The degree/diameter problem in maximal planar bipartite graphs

Cristina Dalfó*, Clemens Huemer*, Julian Salas**

*Departament de Matemàtica Aplicada 4
Universitat Politècnica de Catalunya, BarcelonaTech, Barcelona, Catalonia
**IIIA-CSIC, Bellaterra, Catalonia

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Outlook

Introduction

- $G = G(V, E)$: simple, bipartite, planar, maximal graph.

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  $|E| = 2n - 4$,
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- A maximal planar bipartite graph divides the plane only into quadrangles (Ringel, 1993).
- $|V| - |E| + |F| = 2$ (Euler characteristic),
  $|E| = 2n - 4$,
  $|F| = n - 2$.
- (Δ, D) problem: It consists of finding the maximum possible number of vertices $n = |V|$ in a graph G with maximum degree Δ and diameter D.

The $(\Delta, 2)$ and $(\Delta, 3)$ problems in maximal planar bipartite graphs

- The $(\Delta, 2)$ problem:
  Proposition 1. Consider a maximal planar bipartite graph G with diameter $D = 2$, maximum degree Δ and maximum number of vertices n, then $n = \Delta + 2$. The only graph that satisfies this equation is the complete bipartite graph $K_{2,\Delta}$.

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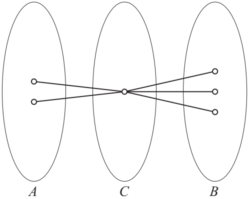
- The $(\Delta, 3)$ problem:
  Theorem 2. Consider a maximal planar bipartite graph G with diameter $D = 3$, maximum degree Δ and maximum number of vertices n, then

$$n = \begin{cases} 3\Delta - 1 & \text{if } \Delta \text{ is odd,} \\ 3\Delta - 2 & \text{if } \Delta \text{ is even.} \end{cases}$$

The (Δ, D) problem in maximal planar bipartite graphs: An upper bound

- **Theorem [Lipton, Tarjan, 1979].** Let G be a planar graph on n vertices containing a spanning tree of radius r. Then $V(G)$ can be partitioned into sets A, B and C such that no edges join vertices in A with vertices in B, $|A| \leq \frac{2}{3}n$, $|B| \leq \frac{2}{3}n$, and $|C| \leq 2r + 1$.



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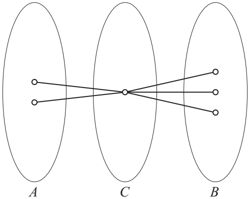

Figure: Sets A, B and C

- **Theorem [Fellows, Hell, Seyffarth, 1995].** Consider a maximal planar graph G with diameter D, maximum degree Δ and maximum number of vertices n, then

$$n = 3(2D + 1)(2\Delta^{\lfloor D/2 \rfloor} + 1).$$

The (Δ, D) problem in maximal planar bipartite graphs: An upper bound

Theorem 3. Let G be a maximal planar bipartite graph on n vertices with maximum degree $\Delta \geq 4$ and diameter $D \geq 4$. Then,

- If $\Delta = 4$: $n \leq 6(2D+1)\left(\left\lfloor \frac{D}{2} \right\rfloor^2 + \left\lfloor \frac{D}{2} \right\rfloor + 1\right)$.

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- If $\Delta > 4$:

$$n \leq 3(2D+1)\left[\frac{\sqrt{\Delta(\Delta-4)}}{2(\Delta-4)^2}\left[(\Delta - 4 + \sqrt{\Delta(\Delta-4)}\,)\left(\frac{\Delta-2-\sqrt{\Delta(\Delta-4)}}{2}\right)^{\lfloor D/2\rfloor+1}\right.\right.$$

$$-2\sqrt{\Delta(\Delta-4)}$$

$$\left.\left. + \ (4 - \Delta + \sqrt{\Delta(\Delta-4)}\,)\left(\frac{\Delta-2+\sqrt{\Delta(\Delta-4)}}{2}\right)^{\lfloor D/2\rfloor+1}\right] + 2\right],$$

which is approximately $3(2D+1)\left[(\Delta-2)^{\lfloor D/2\rfloor} + 1\right]$ if Δ is sufficiently large.

The (Δ, D) problem in maximal planar bipartite graphs: An upper bound

Proof [sketch].

- We compute from each vertex of C the maximum possible number of vertices at distance at most $\lfloor D/2 \rfloor$.

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- We compute from each vertex of C the maximum possible number of vertices at distance at most $\lfloor D/2 \rfloor$.
- We build a subgraph adding vertices at distance i from a given (root) vertex of C in step i $(0 \leq i \leq \lfloor D/2 \rfloor)$, to obtain an almost maximal (its interior faces are quadrangles) planar bipartite graph .

