On Ramsey numbers for wheels and union of graphs

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1. From any 3 persons there are always 2 persons with the same gender.

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1. From any 3 persons there are always 2 persons with the same gender.
2. From any 6 persons, there are at least 3 persons know each others <u>or</u> 3 persons do not know each others. The number 6 is the least integer.

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2. From any 6 persons, there are at least 3 persons know each others or 3 persons do not know each others. The number 6 is the least integer.
3. From any five points in general position in the plane, there are four points forming a convex 4-gon. The number 5 is the smallest.

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3. From any five points in general position in the plane, there are four points forming a convex 4-gon. The number 5 is the smallest.
4. From any 9 persons, there are at least 3 persons know each others <u>or</u> 4 persons do not know each others. The number 9 is the least integer.

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3. From any five points in general position in the plane, there are four points forming a convex 4-gon. The number 5 is the smallest.
4. From any 9 persons, there are at least 3 persons know each others <u>or</u> 4 persons do not know each others. The number 9 is the least integer.
5. In any ordering of the first 101 positive integers, there are always 11 of them forming an increasing or decreasing sequence.

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However, if you order the first 100 positive integers in the following way then there is no increasing nor decreasing subsequence with 11 elements.

91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90
71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70
51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50
31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30
11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

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- Ramsey theory was initially studied in the context of the problem of finding a regular procedure to determine the consistency of any given logical formula (1928).
- The objective was to give a decision procedure for the sentences of propositional logic.

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- Complete disorder is impossible is the theme of Ramsey Theory, as stated by Theodore S. Motzkin.
- A non-technical interpretation of Ramsey Theory: Every sufficiently large structure, regardless of how disorderly it may appear to be, contains an orderly substructure of any prescribed size.

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- A non-technical interpretation of Ramsey Theory: Every sufficiently large structure, regardless of how disorderly it may appear to be, contains an orderly substructure of any prescribed size.
- The word "Ramsey" is due to Frank Plumpton Ramsey, a student of Bertrand Russell, G.E. Moore, Ludwig Wittgenstein, and John Maynard Keynes.

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Infinite version. For any positive integers k and r, if the collection of all r-element subsets of an infinite set S is colored in k colors, then S contains an infinite subset S_1 such that all r-element subsets of S_1 are assigned the same color.

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Finite version. For any positive integers r, n, and k there is an integer $m_0 = R(r, n, k)$ such that if $m \geq m_0$ and the collection of all r-element subsets of an m-element set S_m is colored in k colors, then S_m contains an n-element subset S_n such that all r-element subsets of S_n are assigned the same color.

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The Ramsey theory becames famous after Paul Erdös and George Szekeres (1935) applied it in graph theory.

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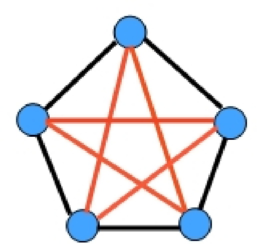
Problem 2.1

For integers n and m, find the smallest integer $r := R(n,m)$ such that in any 2-coloring (red or blue) on the edges of the complete graph K_r on r vertices, there exists either a monochromatic complete graph on n or m vertices. (F.P. Ramsey, 1928)

$R(3,3) = 6$.



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Ramsey Number $R(s,t)$ has the following properties.

Trivially, we have:

- Symmetry: $R(s,t) = R(t,s)$, for all $s,t \geq 2$.
- $R(s,2) = R(2,s) = s$.

Theorem 2.2

The function $R(s,t)$ is finite for all $s,t \geq 2$. If $s > 2, t > 2$, then:

- $R(s,t) \leq R(s-1,t) + R(s,t-1)$, *and*
- $R(s,t) \leq \binom{s+t-2}{s-1}$.

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- $R(s,t) \le R(s-1,t) + R(s,t-1)$.
  Let $n = R(s-1,t) + R(s,t-1)$.
  Consider any red-blue coloring on the edges of K_n. Since the degree of vertex x: $d(x) = R(s-1,t) + R(s,t-1) - 1$, then there are at least $n_1 = R(s-1,t)$ red edges or $n_2 = R(s,t-1)$ blue edges incident to x.
  In any case, we will have a red K_s or a blue K_t.

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  In any case, we will have a red K_s or a blue K_t.

