On the order of cages with a given girth pair

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Definition

A (k, g)-cage is a k-regular graph with girth g and with the least possible number of vertices $n(k, g)$.

Erdős, Sachs '63

(k, g)-cages exist for every k and g.

Moore bound

$$n(k,g) \geq n_0(k,g) = \begin{cases} 1 + k \displaystyle\sum_{i=0}^{(g-3)/2} (k-1)^i & g \text{ odd} \\ 2 \displaystyle\sum_{i=0}^{(g-2)/2} (k-1)^i & g \text{ even} \end{cases}$$

Bannai,Ito '73, Damerell'73

- $g \geq 3$ and $k = 2$.
- $g = 3$.
- $g = 4$.
- $g = 5$ and $k = 3, 7, 57$.
- $g = 6, 8, 12$ and $k - 1$ a prime power.

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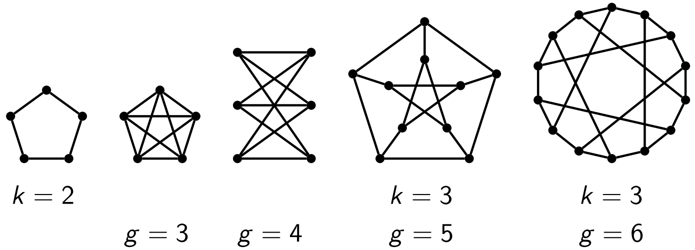

Girth Pair Cage

Minimal $(k; g, h)$-graph

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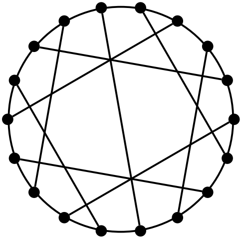

Figure: A $(3; 6, 7)$-cage on 18 vertices.

Girth Pair Cage

Minimal $(k; g, h)$-graph

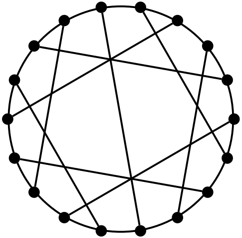

Figure: A $(3; 6, 7)$-cage on 18 vertices.

Few exact values known.

Conjecture W82

All (k, g)-cages with girth g even are bipartite

Theorem BI80

All (k, g)-cages with even girth g and such that $n(k, g) - n_0(k, g) \leq k - 2$ are bipartite.

Corollary [BS14]

$n(k; g, h) \geq n_0(k, g) + k - 1$ for even $g \geq 6$ and $h > g$ odd.

Theorem C97

$n(3; 6, h) \geq (7h + 1)/3$ for h odd.

Objective

Conjecture HK83

$n(k; g, h) \leq n(k, h)$

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Conjecture HK83

$$n(k; g, h) \leq n(k, h)$$

Theorem XWW02

$$n(k; h-1, h) < n(k, h)$$

Tools

Theorem Monotonicity ES63

Let $k \geq 2$, $3 \leq g_1 < g_2$ be integers. Then $n(k, g_1) < n(k, g_2)$.

Lemma J01

Every edge of a (k, g)-cage lies on at least $k - 1$ cycles of length at most $g + 1$.

Constructions for g odd: $k \geq 3$ and $h \geq 6$

- [BJLMM05]: $n(k; g, g+3) > k + k(k-1)^{(g-1)/2}$.
- BI80: There is no (k, h)-graph with even $h \geq 8$ and excess 2.

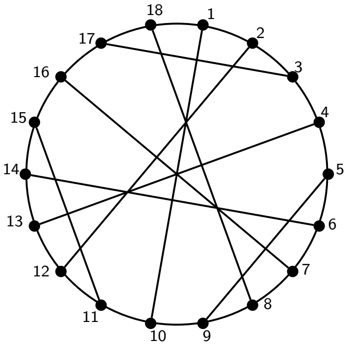


Figure: A $(3; 5, 8)$-graph of 18 vertices.

Theorem

Let $g \geq 5$ odd, such that $h/2 + 1 \leq g < h$. Then

$$n(k;g,h) \leq \begin{cases} n(k,h) - 2\displaystyle\sum_{i=0}^{(h-g-3)/2}(k-1)^i - (k-1)^{(h-g-1)/2} & h \geq 8; \\ n(k,h) - 1 & h = 6. \end{cases}$$

Theorem

Let $h \geq 6$ even, $k \geq 3$ and g and odd number $g \leq h/2 - 1$. Suppose that there is a bipartite (k, h)-cage. Then $n(k; g, h) < n(k, h)$.

Theorem

Let $h \equiv 2 \pmod 4$ and $k \geq 3$. Suppose that there is a bipartite (k, h)-cage. Then $n(k; h/2, h) \leq n(k, h)$.