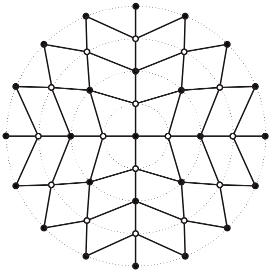

Figure: An almost maximal subgraph for $\Delta = 4$

The (Δ, D) problem in maximal planar bipartite graphs: An upper bound

Proof [sketch].

- Let n_i be the number of vertices at distance i (for $0 \leq i \leq \lfloor D/2 \rfloor$). For $i \geq 3$, n_i follows the recurrence

$$n_i = (\Delta - 2)n_{i-1} - n_{i-2}.$$

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- We use the generating function $G(z) = \frac{\Delta}{\alpha-\beta}\left(\frac{\alpha}{z-\alpha} - \frac{\beta}{z-\beta}\right)$, where $\alpha = \frac{1}{2}(\Delta - 2 + \sqrt{\Delta(\Delta-4)})$ and $\beta = \frac{1}{2}(\Delta - 2 - \sqrt{\Delta(\Delta-4)})$ for $\Delta > 4$.

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- We obtain for $\Delta > 4$

$$n_i = \frac{\Delta}{\sqrt{\Delta(\Delta-4)}}\left[\left(\frac{\Delta-2+\sqrt{\Delta(\Delta-4)}}{2}\right)^i - \left(\frac{\Delta-2-\sqrt{\Delta(\Delta-4)}}{2}\right)^i\right].$$

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- The total number of vertices $n = \sum_{i=0}^{\lfloor D/2 \rfloor} n_i$ is obtained as the difference of two geometric series.

The (Δ, D) problem in maximal planar bipartite graphs: An upper bound

- Fellows, Hell, Seyffarth's bound on n for maximal planar graphs:

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$$n \leq 3(2D+1)\left[(\Delta-2)^{\lfloor D/2 \rfloor}+1\right].$$

The (Δ, D) problem in maximal planar bipartite graphs: An upper bound

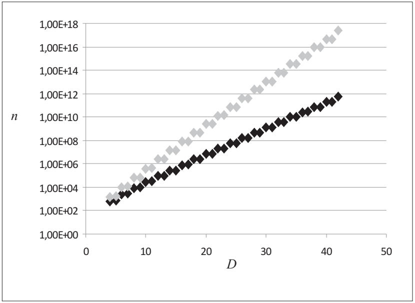

Figure: Plot of the log (base 10) of the number of vertices n with respect to the diameter D (black points: D., Huemer, Salas's bound; grey points: Fellows, Hell and Seyffarth's bound), for $\Delta = 5$ and $4 \leq D \leq 42$

The (Δ, D) problem in maximal planar bipartite graphs: An alternative upper bound

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- **Theorem [Chepoi, Estellon, and Vaxès].** There exists a constant C such that any planar graph G of diameter $D \leq 2k$ can be covered with at most C balls of radius k.
- Lower bound: Gavoille, Peleg, Raspaud, and Sopena presented a family of planar graphs with $C \geq 4$.

The (Δ, D) problem in maximal planar bipartite graphs: An alternative upper bound

- **Corollary 4.** There exists a constant C such that each maximal planar bipartite graph G with maximum degree Δ and diameter D has at most $n \leq C(\Delta - 2)^{\lceil D/2 \rceil}$ vertices.

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- **Corollary 4.** There exists a constant C such that each maximal planar bipartite graph G with maximum degree Δ and diameter D has at most $n \leq C(\Delta - 2)^{\lceil D/2 \rceil}$ vertices.
- For the case D odd and $\Delta \geq D$: We use the N-separator theorem by Tishchenko.

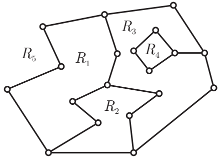

Figure: A 5-separator divides the plane into five regions

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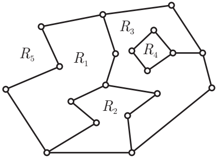

Figure: A 5-separator divides the plane into five regions

- **Theorem 5.** There exists a constant C such that each maximal planar bipartite graph G with maximum degree Δ and odd diameter D, for $\Delta \geq D$, has at most $n \leq C(\Delta - 2)^{\lfloor D/2 \rfloor}$ vertices.

The (Δ, D) problem in maximal planar bipartite graphs: A lower bound

- **Theorem 6.** (a) For any diameter $D = 2k$ $(k \geq 1)$ and maximum degree Δ $(\Delta \geq 5)$, there exists a maximal planar bipartite graph $G_{\Delta,D}$ whose number of vertices $n(G_{\Delta,D})$ is

$$\frac{\Delta\left(\Delta - 2 + \sqrt{\Delta(\Delta-4)}\right)^k + \Delta\left(\Delta - 2 - \sqrt{\Delta(\Delta-4)}\right)^k}{(\Delta-4)2^k} - \frac{8}{\Delta-4},$$

which is approximately $(\Delta-2)^k$, for Δ and D sufficiently large.