- By induction on $s+t$.
  If $s = 2$ or $t = 2$, it holds. Assume it holds for $2 \leq s^* + t^* < s+t$. Then,
  - $R(s,t) \leq R(s-1,t) + R(s,t-1)$
  - $R(s,t) \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}$. $\square$

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Theorem 2.3

$R(3,4) = 9.$

Proof.
Consider any 2-coloring on K_9. Consider the induced subgraph by red edges. Then, this subgraph cannot be 3-regular. Therefore, there are two possibilities:

- Case 1. $\exists v \in V(K_9)$ incident with 4 red edges.
- Case 2. $\exists v \in V(K_9)$ incident with 6 blue edges.

In any case, there will be a red K_3 or a blue K_4. $\square$

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Finding the exact value of $R(n,m)$ has received a lot of attention. However, the results are still far from satisfactory.

- Greenwood & Gleason (1955): $R(3,3)=6, R(3,4)=9, R(3,5)=14,\ R(4,4)=18.$
- Kéry (1964): $R(3,6)=18.$
- Kalbfleisch (1965): $R(3,7)=23$
- Grienstead & Roberts (1982): $R(3,8)=28;\ R(3,9)=36.$
- McKay & Radziszowski (1995): $R(4,5)=25.$

n\m	3	4	5	6	7	8	9	10	11
3	6	9	14	18	23	28	36	40 43	46 51
4		18	25	35 41	49 61	56 84	73 115	92 149	97 191
5			43 49	58 87	80 143	101 216	125 316	143 442	159 633
6				102 165	113 298	130 495	169 780	179 1171	253 1804

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Joel Spencer, 1994: "Erdös asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5,5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6,6)$. In that case, he believes, we should attempt to destroy the aliens."

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- $R(t,t)$ is greatest interest since it is the hardest to estimate. $R(t,t) \le \binom{2t-2}{t-1} \le \frac{2^{2t-2}}{\sqrt{t}}$.

- It is hardly to improve this bound. The best improvement was due to Thomason (1988): $R(t,t) \le \frac{2^{2t}}{t}$, if t is big.

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- $R(t,t)$ grows exponentially: $R(t,t) \geq 2^{t/2}$.
- It is widely believed that there is a constant c, perhaps even $c = 1$, such that:

$$R(t,t) = 2^{(c+o(1))t},$$

but this is very far from being proved.
- By replacing the first two colors by a new color (in a k-coloring $R_k(t_1, t_2, \cdots, t_k)$), we have:

$$R_k(t_1, t_2, \cdots, t_k) \leq R_{k-1}(R(t_1, t_2), t_3, \cdots, t_k).$$

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1. In the fields: number theory, harmonic analysis, algebra, computational geometry, topology, set theory, logic, ergodic theory, information theory and computer science.
2. In particular, in information theory and computer science: it is used for coding, parallel and distributed computing, boolean function computation, automated theorem, approximation algorithm and complexity.

Please refer to:
Vera Rosta, Ramsey Theory Applications, *Electronic Journal of Combinatorics*, Dec 2004, #DS13.

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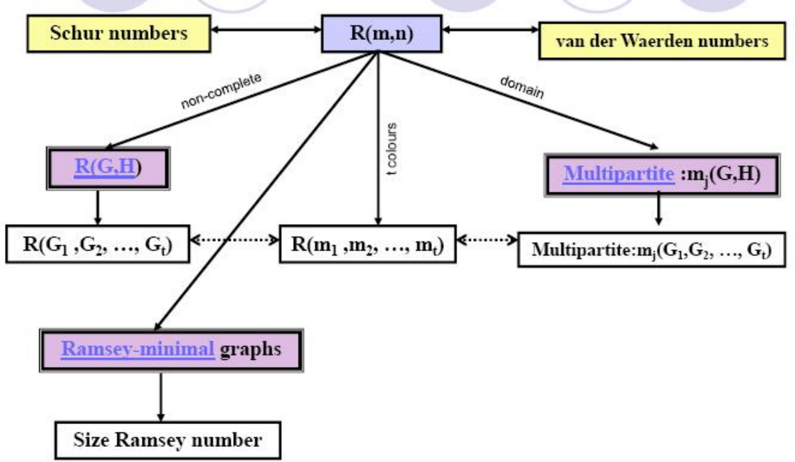

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Graph Ramsey theory has grown enormously in the last five decades to become presently one of the most active areas in Ramsey theory.

