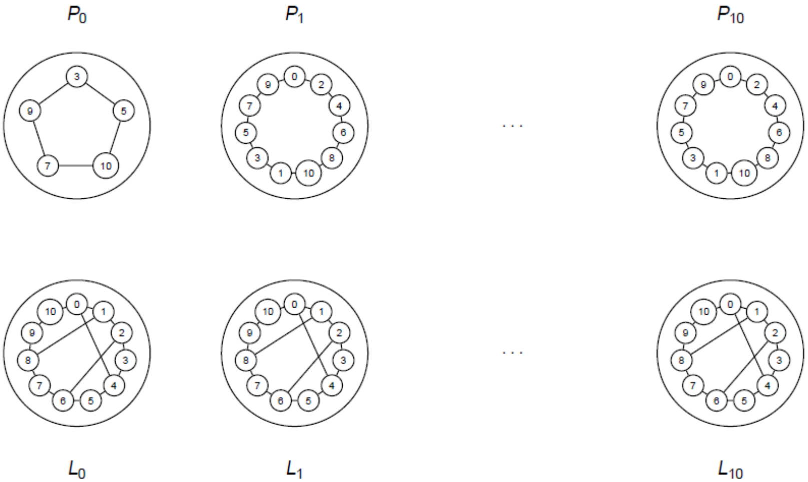

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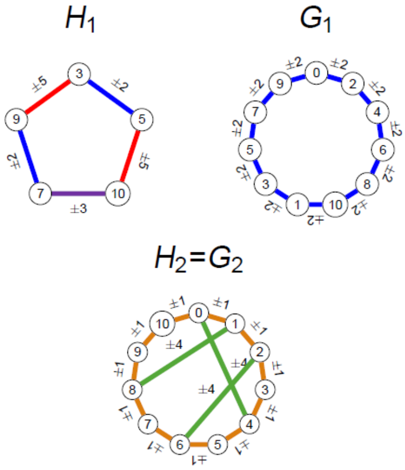

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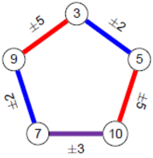

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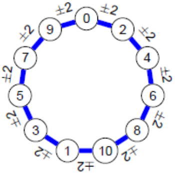

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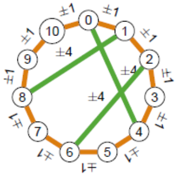

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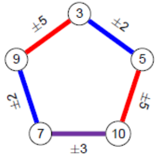

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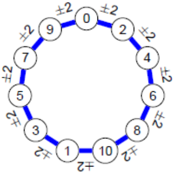

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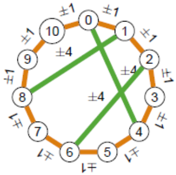

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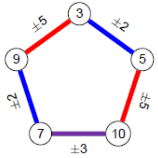

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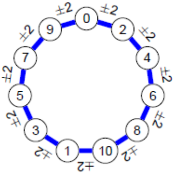

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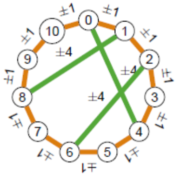

$B^*_{11}(S, T, u)$, for $u = 1, \ldots, 10$, is $(13 - u)$–regular of girth 5 and order $236 - 22u$

Families of small regular graphs of girth 5

[Abreu,Araujo,Balbuena,DL; Discrete Math. - 2012]

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Theorem (Abreu,Araujo,Balbuena,DL,López; El.J.Combin. - 2013)

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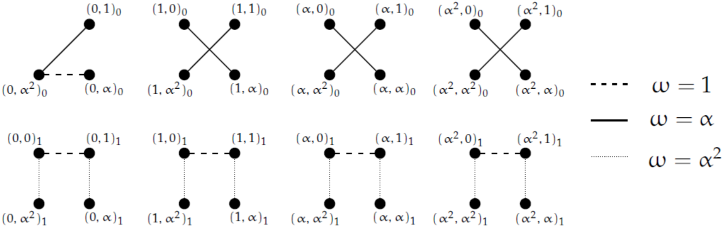

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Results:

$\diamond$ Small regular graphs of girth 7 [Abreu, Araujo–Pardo, Balbuena, DL, Salas - 2014]

Small regular graphs of girth 7: Even prime power

Theorem (Abreu, Araujo–Pardo, Balbuena, DL, Salas - 2014)

Let $q \geqslant 4$ be an even prime power. Then, there is a $(q+1)$–regular graph of girth 7 and order $2q^3 + q^2 + 2q$.

Proof. (Idea)

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- This proof is purely combinatorial (no coordinates are needed)

Small regular graphs of girth 7: Odd prime power

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Let $q \geqslant 5$ be an odd prime power. Then, there is a $(q+1)$–regular graph of girth 7 and order $2q^3 + 2q^2 - q + 1$.

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[A construction of small $(q-1)$–regular graphs of girth 8 - Abreu, Araujo, Balbuena, DL - 2014]

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- Further details in M.Abreu's talk.

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THANK YOU

Golomb Rulers

A *Golomb ruler* of order k is a set $S = \{\alpha_1, \ldots, \alpha_k\}$ of k integers such that the differences $|\alpha_i - \alpha_j|$ are all distinct for all $i \neq j$.

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In 1990, Gropp proved that for all $k \geqslant 3$ there exists an integer $n_0(k)$ such that n_k configurations exist for all $n \geqslant n_0(k)$, and that $n_0(k) = 2l_k + 1$.

A Golomb Configuration is a $(2l_k + 1)_k$ configuration.