The (Δ, D) problem in maximal planar bipartite graphs: A lower bound

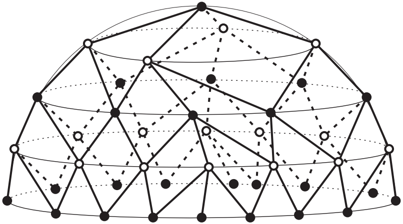

Figure: The superior half of a maximal planar bipartite graph drawn on a sphere for $\Delta = 4$

The (Δ, D) problem in maximal planar bipartite graphs: A lower bound

- **Theorem 6.** (b) For any diameter $D = 2k+1$ $(k \geq 1)$ and odd maximum degree Δ $(\Delta \geq 9)$, there exists a maximal planar bipartite graph $G_{\Delta,D}$ whose number of vertices $n(G_{\Delta,D})$ is

$$\begin{array}{ll} n(G_{\Delta,3}) = 3\Delta - 1 & \text{for } D = 3, \\ n(G_{\Delta,5}) = 3\Delta^2 - 21\Delta + 26 & \text{for } D = 5, \\ n(G_{\Delta,2k+1}) = 3\Delta^2 - 21\Delta + 26 + \frac{3(\Delta-7)(\Delta-2)^2((\Delta-3)^{k-2}-1)}{(\Delta-4)} & \text{for } D = 2k+1 \\ & \text{and } k > 2, \end{array}$$

which is approximately $3(\Delta-3)^k$, for Δ and D sufficiently large.

The (Δ, D) problem in maximal planar bipartite graphs: A lower bound

- **Theorem 6.** (c) For any diameter $D = 2k+1$ $(k \geq 1)$ and even maximum degree Δ $(\Delta \geq 10)$, there exists a maximal planar bipartite graph $G_{\Delta,D}$ whose number of vertices $n(G_{\Delta,D})$ is

$$
\begin{array}{ll}
n(G_{\Delta,3}) = 3\Delta - 2 & \text{for } D = 3, \\
n(G_{\Delta,5}) = 3\Delta^2 - 22\Delta + 26 & \text{for } D = 5, \\
n(G_{\Delta,2k+1}) = 3\Delta^2 - 22\Delta + 26 + \frac{(3\Delta-22)(\Delta-2)^2((\Delta-3)^{k-2}-1)}{(\Delta-4)} & \text{for } D = 2k+1 \\
 & \text{and } k > 2,
\end{array}
$$

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The (Δ, D) problem in maximal planar bipartite graphs: A lower bound

- Iterative construction for **Theorem 6.** (b), (c):

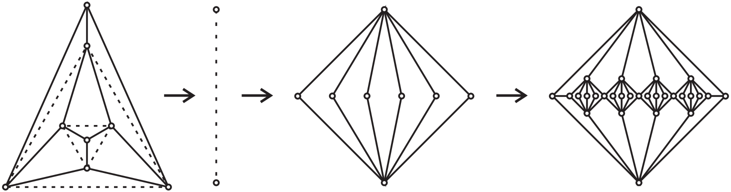

Figure: The iterative construction

- V. Chepoi, B. Estellon, Y. Vaxès. On covering planar graphs with a fixed number of balls. *Discrete Comput. Geom.* 37:237–244, 2007.
- M. Fellows, P. Hell, K. Seyffarth. Large planar graphs with given diameter and maximum degree. *Discrete Appl. Math.* 61:133–153, 1995.
- R. Feria-Puron, G. Pineda-Villavicencio. Constructions of large graphs on surfaces. *Graph. Combinator.*, DOI 10.1007/s00373-013-1323-y, 2013.
- C. Gavoille, D. Peleg, A. Raspaud, E. Sopena. Small k-dominating sets in planar graphs with applications. *Lect. Notes Comput. Sc.* 2204:201–216, 2001.
- R.J. Lipton, R.E. Tarjan. A separator theorem for planar graphs. *SIAM J. Appl. Math.* 36:177–189, 1979.
- E. Loz, H. Pérez-Rosés, G. Pineda-Villavicencio. The degree/diameter problem for planar graphs. http://combinatoricswiki.org/wiki.
- M. Miller, J. Širáň. Moore graphs and beyond: A survey of the degree/diameter problem. *Electron. J. Combin.* 20(2), #DS14v2, 2013.
- G. Pineda-Villavicencio, D.R. Wood. The degree-diameter problem for sparse graph classes. arXiv:1307.4456, 2013.
- G. Ringel. Two trees in maximal planar bipartite graphs. *J. Graph Theory* 17:755–758, 1993.
- S.A. Tishchenko. N-separators in planar graphs. *Eur. J. Combin.* 33:397–407, 2012.

Thank you for your attention.