Definition. Let G and H be two graphs. The Ramsey number $R(G, H)$ is the smallest integer r such that in any red-blue coloring on the edges of K_r on r vertices, there exists either a red G or a blue H as a subgraph.

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This definition is equivalent to the following:

Definition. Let G and H be two graphs. The Ramsey number $R(G,H)$ is the smallest integer r such that every graph F of order r will satisfy the following condition: either $F \supseteq G$ or $\overline{F} \supseteq H$.

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Let $\chi(H)$ be the chromatic number of graph H, and $c(G)$ be the order of the largest component of G.

Chvátal and Harary (1972):

$$R(G,H) \geq (\chi(H)-1)(c(G)-1)+1,$$

since $F = (\chi(H)-1)K_{(c(G)-1)} \not\supseteq G$ and $\overline{F} \not\supseteq H$.

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Example 1. Show that $R(P_3, C_3) = 5$.

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Example 1. Show that $R(P_3, C_3) = 5$.

Proof.

- Consider $F = 2K_2$. Then, $F \not\supseteq P_3$ and $\overline{F} = C_4 \not\supseteq C_3$. So, $R(P_3, C_3) \geq 5$.

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Proof.

- Consider $F = 2K_2$. Then, $F \not\supseteq P_3$ and $\overline{F} = C_4 \not\supseteq C_3$. So, $R(P_3, C_3) \geq 5$.
- To show the upper bound, consider any 2-coloring on the edges of K_5 so that no red P_3. This means that there are at most two red edges. This implies that there exists a blue C_3 in K_5. So, $R(P_3, C_3) \leq 5$.

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- To show the upper bound, consider any 2-coloring on the edges of K_5 so that no red P_3. This means that there are at most two red edges. This implies that there exists a blue C_3 in K_5. So, $R(P_3, C_3) \leq 5$.
- Thus, $R(P_3, C_3) = 5$. □

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Example 2. Show $R(K_{1,3}, C_3) = 7$.

Proof.

- Consider $F = 2K_3$. Then, $F \not\supseteq K_{1,3}$ and $\overline{F} = K_{3,3} \not\supseteq C_3$. So, $R(K_{1,3}, C_3) \geq 7$.

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Example 2. Show $R(K_{1,3}, C_3) = 7$.

Proof.

- Consider $F = 2K_3$. Then, $F \not\supseteq K_{1,3}$ and $\overline{F} = K_{3,3} \not\supseteq C_3$. So, $R(K_{1,3}, C_3) \geq 7$.
- To show the upper bound, consider any 2-coloring on the edges of K_7. If there is no blue C_3 then we have a red C_3 (by $R(3,3) = 6$). But if we have a red C_3 then each vertex v in this C_3 is adjacent to the four remaining vertices with blue edges. This forces either a red $K_{1,3}$ or a blue C_3. So, $R(K_{1,3}, C_3) \leq 7$.
- Thus, $R(K_{1,3}, C_3) = 7$. □

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The determination of Ramsey numbers $R(G,H)$ has been studied for various combinations of graphs G and H.

- $R(P_n, P_m) = n + \lfloor m/2 \rfloor - 1,\ n \geq m \geq 2.$
  (L. Geréncser, A. Gyárfás 1967)

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- $R(P_n, P_m) = n + \lfloor m/2 \rfloor - 1,\ n \geq m \geq 2.$
  (L. Geréncser, A. Gyárfás 1967)

- $R(C_n, C_m) =$
  $$\begin{cases} 2n-1, & \text{for } 3 \leq m \leq n, m \text{ odd}; (n,m) \neq (3,3); \\ n-1+\frac{m}{2}, & \text{for } 4 \leq m \leq n;\ m \text{ and } n \text{ even}; (n,m) \neq (4,4); \\ \max\{n-1+\frac{m}{2}, 2m-1\}, & \text{for } 4 \leq m \leq n, m \text{ even and } n \text{ odd}. \end{cases}$$
  (V. Rosta 1973, R.J. Faudree and R.H. Schelp 1974, Karolyi & Rosta 2001)

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- $R(C_n, K_m) = (n-1)(m-1)+1$,

  for $n \geq m^2 - 2$ [Bondy, Erdös 1972],

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  for $n \geq m^2 - 2$ [Bondy, Erdös 1972],
  for $n > 3 = m$ [Faudree, Schelp 1974],
  for $n \geq 4 = m$ [Yang, Huang, Zhang 1999],
  for $n \geq 5 = m$ [Bollobás, Jayawardene,
  Yang, Huang, Rousseau, Zhang 2000]
  for $n \geq 6 = m$ [Schiermeyer 2003],
  for $n \geq m \geq 7$ with $n \geq m(m-2)$ [Schiermeyer 2003],
  for $n \geq 7 = m$ [Chen, Zhang 2006],
  for $n \geq 4m+2$ and $m \geq 3$ [Nikiforov 2005].

