Cayley Graphs of Dihedral Groups

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Notation 1

Cayley graphs

- Group G
- Set $S \subseteq G \setminus \{1\}$ with $S = S^{-1}$
- $\mathrm{Cay}(G,S)$ has G as vertex set and uv an undirected edge if $uv^{-1} \in S$.
- We usually insist $G = \langle S \rangle$.
- The diameter of $\mathrm{Cay}(G,S)$ is at most k if every $g \in G$ can be expressed as a multiple of no more than k elements of S.

Notation 2

Groups

$\mathbb{Z}_n$ – Cyclic group of order n
C_2 – Multiplicative cyclic group of order 2 $= \{1, -1\}$
D_{2n} – Dihedral group of order $2n$
We view D_{2n} as $\mathbb{Z}_n \rtimes C_2$ where the action is via the inversion automorphism.

Bounds

$M(d,k)$ – Moore bound for degree d, diameter k
$C(d,k)$ – Largest order of Cayley graph of degree d, diameter k
$AC(d,k)$ – Largest order of Cayley graph of abelian group
$DC(d,k)$ – Largest order of Cayley graph of dihedral group

What do we know?

Diameter 2 bounds

$M(d,2) = d^2 + 1$

$$\limsup_{d\to\infty} \frac{AC(d,2)}{d^2} \leq \frac{1}{2}$$

Known constructions

$$\limsup_{d\to\infty} \frac{C(d,2)}{d^2} = 1$$ (Šiagiová,Širáň 2012)

$$\limsup_{d\to\infty} \frac{AC(d,2)}{d^2} \geq \frac{3}{8}$$ (Macbeth,Šiagiová,Širáň 2012)

Dihedral Groups

Result

$$\lim_{d\to\infty} \frac{DC(d,2)}{d^2} = \frac{1}{2}$$

NB this result is not restricted to particular values of d

Proof strategy

- Find a construction with asymptotic order $d^2/2$ for special values of d based on finite fields
- Extend to all values of d by estimating how close we are to a special value and adding edges
- Count the ways we can form a reduced word of length 2 in the generating set to bound the asymptotic order at $d^2/2$

Proof 1 - Construction

Dihedral group

$F = GF(p), G = (F^+ \times F^*) \rtimes C_2, G \cong D_n, n = 2p(p-1)$
$(a,b,c)(\alpha,\beta,\gamma) = (a + \alpha c, b\beta^c, c\gamma)$

Generating set

$v = (0,1,-1)$ (1 element)
$a_x = (0,x,1), x \in F^* \setminus \{1\}$ ($p-2$ elements)
$b_x = (x,x,-1), x \in F^*$ ($p-1$ elements)
All elements of G can be expressed as a product of at most 2 of these generators, except elements of the form $(x,1,1)$ which are a cyclic subgroup of order p so can be covered by $2\lceil\sqrt{p}\rceil$. Generating set has cardinality $2(p + \lceil\sqrt{p}\rceil - 1)$.

$$\limsup_{d\to\infty} \frac{DC(d,2)}{d^2} \geq \frac{1}{2}$$

Proof 2 - Extension to all d

The idea

(Šiagiová,Širáň,Ždímalová 2011.)
A diameter 2 Cayley graph of D_n with degree D can be extended to one of the same order but degree $d > D$ by adding $d - D$ involutions to the generating set.
Then for all d, $DC(d,2) \geq 2p(p-1)$ where p is the largest prime with $D = 2(p + \lceil\sqrt{p}\rceil - 1) \leq d$.

Finding p

For all sufficiently large N there is a prime p satisfying $N - N^{0.525} \leq p \leq N$ (Baker,Harmon,Pintz 2001).
So for all sufficiently large d, $DC(d,2) \geq 0.5d^2 - 1.39d^{1.525}$.

$$\lim_{d\to\infty} \frac{DC(d,2)}{d^2} \geq \frac{1}{2}$$

Proof 3 - Counting

The idea

A diameter 2 Cayley graph of D_{2n} must have degree $d \geq 2\sqrt{n} - 1$. Idea - if a, b are in the generating set S with a in the cyclic subgroup and b an involution then $ab = ba^{-1}$. Since S is inverse-closed each involution in $D_{2n} \setminus S$ has 2 distinct representations as a length 2 word in the generators.

The result

The largest possible order of a Cayley graph of a dihedral group with degree d diameter 2 is asymptotically $d^2/2$. Combined with previous results this gives

$$\lim_{d\to\infty} \frac{DC(d,2)}{d^2} = \frac{1}{2}$$

Other ideas

Questions

1. Does this extend to Cayley graphs of other split extensions of cyclic or abelian groups?
2. Does this extend to higher diameters?
3. Why can't we find a robust bound for cyclic or general abelian groups?