Construction of Small Regular Graphs of Girth 7

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joint work with
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Cages

- A (k, g)–cage is a k–regular graph of girth g with minimum number of vertices
- (Sachs; 1963): Existence of (k, g)–graphs
  for each $k \geqslant 3$ and $g \geqslant 5$
- Moore's bound is obtained when counting the minimum number of vertices necessary to construct a (k, g)–graph
- A (k, g)–graph whose order attains Moore's bound is, by definition, also a Moore graph

Moore Graphs

- The only Moore graphs:
  - Girth 5 and $k = 2, 3, 7$ and maybe 57
  - Girth $6, 8$ or 12 and they are incidence graphs of finite projective planes, generalized quadrangles or generalized hexagons, respectively
- $('60–'70)$ Hoffman, Singleton, Feit, Higman, Damerell, Bannai and Ito $\Longrightarrow$ there are no further Moore graphs
- This means that in most cases the number of vertices in a (k, g)–cage is strictly greater than Moore's bound
- Many authors are trying to construct cages, or at least smaller (k, g)–graphs than previously known ones.

Results [M.A., Araujo, Balbuena, Labbate, Salas - 2014]

Here, we will show how to construct the smallest $(q+1)$–regular graphs of girth 7 known so far, where $q \geqslant 4$ is a prime power.

Theorem 1

Let $q \geqslant 4$ be an even prime power. Then, there is a $(q+1)$–regular graph of girth 7 and order $2q^3 + q^2 + 2q$.

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Theorem 2

Let $q \geqslant 5$ be an odd prime power. Then, there is a $(q+1)$–regular graph of girth 7 and order $2q^3 + 2q^2 - q + 1$.

Construction for even prime powers: the graph H

Let Γ_q be a $(q+1, 8)$–cage, for an even prime power, $q \geqslant 4$.

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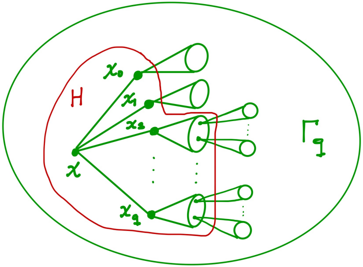
Let H be a subgraph of Γ_q consisting of the neighbourhood of a vertex $x \in V(\Gamma_q)$ and the neighbourhods of all but two of its neighbours, i. e.

$$H := N(x) \cup \bigcup_{i=2}^{q} N(x_i) \quad \text{where } N(x) = x_0, x_1, x_2, \ldots, x_q.$$



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Observe that $|V(\Gamma_q)| = 2(q^3+q^2+q+1)$
and $|H| = 1+q+1+q(q-1) = q^2+2$, since Γ_q has girth 8.

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and $|H| = 1 + q + 1 + q(q-1) = q^2 + 2$, since Γ_q has girth 8.
Hence, $V(\Gamma_q \setminus H) = 2q^3 + q^2 + 2q$, the number of vertices that our final graph will have.

Construction for even prime powers: the graph $\Gamma_q \setminus H$

However, $\Gamma_q \setminus H$ is not regular. It is indeed biregular of degrees q and $q+1$.

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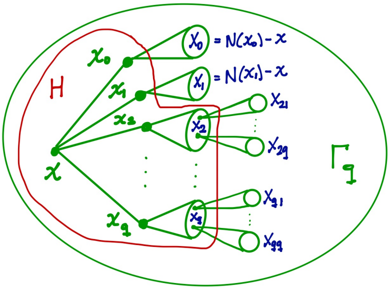
where x_{ij} is the j^{th} neighbour of x_i for $i = 2, \ldots, q$ and $j = 1, \ldots, q$



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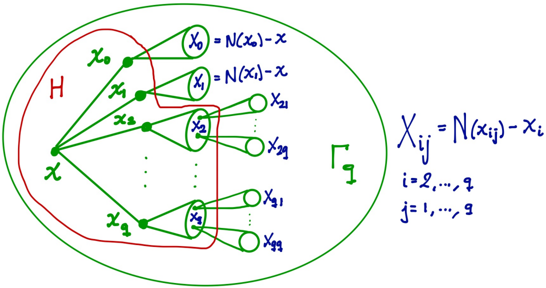

All these sets have even cardinality

Construction for even prime powers: the graph Γ_q^1

Let $\mathcal{Z} = \{X_0, X_1, X_{ij} : i = 2, \ldots, q, j = 1, \ldots, q\}$.

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Definition

We define Γ_q^1 to be the graph with:

$V(\Gamma_q^1) := V(\Gamma_q - H)$ and $E(\Gamma_q^1) := E(\Gamma_q - H) \cup \bigcup_{Z \in \mathcal{Z}} M_Z$.