- It was conjectured $R(C_n, K_m) = (n-1)(m-1)+1$, for all $n \geq m \geq 3$ except $n = m = 3$.

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- Chvátal (1977) proved that:

$$R(T_n, K_m) = (n-1)(m-1)+1.$$

- Now, if graph K_m is replaced by a graph G of diameter 2, for instance a wheel W_m, then what is $R(T_n, G)$?

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Let W_n be a wheel of $n+1$ vertices, namely $W_n = K_1 + C_n$.

Surahmat, etb 2001:
For all $n \geq 3$, $R(P_n, W_4) = 2n - 1$ and $R(P_n, W_5) = 3n - 2$.

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For all $n \geq 3$, $R(P_n, W_4) = 2n-1$ and $R(P_n, W_5) = 3n-2$.

Proof:
* Suppose F be a $(P_n, W_4, 2n-1)$-good graph.

* Let P be a longest path in F with endpoints p_1 and p_2.
Obviously, $zp_1, zp_2 \notin E(F)$ for each $z \in V(F) \setminus V(P)$.

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* Let $X = V(F) \backslash V(P)$ and Q be a longest path in $F[X]$.

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* Let P be a longest path in F with endpoints p_1 and p_2. Obviously, $zp_1, zp_2 \notin E(F)$ for each $z \in V(F) \backslash V(P)$.

* Let $X = V(F) \backslash V(P)$ and Q be a longest path in $F[X]$.

* Let q_1 and q_2 be its endpoints. Since $|V(F)| = 2n-1$ and the longest path in F is of length $\leq n-1$ then there exists a vertex $w \notin V(P) \cup V(Q)$ such that w is independent to all endpoints p_1, p_2, q_1 and q_2.

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Let W_n be a wheel of $n+1$ vertices, namely $W_n = K_1 + C_n$.

Surahmat, etb 2001:
For all $n \geq 3$, $R(P_n, W_4) = 2n-1$ and $R(P_n, W_5) = 3n-2$.

Proof:
* Suppose F be a $(P_n, W_4, 2n-1)$-good graph.

* Let P be a longest path in F with endpoints p_1 and p_2. Obviously, $zp_1, zp_2 \notin E(F)$ for each $z \in V(F) \setminus V(P)$.

* Let $X = V(F) \setminus V(P)$ and Q be a longest path in $F[X]$.

* Let q_1 and q_2 be its endpoints. Since $|V(F)| = 2n-1$ and the longest path in F is of length $\leq n-1$ then there exists a vertex $w \notin V(P) \cup V(Q)$ such that w is independent to all endpoints p_1, p_2, q_1 and q_2.

* Thus, we have W_4 with w as a hub and $\{p_1, p_2, q_1, q_2\}$ as rims, a contradiction. This concludes the proof. $\square$

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These Ramsey numbers remain the same when we replace W_4 and W_5 by W_6 and W_7 respectively. Preciously, we have the following theorem.

etb, 2002:
$R(P_n, W_6) = 2n - 1$ if $n \geq 6$ and
$R(P_n, W_7) = 3n - 2$ if $n \geq 7$.

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$R(P_n, W_7) = 3n - 2$ if $n \geq 7$.

By employing a generalised version of the previous method, we could show that the above assertion is true if $n \geq \frac{m}{2}(m-2)$. Precisely, we have:

etb, Surahmat 2001:
1) If $n \geq \frac{m}{2}(m-2), m \geq 4$ even then $R(P_n, W_m) = 2n - 1$.
2) If $n \geq \frac{m-1}{2}(m-3), m \geq 5$ odd then $R(P_n, W_m) = 3n - 2$.

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This result has been refined by Yaojun Chen, Yunqing Zhang and Kemin Zhang (2002) by showing that:

1) $R(P_n, W_m) = 2n-1$ for even m and $n \geq m-1 \geq 3$,

2) $R(P_n, W_m) = 3n-2$ for odd m and $n \geq m-1 \geq 2$.

