Communicability in Cubic Generalized Moore Graphs

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Motivation: Optimal interconnection networks

Parallel architectures / Small-world networks

- ⋆ small maximum degree (hardware constraints)
- ⋆ small average distance (fast transmission of messages)
- ⋆ large number of nodes (computational/social power)
- ⋆ high connectivity (fault tolerance)

(or just a mathematical recreation *à la* Martin Gardner ?)

Generalized Moore bound

$G(V, E)$. $n = |V|$. Δ max.degree. $d(u, w)$ distance. D diameter.

Transmission $\sigma = \sum_{u,w \in V} d(u, w)$. Average dist. $\frac{\sigma}{n \cdot (n-1)}$.

Moore bound (maximum order for a graph with maximum degree Δ and diameter D)

$$n(\Delta, D) \leq n_M(\Delta, D) = 1 + \Delta \sum_{i=1}^{D} (\Delta - 1)^{i-1}$$

Question: Smallest transmission $\sigma(n, \Delta)$ of a graph with order n and maximum degree Δ.

generalized Moore bound

$$\sigma(n, \Delta) \geq \sigma_0(n, \Delta) = n(\Delta \sum_{i=1}^{k-1} i(\Delta - 1)^{i-1} + kR),$$

where $R = n - n_M(\Delta, k - 1) \geq 0$

$k =$ largest integer such that last inequality holds

V.G. Cerf, D.D. Cowan, R.C. Mullin, R.G. Stanton. A lower bound on the average shortest path length in regular graphs. Networks, 4 (1974), pp. 335342

P. Dankelmann, Mittlere Entfernung in Graphen. Dissertation,RWTH Aachen, Germany, 1993.

Moore graphs are generalized Moore graphs

There are over a hundred of known GMG, mostly cubic.

- ⋆ g-cages for $g \leq 8$
- ⋆ optimal degree-diameter graphs for $D \leq 4$.

Δ / D	2	3	4	5
3	*P* **10**	$C_5 * F_4$ **20**	*vC* **38**	*vC* 70
4	$K_3 * C_5$ **15**	*Allwr* 41	*Exoo* 98	$H'_3 d$ 364
5	$K_3 * X_8$ **24**	*Exoo* 72	*Exoo* 212	*Loz* 624
6	$K_4 * X_8$ **32**	*Exoo* 111	*Loz* 390	*Loz* 1404
7	*HS* **50**	*Exoo* 168	*Sampels* 672	*DinHaf* 2756

Largest known (Δ, D)-graph

More on Moore graphs as generalized Moore graphs

Buset (2000): Two maximal cubic graphs with diameter 4 and order 38:

- ⋆ von Conta graph (1982). (Alegre, Fiol & Yebra, 1986). Girth 7. # vertices 1,3,6,12,16 (dist. 0,1,2,3,4)
- ⋆ Doty graph (1982). Two 6-cycles. # vertices 1,3,6,11,17 (dist. 0,1,2,3,4)

The von Conta graph has the smallest average distance.

D. Buset; Maximal cubic graphs with diameter 4; Discrete Math.,101 (2000), pp. 53–61.

C. von Conta; Torus and other networks as communication networks with up to some hundred points; IEEE Trans Comp,c-32 (1983), pp. 657-666.

Karl W. Doty: Large Regular Interconnection Networks. 312-317.Proceedings of the 3rd International Conference on Distributed Computing Systems, Miami/Ft. Lauderdale, Florida, USA, October 18-22, 1982. IEEE Computer Society 1982

Number of cubic generalized Moore graphs - Integer sequence

The On-Line Encyclopedia of Integer Sequences.
N.J.A. Sloane
A005007

Number of cubic (i.e. regular of degree 3) generalized Moore graphs with 2n nodes.

Order	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
# GMG	0	1	2	2	1	2	7	6	1	1	0	1	2	9	40

More on known cubic generalized Moore graphs

- Cerf, Cowan, Mullin and Stanton (1973, 1974) characterize all small cubic GMG graphs up to order 24.
- McKay & Stanton (1979) provide more details on these graphs and characterize those of orders 24, 26, 28 and 30
- There are no cubic generalized Moore graphs of order $38 < n < 60$, because 9-cages have order ≥ 58 and all of them have girth 6. Brinkmann, McKay and Saager (1995).

Yet more on known cubic generalized Moore graphs

- ⋆ Recently McKay and Menon proved (through an unpublished computational study) that there are 6 GMG of order 60, and 1 of order 62 and 64.
- ⋆ Another cubic GMG has 70 vertices -again a candidate solution for the degree-diameter problem-
- ⋆ There is also one GMG with order 126: the Tutte 12-cage.