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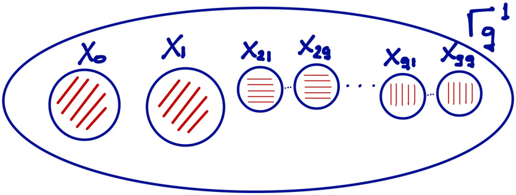

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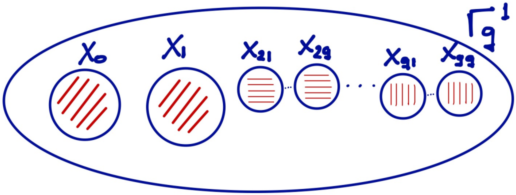

The graph Γ_q^1 is $(q+1)$-regular

Construction for even prime powers: Condition on Matchings

Lemma

Γ_q^1 has girth 7 if the following condition holds:

For each $uv \in M_{X_{ij}}$ and X_{kl}, where $i, k \in \{0, \ldots, q-2\}$, $j, l \in \{1, \ldots, q\}$

$$E(\Gamma_q^1[N_2(uv) \cap X_{kl}]) \cap M_{X_{kl}} = \emptyset.$$

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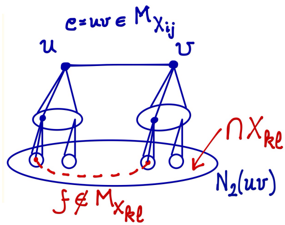
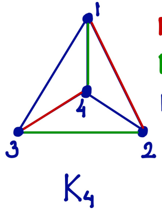


Construction for even prime powers: Matchings satisfying the Condition

Lemma

There exist $q^2 - q$ matchings $M_{X_{ij}}$ satisfying the previous condition.

Idea of the proof: Let $F_1, \ldots, F_{q-1}$ be a 1–factorization of K_q with vertices $h \in \{1, \ldots, q.\}$



$F_1 = (12)(34)$
$F_2 = (14)(23)$
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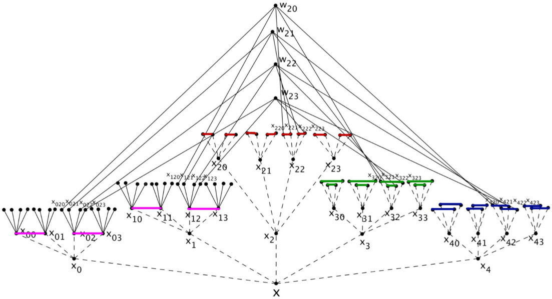

$(5, 7)$–graph obtained from the construction for even prime powers, $q = 4$

Construction for odd prime powers: the graph H

Definition

Let $x, y \in V(\Gamma_q)$ be vertices at distance four in Γ_q, and let $xx_is_iy_iy$ be the edge disjoint xy-paths for $i = 0, \ldots, q$. We define the following sets:

$$
\begin{aligned}
H &= \{x, y, s_3, s_4, \ldots, s_q\} \cup N(x) \cup N(y) \subset V(\Gamma_q); \\
X_i &= N(x_i) \cap V(\Gamma_q - H), \quad i = 0, \ldots, q; \\
Y_i &= N(y_i) \cap V(\Gamma_q - H), \quad i = 0, \ldots, q; \\
S_i &= N(s_i) \cap V(\Gamma_q - H), \quad i = 3, \ldots, q.
\end{aligned}
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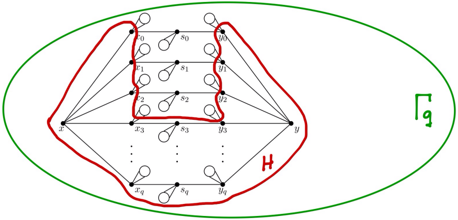

Construction for odd prime powers: the graph $\Gamma_q - H$

The graph $\Gamma_q - H$ has order $2q^3 + 2q^2 - q + 1$ but it is not regular. Its degrees are $q-1, q$ and $q+1$.

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Indeed, the vertices s_0, s_1, s_2 have degree $q-1$,
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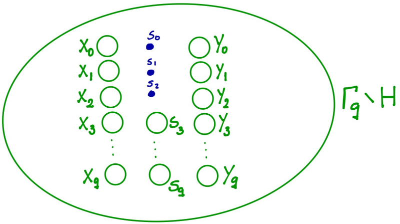

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Therefore, in order to complete the degrees of such vertices it is necessary to add edges to $\Gamma_q - H$ being careful to avoid cycles of length smaller than seven.

Construction for odd prime powers: the graphs Γ_q^1 and Γ_q^2

As before, let $\mathcal{Z}$ be the family of all the sets X_i, Y_i, S_i. Note that, all sets in $\mathcal{Z}$ have even cardinality.