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2) $R(P_n, W_m) = 3n-2$ for odd m and $n \geq m-1 \geq 2$.

However, for $n < m$ the situation is different. Here we present our knowledge on this.

Salman, Broersma, 2007:
For all $m \geq 6$,

$$R(P_4, W_m) = \begin{cases} m+2 & \text{if } m \equiv 0,2 \mod 3, \\ m+3 & \text{if } m \equiv 1 \mod 3. \end{cases}$$

For all $m \geq 8$,

$$R(P_5, W_m) = \begin{cases} m+3 & \text{if } m \equiv 0,2,3 \mod 4, \\ m+4 & \text{if } m \equiv 1 \mod 4. \end{cases}$$

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Surprisingly, for $n \geq 3$ the Ramsey numbers $R(S_n, W_5) = R(P_n, W_5)$, but the $R(S_n, W_4) \neq R(P_n, W_4)$.

Surahmat, etb 2001:
For all $n \geq 3$,

$$R(S_n, W_4) = \begin{cases} 2n-1 & \text{if } n \text{ is odd,} \\ 2n+1 & \text{if } n \text{ is even.} \end{cases}$$

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However, ...
Chen, Zhang, Zhang, 2004:
$R(S_n, W_6) = 2n+1$, for all $n \geq 3$.

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$R(S_n, W_6) = 2n+1$, for all $n \geq 3$.

Let $m \geq 6$ be even, $n = km/2 + 2$ $k \geq 2$. Let $G = H \cup K_{n-1}$, where $\overline{H} = (k+1)K_{m/2}$. Obviously, G has order $2n + m/2 - 3$ and $\Delta(G) = n-2$ and hence $G \not\supseteq S_n$. It is not difficult to see $\overline{G} \not\supseteq W_m$. Thus, $R(S_n, W_m) \geq 2n + m/2 - 2$ if $n = km/2 + 2$, for $k \geq 2$.

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Furthermore, Surahmat, etb and Broersma (2002) showed that the following theorem holds for stars and **odd** wheels:

Surahmat, etb, Broersma, 2002:
For all $n \geq 2m-4$, $m \geq 5$ and m odd, $R(S_n, W_m) = 3n-2$.

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This result was improved by Chen, Zhang, Zhang, *European Journal of Combinatorics* 25 (2004) 1067-1075:

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For all $n \geq m-1 \geq 2$ and m odd, $R(S_n, W_m) = 3n-2$.

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Hasmawati, etb, Assiyatun, *JCMCC* 55 (2005), 123-128: improved...

Hasmawati, etb, Assiyatun, 2005:
For all $n \geq (m+1)/2$, m odd and $m \geq 5$, $R(S_n, W_m) = 3n-2$.

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With a star-like tree we mean a subdivided star (which is not a path), i.e., a tree with exactly one vertex of degree exceeding two.

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With a star-like tree we mean a subdivided star (which is not a path), i.e., a tree with exactly one vertex of degree exceeding two.

We denote by $Y_{n,l_1,l_2,\ldots,l_k}$ the star-like tree consisting of a P_n, and k additional mutually disjoint paths $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$ all attached by one edge from one of their end vertices to the same end vertex of the P_n. Then, we have the following theorem.

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Surahmat, etb, Broersma, 2002:

$R(Y_{n,l_1,l_2,\ldots,l_k}, W_m) = 3(n + \sum_{i=1}^{k} l_i) - 2$ for $n \geq 2m-4, n \geq l_i$ for each $i = 1,2,\ldots,k$, $m \geq 5$ **odd**, and $\lfloor \frac{m}{2} \rfloor + 1 \leq \sum_{i=1}^{k} l_i$.

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etb, Surahmat, Nababan, Miller (2002):

- Let $n \geq 4$ and assume that we are given a particular tree T_n of n vertices other than a star. Then, the Ramsey number $R(T_n, W_4) = 2n - 1$.
- Let $n \geq 3$ and assume that we are given a particular tree T_n of n vertices. Then the Ramsey number $R(T_n, W_5) = 3n - 2$.

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- Let $n \geq 3$ and assume that we are given a particular tree T_n of n vertices. Then the Ramsey number $R(T_n, W_5) = 3n - 2$.

