Graphs with Excess or Defect 2

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Moore Bound

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Definition

A graph G with minimal degree d and girth $g = 2k+1$ on $m(d,g)$ vertices is called a Moore graph of type (d,g).

Moore Graphs

Example

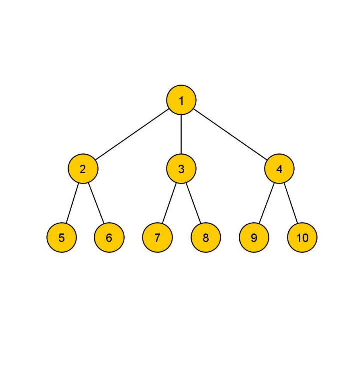

Figure : The unique Moore graph of type $(3, 5)$.

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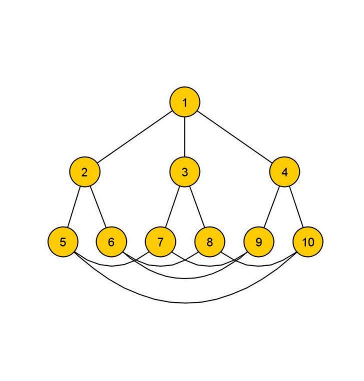

Figure : The Peterson graph is the unique Moore graph of type $(3, 5)$.

Moore Graphs

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- For $d = 3$ the unique Moore graph is the Peterson graph.
- For $d = 7$ the unique Moore graph is the Hoffmann-Singleton graph.
- For $d = 57$ it is still unknown, if there exists a Moore graph.

Excess

Definition

A connected graph G with minimal degree d and girth $2k+1$ on $m(d, 2k+1)+\epsilon$ vertices is called a graph of type $(d, 2k+1, \epsilon)$. The number ϵ is called the excess of G.

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Let $\epsilon < \sum_{i=0}^{k-1}(d-1)^i$. Then a $(d, 2k+1, \epsilon)$-graph is regular.

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Let $\epsilon < \sum_{i=0}^{k-1}(d-1)^i$. Then a $(d,2k+1,\epsilon)$-graph is regular.

Theorem (Brown, Bannai, Ito, 1967-81)

There is no graph of type $(d,2k+1,1)$.

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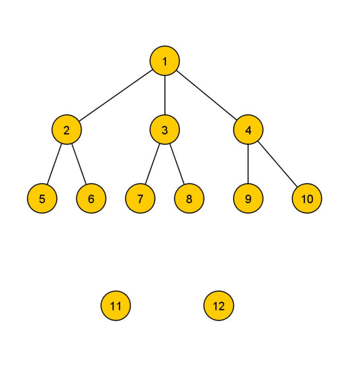

Figure : A graph of type $(3, 5, 2)$.

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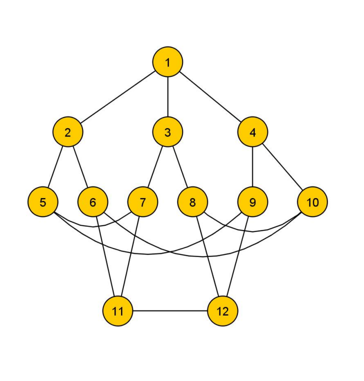

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Let $d \in \mathbb{N}$ be odd, $d \neq l^2 + l + 3$ and $d \neq l^2 + l - 1$ for every $l \in \mathbb{Z}_{\geq 0}$. Then there is no graph of type $(d, 5, 2)$.

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- One graph of type $(3, 7, 2)$.

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Theorem (G., 2013)

Let $d \in \mathbb{N}$ be odd. Then there is no graph of type $(d, 9, 2)$.

We will use some algebraic properties of graphs with excess or defect 2 and introduce the following polynomials.

Definition

Define the polynomials $F_{d,k}(x)$ by

$$F_{d,0}(x) = 1$$
$$F_{d,1}(x) = x + 1$$
$$F_{d,k+1}(x) = xF_{d,k}(x) - (d-1)F_{d,k-1}(x) \ .$$

Lemma 1

Lemma (1)

Let G be a graph of type $(d, 2k+1, 2)$ and A its adjacency matrix. Then

$$F_{d,k}(A) = J - B \ ,$$

where B consists of a disjoint union of c cycles C_i of length r_i with $1 \leq i \leq c$ and J is the all one matrix.

Corollary

Let G be a graph of type $(d, 2k+1, 2)$ and B be the matrix from Lemma 1. Furthermore, let c be the number of cycles of B and c_2 be the number of cycles of B whose length is divisible by 2.

(i) *If $F_{d,k}(x) - 2$ is irreducible, then $c - 1 \equiv 0 \mod k$.*

(ii) *If $F_{d,k}(x) + 2$ is irreducible, then $c_2 \equiv 0 \mod k$.*

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Proof.

$F_{d,k}(A) = J - B \Rightarrow$ The eigenvalues of A are d and for $\sigma(B) = \{2, \mu_2, \cdots, \mu_n\}$ one root of every

$$F_{d,k}(x) = -\mu_i, \; 2 \leq i \leq n.$$

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Exactly $c - 1$ eigenvalues are roots of $F_{d,k}(x) - 2$. Exactly c_2 eigenvalues are roots of $F_{d,k}(x) + 2$. □

Lemma 2

Lemma (2)

Let d be odd and k be even. If $F_{d,k}(x)+2$ and $F_{d,k}(x)-2$ have both no integer root, then there is no graph of type $(d, 2k+1, 2)$.