Known cubic generalized Moore graphs

order	#	v-t	D	σ	name
4	1	1	2	12	*Tetra*
6	2	1	2	42	*Utility /Prism*
8	2	1	2	88	*Wagner*
10	1	1	2	150	*Petersen*
12	2	0	3	252	
14	7	1	3	378	*Heawood*
16	6	0	3	528	
18	1	0	3	702	
20	1	0	3	900	$C_5 * F_4$
22	0	0	3	1122	
24	1	0	4	1416	*McGee*
26	2	0	4	1742	*10cross#/ genPet*
28	9	1	4	2100	*Coxeter*
30	40	1	4	2490	*Levi*
32	56	0	4	2912	

cubic generalized Moore graphs

Known cubic generalized Moore graphs

order	#	v-t	D	σ	name
34	3	0	4	3366	
36	0	0	4	3852	
38	1	0	4	4370	*von Conta*
40	0	0	4	4920	
$\vdots$	0	$\vdots$	$\vdots$	$\vdots$	
58	0	0	5	12006	
60	6	$\geq$ 2?	5	13020	*von Conta*
62	1	?	5	14074	
64	1	?	5	15168	
$\vdots$	?	$\vdots$	$\vdots$	$\vdots$	
70	$\geq$1 ?	?	5	18690	*von Conta*
$\vdots$	?	$\vdots$	$\vdots$	$\vdots$	
126	$\geq$1 ?	?	6	72954	*Tutte12-cage*

cubic generalized Moore graphs

Betweenness centrality

Definitions

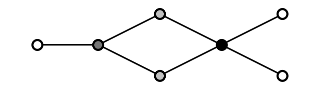
Betweenness centrality (BC) of node w with respect to nodes u,v:

$$b_w(u,v) = \frac{\sigma_{uv}(w)}{\sigma_{uv}}$$

Normalized betweenness centrality of vertex w:

$$\beta_w = \frac{1}{(n-1)(n-2)} \sum_{u,v \neq w} b_w(u,v)$$

BC of a graph of order n is the set of values $\{\beta_1, \beta_2, \ldots, \beta_n\}$.



$$BC = \{0, \tfrac{11}{30}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{19}{30}, 0, 0\}$$

Average normalized BC of a graph of order n:

$$\bar{BC} = \overline{\beta} = \frac{\sum_{u \in V} \beta_u}{n}$$

$\overline{\beta}$ is related to the average distance as $\bar{l} = (n-2)\overline{\beta} + 1$.

Communicability centrality

$G = (V, E)$, adjacency matrix A

Communicability centrality: Measures the centrality of a vertex from the number of weighted walks that start and end on it. (Longer walks contribute less.)

Communicability centrality

$$C = \sum_{i=1}^{\infty} \frac{A^k}{k!} = e^A, \quad CC = \{C_{ii}\}$$

Communicability distance index of G: Measures the "closeness" of vertices. A small value means they are closer.

Υ: Communicability distance of G

$$\Upsilon(G) = \frac{1}{2} \sum_{u,w} (C_{uu} + C_{ww} - 2C_{uw})^{\frac{1}{2}}$$

Walk entropy

$G = (V, E)$, adjacency matrix A

Walk entropy

$$S^V(G,\beta) = -\sum_{i=1}^{n} p_i(\beta) \ln p_i(\beta)$$

with $p_i(\beta) = \frac{(e^{\beta A})_{ii}}{Tr(e^{\beta A})}$, $\beta = \frac{1}{k_B}$, k_B Boltzmann cons., $T > 0$ absolute temp.).

Theorem (Benzi, 2014)

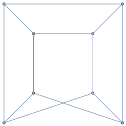
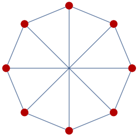
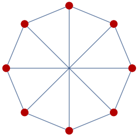
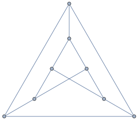
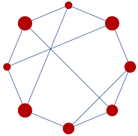
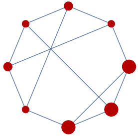
A graph G is walk-regular if and only if $S^V(G,\beta) = \ln n$ for all $\beta \geq 0$.