These results proved by:
* Consider the largest independent set.
* Lemmas:
For odd $n \geq 3$, $n = 2t + 1$, the graph $H_t + K_1$ contains all trees T_n
For even $n \geq 4$, $n = 2t$, the graph H_t contains all trees T_n other than a star.

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Chen, Zhang, Zhang, 2004:
for a tree T_n with $\Delta(T_n) \geq n-3$, we have:

- $R(S_n(1,1), W_6) = 2n$, for $n \geq 4$
- $R(S_n(1,2), W_6) = 2n$, for $n \geq 6$ and $n \equiv 0 \pmod 3$.
- $R(S_n(3), W_6) = R(S_n(2,1), W_6) = 2n-1$, for $n \geq 6$.
- $R(S_n(1,2), W_6) = 2n-1$ for $n \geq 6$ and $n \not\equiv 0 \pmod 3$.

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In general, by modifying the examples above, we can show for even m, $R(T_n, W_m)$ depends on the values of n and m if $\Delta(T_n)$ is large enough. Since $R(P_n, W_m) = 2n-1$ for even m and $n \geq m-1 \geq 3$, we believe $R(T_n, W_m) = 2n-1$ for m even and $n \geq m-1$ if $\Delta(T_n)$ is small.

Problem 4.1

Characterize all trees T_n with $R(T_n, W_m) = 2n-1$ for m even and $n \geq m-1$.

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Cycles behave like paths in their Ramsey numbers with respect to Wheels.

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Cycles behave like paths in their Ramsey numbers with respect to Wheels.
However, for proving it we have to employ different techniques and utilize the results in Hamiltonicity.

Large cycles vs. small wheels:
Surahmat, etb, Broersma 2004:
$R(C_n, W_4) = 2n-1, n \geq 5,$
$R(C_n, W_5) = 3n-2,\ n \geq 5.$

Surahmat, etb, Tomescu, (2006):
$R(C_n, W_m) = 2n-1$, for even m, and $n \geq 5m/2-1$.

Surahmat, etb, Tomescu, (2008):
$R(C_n, W_m) = 3n-2$, for $m \geq 5$ odd, and $n > \frac{5m-9}{2}$.

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Small cycles vs. large wheels:

Surahmat, etb, Nababan 2002:
$R(C_4, W_m) = 9, 10, 9$ for $m = 4, 5, 6$.

Tse 2003:
$R(C_4, W_m) = 11, 12, 13, 14, 16$ and 17 for $m = 7, 8, 9, 10, 11$, and 12.

Surahmat, etb, Uttunggadewa, Broersma 2005:
$R(C_4, W_m) \leq m + \lceil \frac{m}{2} \rceil + 1$, for $m \geq 13$.

Dybizbanski and Dzido 2013:
$R(C_4, W_m) = m + 4$ for $14 \leq m \leq 16$.
$R(C_4, W_{q^2+1}) = q^2 + q + 1$ for a prime power $q \geq 4$.
$R(C_4, W_m) \leq m + \lfloor \sqrt{m-2} \rfloor + 1$, for $m \geq 11$.

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$R(C_4, W_m)$ relates to Finding the largest C_4-free graph whose its component containing no W_m.

Note that:
For $k \geq 5$, a (k, g)-graph of order n provides a lower bound of $R(C_4, W_{n-k})$.

Open Problem: Find the general formula of $R(C_4, W_m)$, for a bigger m.

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Question.
Is there any relation between $R(G_i, H_j)$ with $R(\cup G_i, H_j)$, $R(G_i, \cup H_j)$ or $R(\cup G_i, \cup H_j)$?

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An upper bound for the Ramsey number $R(\cup G_i, H)$ is given by **Hasmawati, etb, Assiyatun, 2008**:

For any connected graphs G and H, and $k \geq 1$, we have $R(kG, H) \leq R(G, H) + (k-1)|V(G)|$.

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- $k = 1$ it is trivial. Assume the theorem holds for any $r < k$.

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- Let F be a graph with order $R(G, H) + (k-1)|V(G)|$. Suppose $\overline{F} \not\supseteq H$. By induction hypothesis, $F \supseteq (k-1)G$.

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- Now, write $T = F \backslash (k-1)G$. Thus, $|V(T)| = R(G, H)$. Since $\overline{T} \not\supseteq H$, then T must contain G. Hence, $F \supseteq (k-1)G \cup G$.

