A Spectral Characterization of Strongly Distance-Regular Graphs with Diameter Four.

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Outline

Introduction and motivation

A known characterization

A graph G with diameter d and distance matrices

$$\boldsymbol{A}_0(=\boldsymbol{I}), \boldsymbol{A}_1(=\boldsymbol{A}), \boldsymbol{A}_2, \ldots, \boldsymbol{A}_d,$$

is distance-regular if and only if there exists a sequence of (orthogonal) polynomials $p_0, p_1, p_2, \ldots, p_d$, $\deg p_i = i$, such that

$$\boldsymbol{A}_i = p_i(\boldsymbol{A}), \qquad i = 0, 1, \ldots, d.$$

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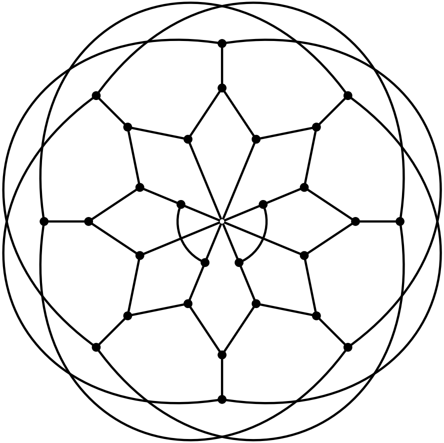
In particular, G (connected) is strongly regular when it is distance-regular with diameter $d = 2$. (in general, nK_s or $K_{s,\ldots,s}$ are also allowed.)

A distance-regular graph



The Coxeter graph (according to Charles Delorme)

A spectral characterization

A graph (connected) G is strongly regular if and only if it is regular (a poperty that can be deduced from the spectrum) and it has three distinct eigenvalues.

Strongly distance-regular graphs

A *strongly distance-regular graph* is a distance-regular graph G (of diameter d, say) with the property that its distance-d graph G_d is strongly regular.

(See the blackboard...)

Examples

The known examples of strongly distance-regular graphs are:

- The strongly regular graphs (since G_d is the complement of G),
- The antipodal distance-regular graphs (where G_d is a disjoint union of complete graphs), and
- All the distance-regular graphs with $d = 3$ and third largest eigenvalue $\lambda_2 = -1$. (there are infinite families of this type, such as the generalized hexagons and the Brouwer graphs).

The problem

Given a distance-regular graph G, decides, from its spectrum,

$$\operatorname{sp} G = \operatorname{sp} \boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\},$$

where

$$\lambda_0 > \lambda_1 > \cdots > \lambda_d,$$

and the superscripts stand for the multiplicities $m_i = m(\lambda_i)$, whether or not G is strongly distance-regular.

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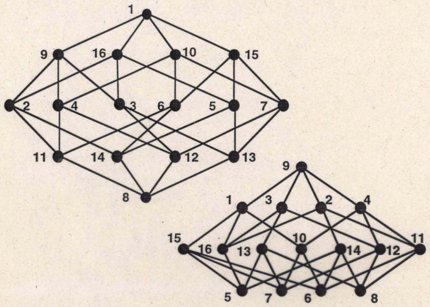
Is the spectrum enough?

NOT!

The Hoffman graph H

For instance

The Hoffman graph



- The Hoffman graph is cospectral with the hypercube Q_4, but it is **not** distance-regular

An open problem

Why do we need such a characterization?

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Solve the (open) problem of deciding whether or not the above known families of strongly distance-regular graphs exhaust all the possibilities.

One step in this direction

Here we prove that a distance-regular graph G with five distinct eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_4$ (the case of diameter four) is strongly distance regular if and only an equality involving them, and the intersection parameters a_1 or b_1, is satisfied.

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Here we prove that a distance-regular graph G with five distinct eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_4$ (the case of diameter four) is strongly distance regular if and only an equality involving them, and the intersection parameters a_1 or b_1, is satisfied. Then, as a consequence, it is shown that all bipartite strongly distance-regular graphs with such a diameter are antipodal.