Cubic generalized Moore graphs: $|V| = 8$

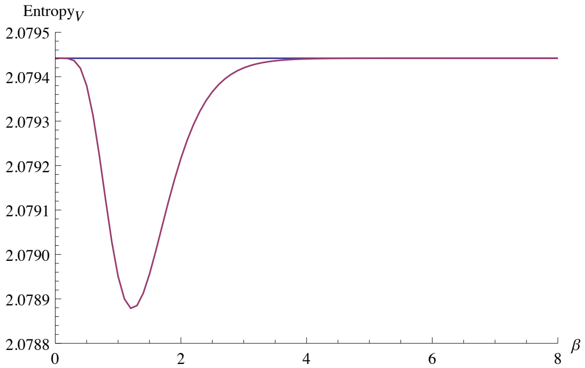
order	#	v-t	D	σ	name
4	1	1	2	12	*Tetra*
6	2	1	2	42	*Utility /Prism*
8	**2**	**1**	**2**	**88**	**Wagner**
10	1	1	2	150	*Petersen*
12	2	0	3	252	
14	7	1	3	378	*Heawood*
16	6	0	3	528	
18	1	0	3	702	
20	1	0	3	900	$C_5 * F_4$
22	0	0	3	1122	
24	1	0	4	1416	*McGee*
26	2	0	4	1742	*10cross#/ genPet*
28	9	1	4	2100	*Coxeter*
30	40	1	4	2490	*Levi*
32	56	0	4	2912	

cubic generalized Moore graphs

Cubic generalized Moore graphs: $|V| = 8$

Wagner graph	 $D = 2,\ \sigma = 88$	 $BC = 16$ $\{2\}$	 $\Upsilon = 44.4272$ $\{3.6369\}$
Cubic 8,3 (wolfram)	 $D = 2,\ \sigma = 88$	 $\mathrm{BC} = 16$ $\{1.5, 2., 2.3333\}$	 $\Upsilon = 46.2551$ $\{3.9029, 3.6387, 3.7056\}$

Entropy cubic GMG (Wagner / Cubic 8,3)

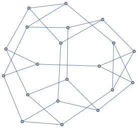
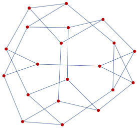
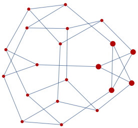


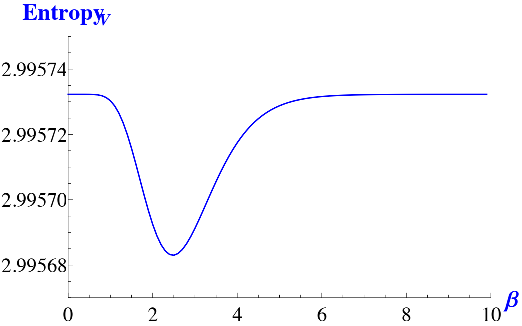
Cubic generalized Moore graphs: $|V| = 20$

order	#	v-t	D	σ	name
4	1	1	2	12	*Tetra*
6	2	1	2	42	*Utility /Prism*
8	2	1	2	88	*Wagner*
10	1	1	2	150	*Petersen*
12	2	0	3	252	
14	7	1	3	378	*Heawood*
16	6	0	3	528	
18	1	0	3	702	
20	**1**	**0**	**3**	**900**	$C_5 * F_4$
22	0	0	3	1122	
24	1	0	4	1416	*McGee*
26	2	0	4	1742	*10cross#/ genPet*
28	9	1	4	2100	*Coxeter*
30	40	1	4	2490	*Levi*
32	56	0	4	2912	

cubic generalized Moore graphs

Cubic GMG with 20 vertices, optimal Moore

			
$C_5{*}F_4$ graph	$D = 3,\ \sigma = 900$	$BC = 260$ $\{13\}$	$\Upsilon = 411.555$ $\{3.27387, 3.27383, 3.28889, \}$



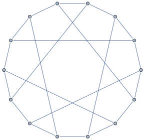
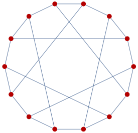
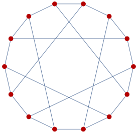
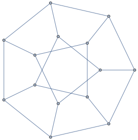
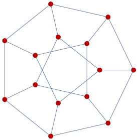
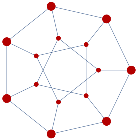
Cubic generalised Moore graphs: $|V| = 14$

order	$\#$	v-t	D	σ	name
4	1	1	2	12	*Tetra*
6	2	1	2	42	*Utility /Prism*
8	2	1	2	88	*Wagner*
10	1	1	2	150	*Petersen*
12	2	0	3	252	
14	**7**	**1**	**3**	**378**	**Heawood**
16	6	0	3	528	
18	1	0	3	702	
20	1	0	3	900	$C_5 * F_4$
22	0	0	3	1122	
24	1	0	4	1416	*McGee*
26	2	0	4	1742	*10cross#/ genPet*
28	9	1	4	2100	*Coxeter*
30	40	1	4	2490	*Levi*
32	56	0	4	2912	