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- Now, write $T = F \backslash (k-1)G$. Thus, $|V(T)| = R(G, H)$. Since $\overline{T} \not\supseteq H$, then T must contain G. Hence, $F \supseteq (k-1)G \cup G$.
- Therefore, we have $R(kG, H) \leq R(G, H) + (k-1)|V(G)|$.

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etb, Hasmawati, Assiyatun, 2006:
$R(kS_n, W_m) = 3n - 2 + (k-1)n$, if m is odd, $n \geq \frac{m+1}{2} \geq 3$.

The good graph: $F_1 = K_{kn-1} \cup 2K_{n-1}$.

For $n \geq 3$,

$$R(kS_n, W_4) = \begin{cases} (k+1)n & \text{if } n \text{ is even and } k \geq 2, \\ (k+1)n - 1 & \text{if } n \text{ is odd and } k \geq 1. \end{cases}$$

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$$R(kS_n, W_4) = \begin{cases} (k+1)n & \text{if } n \text{ is even and } k \geq 2, \\ (k+1)n - 1 & \text{if } n \text{ is odd and } k \geq 1. \end{cases}$$

The good graph: $F_1 = (H_{\frac{kn-2}{2}} + K_1) \cup H_{\frac{n}{2}}$ and;
$F_2 = K_{kn-1} \cup K_{n-1}$ (n odd).

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etb, Hasmawati, Assiyatun, 2006:

Let $n_i \geq n_{i+1}$, $i = 1, 2, \ldots, k-1$. If $n_i \geq (n_i - n_{i+1})(m-1)$ then $R(\bigcup_{i=1}^{k} T_{n_i}, K_m) = R(T_{n_k}, K_m) + \sum_{i=1}^{k-1} n_i$.

\# The good graph: $F = (m-2)K_{n_k - 1} \cup K_{\sum_{i=1}^{k} n_i - 1}$.

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The good graph: $F = (m-2)K_{n_k - 1} \cup K_{\sum_{i=1}^{k} n_i - 1}$.

Hasmawati, etb, Assiyatun, 2008:
If $n \geq 5$ odd, then
$R(kS_n, W_m) = R(S_n, W_m) + (k-1)n$, for $m = 2n-4, 2n-6$ or $2n-8$.

The good graph: $F_1 \simeq K_{kn-1} \cup K_{n-2,n-2}$, for $m = 2n-4$.
The good graph: $F_2 \simeq K_{kn-1} \cup [\binom{\frac{n-3}{2}}{} K_2 + \binom{\frac{n-3}{2}}{} K_2]$, for $m = 2n-6$ or $2n-8$.

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Hasmawati, etb, Assiyatun, 2008:
Let H and G_i be connected graphs with $|G_i| \geq |G_{i+1}|$, $i = 1, 2, \ldots, k-1$. If $|G_i| > (|G_i| - |G_{i+1}|)(\chi(H) - 1)$ and $R(G_i, H) = (\chi(H) - 1)(|G_i| - 1) + 1$, then $R(\bigcup_{i=1}^{k} G_i, H) = R(G_k, H) + \sum_{i=1}^{k-1} |G_i|$.

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Burr 1981:
Let H be a graph with chromatic number h and chromatic surplus s (namely, the minimum cardinality of a color class taken over all proper $\chi(H)$-colorings of H), and G a graph with n vertices and if $n \geq s$ then: $R(G, H) \geq (h-1)(n-1) + s$.

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Hasmawati, etb, Assiyatun, 2008:
Let H and G_i be connected graphs with $|G_i| \geq |G_{i+1}|$, $i = 1, 2, \ldots, k-1$. If $|G_i| > (|G_i| - |G_{i+1}|)(\chi(H) - 1)$ and $R(G_i, H) = (\chi(H) - 1)(|G_i| - 1) + 1$, then $R(\bigcup_{i=1}^{k} G_i, H) = R(G_k, H) + \sum_{i=1}^{k-1} |G_i|$.

Burr 1981:
Let H be a graph with chromatic number h and chromatic surplus s (namely, the minimum cardinality of a color class taken over all proper $\chi(H)$-colorings of H), and G a graph with n vertices and if $n \geq s$ then: $R(G, H) \geq (h-1)(n-1) + s$.

Good graph: $F = (h-1)K_{n-1} \cup K_{s-1}$.

Definition:
Graph G is called to be H-good if $R(G, H) = (h-1)(n-1) + s$.