Conclusion

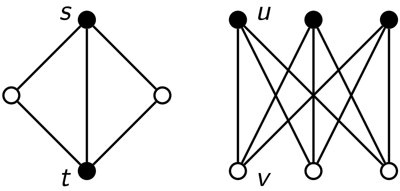
If $g < h$ with g odd, then $n(k; g, h) < n(k; h)$ for $g \neq h/2$ and $n(k; h/2, h) \leq n(k, h)$ provided that there is a bipartite (k, h)-cage.

Constructions for g even and h odd

Graph $G^{uv}\Gamma_{st}H$ the *insertion* of (G, uv) into (H, st)

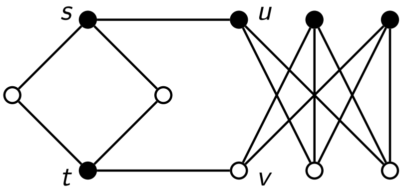
Let G, H be two vertex-disjoint graphs, $uv \in E(G)$ and $st \in E(H)$.

- $V(G^{uv}\Gamma_{st}H) = V(G) \cup V(H)$
- $E(G^{uv}\Gamma_{st}H) = (E(G) \setminus \{uv\}) \cup (E(H) \setminus \{st\}) \cup \{us, vt\}$.



Graph H Graph G

Observe that if G and H are k-regular and bipartite then $G^{uv}\Gamma_{st}H$ is k-regular and bipartite.

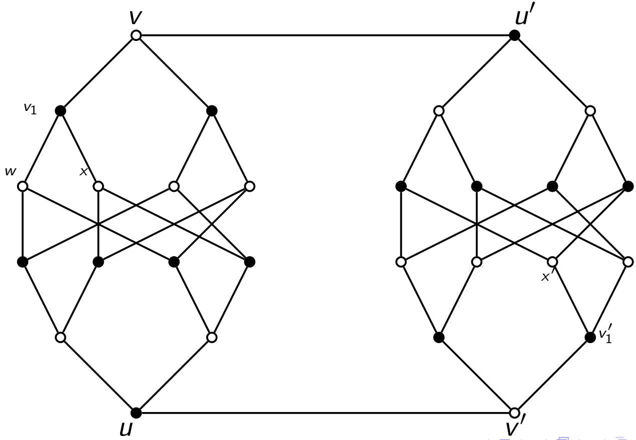


Graph $G^{uv}\Gamma_{st}H$

Observe that if G and H are k-regular and bipartite then $G^{uv}\Gamma_{st}H$ is k-regular and bipartite.

Theorem

Let $k \geq 3$ and $g \geq 6$ even. Then
$n(k; g, 2g-1) \leq 2n(k; g) < n(k, 2g-1)$.



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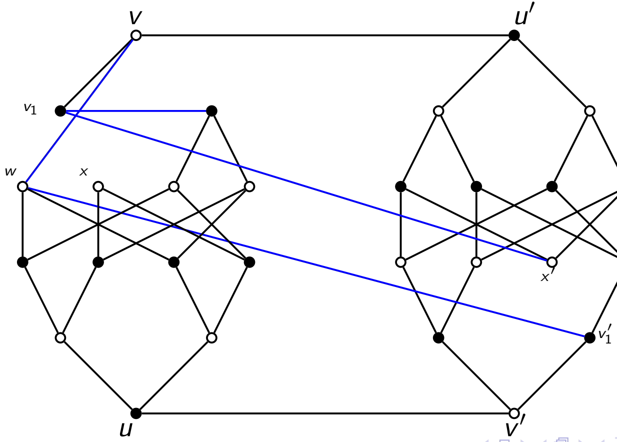

Let $k \geq 3$, $g \geq 6$ even, and suppose that there is a bipartite (k,g)-cage. Let r an odd number such that $1 \leq r \leq g-3$.

Theorem

$n(k; g, g+r) \leq 4n(k; g)$

Lemma

$n(k; g, mg+r) \leq 4n(k; g) + k(m-1)n(k; g)$, for $m \geq 1$.

Theorem

$n(k; g, h) < n(k, h)$, for $h \geq 2g - 1$.

corollary

For $g = 6, 8, 12$ and q a prime power and r odd such that $3 \leq r \leq g - 1$. $n(q + 1; g, g + r) < n(q + 1; g + r)$.

corollary

For q a prime power, $g = 6, 8, 12$, $m \geq 2$, and r odd such that $1 \leq r \leq g - 1$ and $h = mg + r$, $h > g + 1$.
$n(q + 1; g, h) < n(q + 1; h)$.

Corollary

Suppose that exists a (k, g)-cage with girth g even,such that $n(k, g) - n_0(k, g) \leq k - 2$. Then:

(i) $n(k+1; g, g+1) < n(k+1; g+1)$, for all k;

(ii) $n(k+1; g, g+r) < n(k+1; g+r)$, for every odd number such that $2 \leq r \leq g-1$.

Corollary

Suppose that exists a (k, g)-cage with girth g even,such that $n(k, g) - n_0(k, g) \leq k - 2$. Then: $n(k+1; g, h) < n(k+1; h)$.

THANKS