The main result

Some tools

A scalar product:

$$\langle p, q \rangle_G = \frac{1}{n} \operatorname{tr}(p(\boldsymbol{A})q(\boldsymbol{A})) = \frac{1}{n} \sum_{i=0}^{d} m_i p(\lambda_i) q(\lambda_i), \qquad p, q \in \mathbb{R}_d[x], \tag{1}$$

A basic result:

Lemma

Let G be a distance-regular graph with spectrum $\operatorname{sp} G = \{\lambda_0, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, where $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. Then,

(a) *G is r-antipodal if and only if*

$$m_i = \frac{\pi_0}{\pi_i} \quad (i \text{ even}), \qquad m_i = (r-1)\frac{\pi_0}{\pi_i} \quad (i \text{ odd}).$$

(b) *G is strongly distance-regular if and only if, for some positive constants α, β,*

$$m_i \pi_i = \alpha \quad (i \text{ odd}), \qquad m_i \pi_i = \beta \quad (i \text{ even},\ i \neq 0).$$

Theorem

Let G be a distance-regular graph with n vertices, diameter $d = 4$, and distinct eigenvalues $\lambda_0 (= k) > \lambda_1 > \cdots > \lambda_4$. Then G is strongly distance-regular if and only if

$$(1 + \lambda_1)(1 + \lambda_3) = (1 + \lambda_2)(1 + \lambda_4) = -b_1. \qquad (2)$$

Moreover, in this case, G is antipodal if and only if either,

$$\lambda_1 \lambda_3 = -k, \qquad \text{or} \qquad \lambda_1 + \lambda_3 = a_1 \qquad (3)$$

Some useful facts in the proof

Notice first that the multiplicities $m_0(=1), m_1, \dots, m_4$, satisfy the following equations:

$$\sum_{i=0}^{4} m_i = n, \quad \sum_{i=0}^{4} m_i \lambda_i = 0, \quad \sum_{i=0}^{4} m_i \lambda_i^2 = nk, \quad \sum_{i=0}^{4} m_i \lambda_i^3 = nka_1,$$

or, in terms of the scalar product (1),

$$\langle 1, 1 \rangle_G = 1, \quad \langle x, 1 \rangle_G = 0, \quad \langle x^2, 1 \rangle_G = k, \quad \langle x^3, 1 \rangle_G = ka_1. \quad (4)$$

Examples, consequences, and problems

The case of bipartite graphs

For the case of bipartite graphs, the conditions in (3) clearly hold since $\lambda_3 = -\lambda_1$ and $a_1 = 0$. Thus,

Every bipartite strongly distance-regular graph is antipodal.

Besides, the condition (2) turns to be very simple:

Corollary

A bipartite distance-regular graph G with diameter $d = 4$ is strongly distance-regular if and only if $\lambda_1 = \sqrt{k}$.

The cases $i = 0, 1$

Then, these graphs have spectrum

$$\{k^1, \sqrt{k}^{n/2-k}, 0^{2k-2}, -\sqrt{k}^{n/2-k}, -k^1\}$$

and, in fact, they constitute a well known infinite family (see Brouwer, Cohen and Neumaier (1989): With $n = 2m^2\mu$ and $k = m\mu$, they are precisely the incidence graphs of symmetric (m, μ)-nets, with intersection array

$$\{k, k-1, k-\mu, 1; 1, \mu, k-1, k\}.$$

Strongly distance-regular graphs with $d = 4$

Looking at the table of known (or feasible) distance-regular graphs, it turns out that:

- **Primitive graphs:** none
- **Antipodal (but not bipartite) graphs:** all
- **Antipodal bipartite graphs:** the family presented

Thus,

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All the strongly distance-regular graphs with diameter four are antipodal.

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Are the conditions (2) and (3) somehow related?

Many thanks

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