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Let H be a graph with chromatic number h and chromatic surplus $s \geq 1$.
Graph G has all components which are H-good,
$c(G)$: the order of the largest component in G, and
$k_i(G)$: the number of components of order i. Then:

Bielak, 2009: (Only for $s = 1$)

$$R(G,H) = \max_{1 \leq j \leq c(G)} \left\{ (j-1)(h-2) + \sum_{i=j}^{c(G)} i k_i(G) \right\}.$$

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Sudarsana, etb, Assiyatun, Uttunggadewa, 2010:

$$R(G,H) = \max_{1 \leq j \leq c(G)} \left\{ (j-1)(h-2) + \sum_{i=j}^{c(G)} i k_i(G) \right\} + s - 1.$$

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Sudarsana, etb, Assiyatun, Uttunggadewa, 2010:

$R(P_n, 2K_3) = 2n$.

$R(S_n, 2K_3) = 2n$.

$R(P_n, 2K_4) = 3n - 1$.

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Let $k \geq 1$ and $n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 4$ be integers.
If $G = \bigcup_{i=1}^{k} l_i T_{n_i}$ for $T_{n_i} \simeq P_{n_i}$ or S_{n_i} then:

$$R(G, 2K_3) = \max_{1 \leq i \leq k} \left\{ n_i + \sum_{j=i}^{k} l_i n_j \right\}. \tag{1}$$

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Let $k \geq 1$ and $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 6$ be integers.
1) If $G = \bigcup_{i=1}^{k} l_i P_{n_i}$ then:

$$R(G, 2K_4) = \max_{1 \leq i \leq k} \left\{ 2n_i + \sum_{j=i}^{k} l_j n_j \right\} - 1. \tag{2}$$

2) If $G = \bigcup_{i=1}^{k} l_i P_{n_i}$ and $H = 2K_3 \cup 2K_4$ then:

$$R(G, H) = R(G, 2K_4). \tag{3}$$

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Sudarsana, etb, Assiyatun, Uttunggadewa, 2010:
1) If $n \geq 3$ then

$$R(W_n, tK_2) = \begin{cases} n+t, & \text{for } t \leq \lfloor \frac{n}{2} \rfloor, \\ 2t + \lceil \frac{n}{2} \rceil, & \text{for } t > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

\# Case 2: $K_{\lceil \frac{n}{2} \rceil} + \overline{K}_{2t-1}$ is a (W_n, tK_2)-good graph with $2t + \lceil \frac{n}{2} \rceil - 1$ vertices.

2) If $\lfloor \frac{n}{2} \rfloor \geq t$ then $R(K_t + C_n, tK_2) = n + 2t - 1$.

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Sudarsana, etb, Assiyatun, Uttunggadewa, 2014:

Let H be a graph with chromatic number $h \geq 2$ and chromatic surplus $s \geq 1$. Let $G \simeq \bigcup_{i=1}^{k} G_i$, where G_i is a connected graph of order n_i satisfying $R(G_1, H) \geq R(G_2, H) \geq \ldots \geq R(G_k, H)$. Then,

$$R(G,H) \leq \max_{1 \leq i \leq k} \left\{ R(G_i, H) + \sum_{j=1}^{i-1} n_j \right\}. \tag{4}$$

Furthermore, let the maximum value in the right side of (4) be achieved for i_0. If $n_1 \geq n_2 \geq \ldots \geq n_k \geq s$ and G_{i_0} is H-good then

$$R(G,H) = \max_{1 \leq i \leq k} \left\{ R(G_i, H) + \sum_{j=1}^{i-1} n_j \right\}. \tag{5}$$

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Open Problem 1. Let $G \simeq \bigcup_{i=1}^{k} G_i$, where G_i is a connected graph and H be a graph.

1. Find $R(G,H)$ if the component of G with having the smallest Ramsey number is not H-good.
2. Find $R(G,H)$ if all components of G are not H-good.

Open Problem 2. Let $G \simeq \bigcup_{i=1}^{k} G_i$, and $H \simeq \bigcup_{i=1}^{t} H_i$, where G_i and H_i are connected graphs. Find the Ramsey number $R(G,H)$.

A nice survey paper:
S.P. Radziszowski, Small Ramsey Numbers, Electronic Journal of Combinatorics (2014), DS1.14

****MEMBERS OF OUR RESEARCH DIVISION****
THANK YOU FOR YOUR ATTENTION.


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