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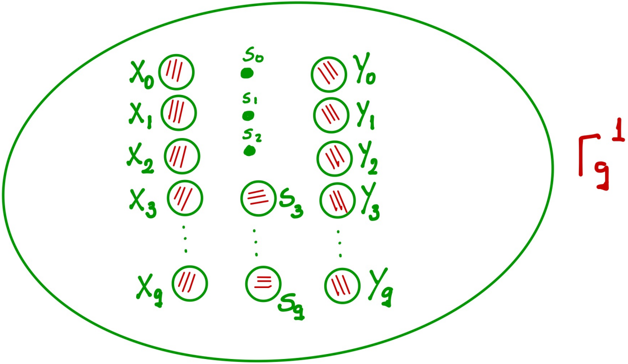

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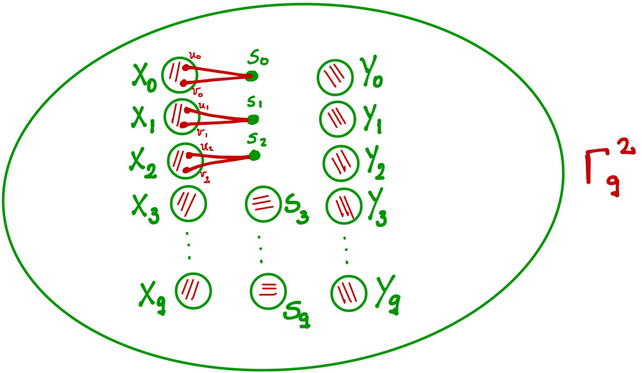

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Let Γ_q be a $(q+1, 8)$-cage for odd prime power $q \geqslant 5$.

- Let Γ_q^1 be the graph with:
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- Define Γ_q^2 as $V(\Gamma_q^2) := V(\Gamma_q^1)$ and
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  the deleted edges u_iv_i belong to M_{X_i} in Γ_q^1 and they are replaced by the paths of length two $u_is_iv_i$, $i \in \{0, 1, 2\}$.

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Γ_q^2 is $(q+1)$–regular.

Construction for odd prime powers: Conditions on the Matchings

Lemma

Γ_q^1 and Γ_q^2 both have girth 7 if the matchings M_{S_i}, M_{X_i} and M_{Y_i} have the following properties:

(a1) *For all $uv \in M_{S_i}$, $E(\Gamma_q^1[N_2(uv) \cap S_j]) \cap M_{s_j} = \emptyset$.*

(a2) *For all $uv \in M_{X_i}$, $E(\Gamma_q^1[N_2(uv) \cap Y_j]) \cap M_{Y_j} = \emptyset$.*

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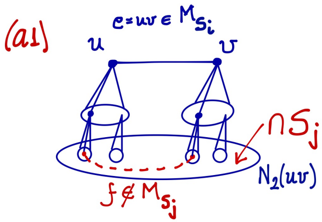

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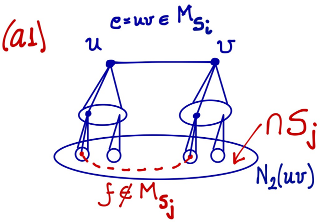

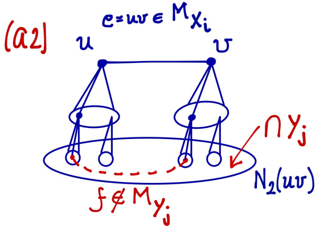

The choice of the matchings M_{S_i}

Lemma

There exist matchings M_{S_i}, for $i = 3, \ldots, q$, such that condition $(a1)$ holds.

The proof follows from the regularity of $W(q)$ which implies that $\{x, y\}^{\perp\perp} = \bigcap_{s \in N_2(x) \cap N_2(y)} N_2(s)$, and hence $|\bigcap_{i=0}^{q} N(S_i)| = q - 1$.

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The choice of the matchings M_{X_i} and M_{Y_i}

In order to find the remaining matchings it was necessary to use a labeling of the vertices of Γ_q according to a coordinatization, using finite fields, of the corresponding $W(q)$.

Lemma

There exist matchings M_{X_i} and M_{Y_i}, for $i = 0, \ldots, q$, such that condition $(a2)$ holds.

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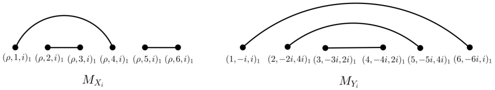

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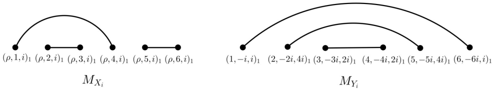
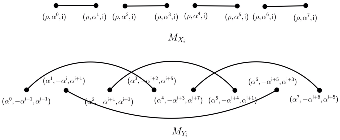


- $q = p^a$, $a > 1$ is a prime power



Thank You