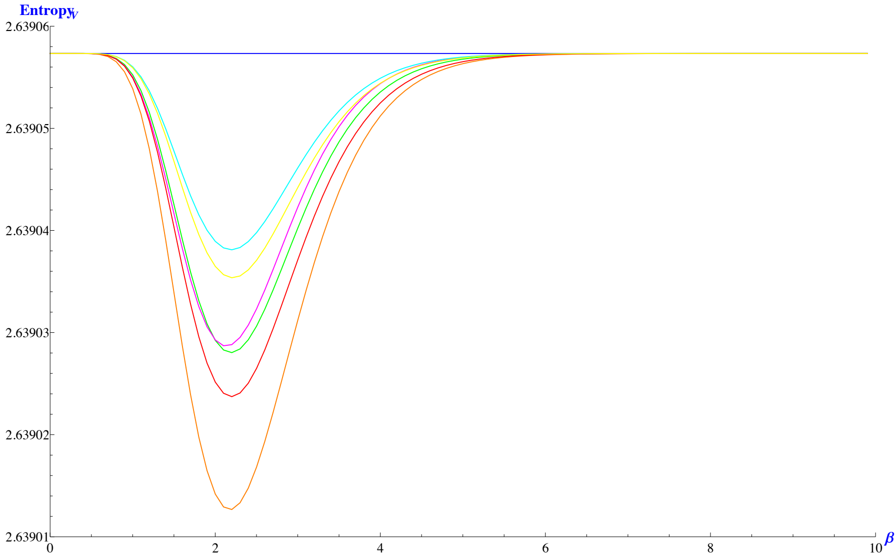
cubic generalized Moore graphs

Cubic generalised Moore graphs: $|V| = 14$

7 graphs: Heawood, (7,2) Generalized Petersen, 3-Crossing Number, wolfram Cubic (501, 503 504, 505) Error in the original paper and review (one of the 8 graphs has $\sigma = 380$).

Heawood graph	 $D = 3$, $\sigma = 378$	 $BC = 98$ $\{7\}$	 $\Upsilon = 180.814$ $\{3.30525\}$
Generalized Petersen	 $D = 3$, $\sigma = 378$	 BC $= 98$ $\{7\}$	 $\Upsilon = 181.878$ $\{3.32681, 3.34113\}$

Entropy cubic GMG with $|V| = 14$



cubic generalised Moore graphs $|V| = 16$

order	#	v-t	D	σ	name
4	1	1	2	12	*Tetra*
6	2	1	2	42	*Utility /Prism*
8	2	1	2	88	*Wagner*
10	1	1	2	150	*Petersen*
12	2	0	3	252	
14	7	1	3	378	*Heawood*
16	**6**	**0**	**3**	**528**	
18	1	0	3	702	
20	1	0	3	900	$C_5 * F_4$
22	0	0	3	1122	
24	1	0	4	1416	*McGee*
26	2	0	4	1742	*10cross#/ genPet*
28	9	1	4	2100	*Coxeter*
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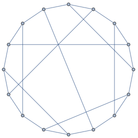
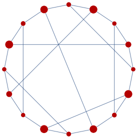
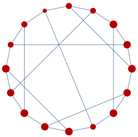
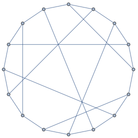
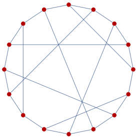
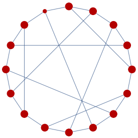
32	56	0	4	2912	

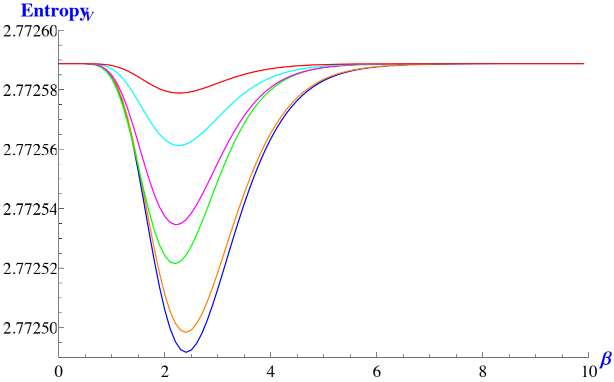
cubic generalized Moore graphs

Cubic generalised Moore graphs: $|V| = 16$

6 non isomorphic graphs (none v-t)

Cerf16A graph	 $D = 3,\ \sigma = 528$	 $BC = 144$ $\{9.7, 8.5, 9.8, 8.3, 9\}$	 $\Upsilon = 248.482$ $\{3.28 - -3.32\}$
Cerf16B Petersen	 $D = 3,\ \sigma = 528$	 $\mathrm{BC} = 144$ $\{9\}$	 $\Upsilon = 248.024$ $\{3.2886, 3.3033\}$

Entropy cubic GMG with $|V| = 16$



Open problems

- Are there infinitely many GMG for a given degree ?
- Find GMG and GMG families for different degrees.
- Communication properties of GMG (synchronizability, broadcasting, gossiping).
- First passage time. Spectral characterisation.

Thanks !

Thanks!