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$F_{d,k}(x) \pm 2 = \sum_{i=0} a_i x^i = x^k + x^{k-1} + (d-1) \cdot (\text{lower order terms}) \pm 2.$

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But $n = 2 + 1 + d\sum_{i=0}^{k-1}(d-1)^{i-1} \equiv 1 + 1 \cdot 1 \equiv 0 \bmod 2$. □

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We examine

$$F_{d,4}(x) \pm 2 = x^4 + x^3 - 3x^2(d-1) - 2x(d-1) + (d-1)^2 \pm 2 = 0\ .$$

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The discriminant of the quadratic polynomial in z has to be rational $\Rightarrow$ integer solution (x, y) of

$$y^2 = 5x^4 + 8x^3 + 4x^2 \mp 8\ .$$

Hence it is sufficient to compute the integer solutions of this equation. □

Proof of the Theorem

Proof.

We transform the first equation under $U = x - 1$ to

$$V^2 = 5U^4 + 28U^3 + 58U^2 + 52U + 9 \; . \qquad \text{(I)}$$

and the second equation under $U = x + 1$ to

$$V^2 = 5U^4 - 12U^3 + 10U^2 - 4U + 9 \; . \qquad \text{(II)}$$

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Integer solutions of (II): $(0, \pm 3), (2, \pm 5)$
Solutions to the original curves: $(1, 2), (1, 5)$ respectively $(1, 1), (1, 6), (-1, 0), (-1, 3)$. Hence there is no graph for $d \neq 3, 5$ by Lemma 2. □

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Remember eigenvalues of A are roots of $F_{5,4}(x) + \mu$, thus roots of $p_\mu(-F_{5,4}(x))$, where $p_\mu(x)$ is the minimal polynomial of the eigenvalue $\mu = 2\cos(\frac{j2\pi}{c})$ of B.

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$$tr(A) = 5 + m_A(1) \cdot 1 + \frac{c - 1 - m_A(1)}{3} \cdot (-2) - \frac{n - c}{4} .$$

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$$tr(A) = 0 \;\Rightarrow\; 0 = 20 m_A(1) - 5c - 1216 \, ,$$

no solution modulo 5. □

Cage Number

This result improves the lower bound for the cage number for d odd and girth 9 to

$$n(d,9) \geq m(d,9) + 4 \, .$$

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Proposition

Let be $D \geq 2$ and $\delta < \sum_{i=0}^{k-1}(d-1)^i$. Then a $(d, D, -\delta)$-graph is regular.

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Corollary

There is no graph with diameter 4 and defect 2.

Problem for girth higher than 9: No general method known to determine the integral points of curves of genus higher than 1. For example $g = 13$ leads to curves of genus 4

$$x^6 + x^5 - 5x^4(d-1) - 4x^3(d-1) + 6x^2(d-1)^2 + 3x(d-1)^2 - (d-1)^3 \pm 2 = 0$$

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But Siegel proved in 1929 that there are only finitely many integral points for curves of genus ≥ 1.

Problem for girth higher than 9: No general method known to determine the integral points of curves of genus higher than 1. For example $g = 13$ leads to curves of genus 4

$$x^6 + x^5 - 5x^4(d-1) - 4x^3(d-1) + 6x^2(d-1)^2 + 3x(d-1)^2 - (d-1)^3 \pm 2 = 0$$

But Siegel proved in 1929 that there are only finitely many integral points for curves of genus ≥ 1.
Using other methods to check for integer solutions leads to:

degree	$\frac{g-1}{2}$ or diameter	result
$d \equiv 0 \bmod 5$	6	no $(d, 6, -2)$, $(d, 13, 2)$
$d \equiv 5 \bmod 7$	6	no $(d, 6, -2)$, $(d, 13, 2)$
$d \equiv 4 \bmod 11$	6	no $(d, 6, -2)$, $(d, 13, 2)$
$d \equiv 2, 5, 10, 12 \bmod 13$	6	no $(d, 6, -2)$, $(d, 13, 2)$
$d \equiv 3 \bmod 5$	8	no $(d, 8, -2)$, $(d, 17, 2)$
$d \equiv 0, 2 \bmod 7$	8	no $(d, 8, -2)$, $(d, 17, 2)$
$d \equiv 9 \bmod 11$	8	no $(d, 8, -2)$, $(d, 17, 2)$
$d \equiv 6 \bmod 13$	8	no $(d, 8, -2)$, $(d, 17, 2)$
$d \equiv 7, 10 \bmod 11$	10	no $(d, 10, -2)$, $(d, 21, 2)$
$d \equiv 2, 3, 4 \bmod 5$	12	no $(d, 12, -2)$, $(d, 25, 2)$
$d \equiv 2, 6 \bmod 7$	12	no $(d, 12, -2)$, $(d, 25, 2)$
$d \equiv 2, 6, 8 \bmod 11$	12	no $(d, 12, -2)$, $(d, 25, 2)$
$d \equiv 4, 7 \bmod 13$	12	no $(d, 12, -2)$, $(d, 25, 2)$
3	$D \equiv 0 \bmod 4$	no $(3, D, -2)$, $(3, 2D+1, 2)$
5	$D \equiv 2 \bmod 4$	no $(5, D, -2)$, $(5, 2D+1, 2)$
7	$D \equiv 0 \bmod 2$	no $(7, D, -2)$, $(7, 2D+1, 2)$
9	$D \equiv 0 \bmod 4$	no $(9, D, -2)$, $(9, 2D+1, 2)$