On the cost of asymmetrizing graphs

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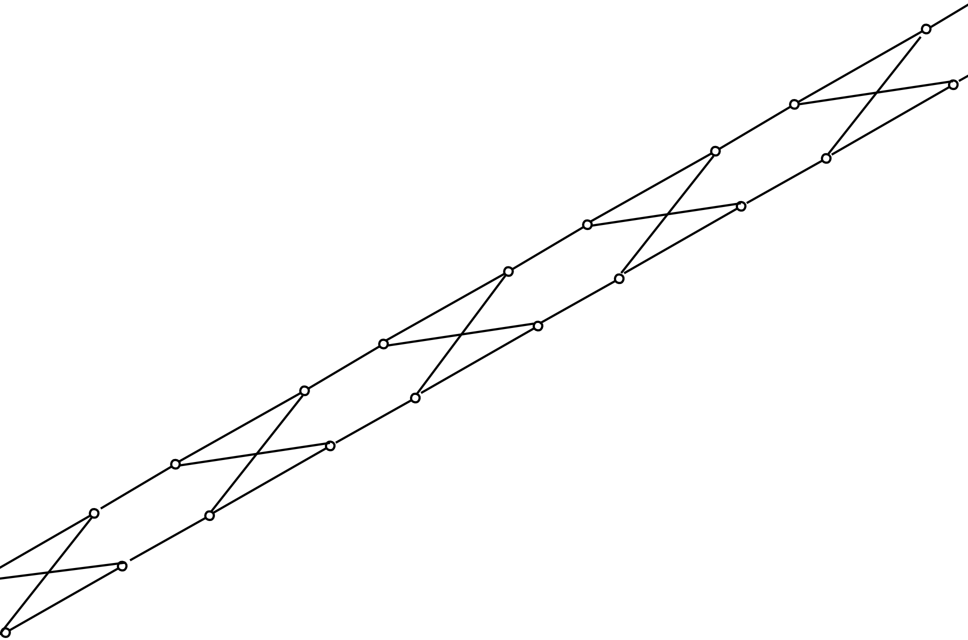
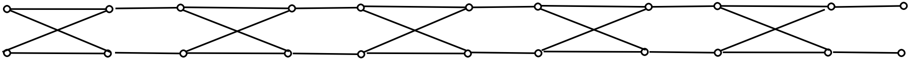




I will mainly speak about joint work with

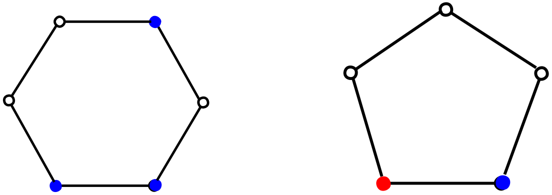
Gundelinde Wiegel
Thomas Lachmann
Thomas Tucker

Gundelinde Wiegel and Thomas Lachmann were students of the
Doctoral Program Discrete Mathematics

of the Technical University Graz, the Karl Franzens University Graz and the
Montanuniversität Leoben.

Automorphism breaking

Pertains to automorphism breaking of graphs by vertex colorings.



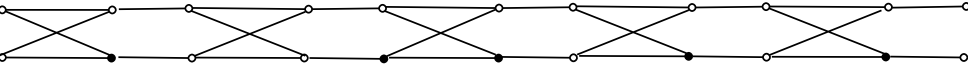
In both graphs each automorphism that preserves the coloring is the identity.
Such a coloring is called distinguishing.

We wish

• to use as few colors as possible,

• and to minimize the amount of "paint" used, but have to be careful with infinite graphs.

Breaking the automorphisms of our chain of quadrangles K



Can we do it by coloring only finitely many vertices?

Can we do it with more colors and only finitely many vertices?

Which percentage of the vertices is colored?

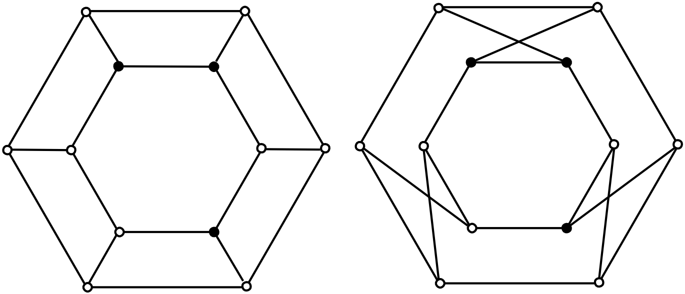
Can we reduce that percentage?

Are there graphs with other percentages?

Now many vertices are moved by each automorphism?

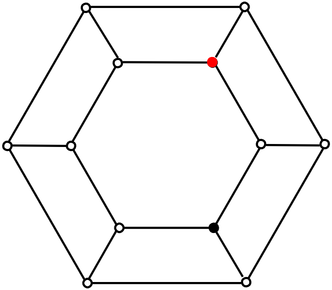
What is the size of $\mathrm{Aut}(K)$?

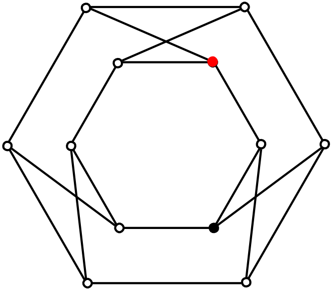
Now two finite analogues



How much paint would we need if we had two colors?

More colors





On the left we can do with less paint, but not on the right.

How many vertices are moved by each automorphism?

On the left at least 8, on the right only 4.

The distinguishing number and the distinguishing cost

We wished to
• to use as few colors as possible,
• and to minimize the amount of "paint".

• For the smallest number of colors needed to distinguish a graph G by a vertex coloring, Albertson and Collins (1996) introduced the term distinguishing number and denoted it by $D(G)$.

$D(G)$ is 1 for asymmetric graphs and 2 for almost all other finite graphs, because almost all finite graphs that are not asymmetric have just one automorphism, which is an involution.

Such graphs can be distinguished by coloring a single unfixed vertex black.

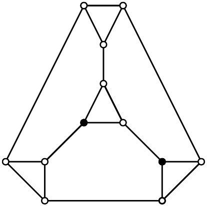
• We primarily consider graphs with distinguishing number 2.

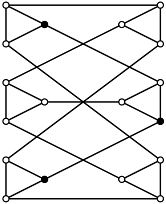
Then each distinguishing 2-coloring has two color classes.

For the size of a smallest such class among all distinguishing 2-colorings Boutin (2008) introduced the term 2-distinguishing cost and denoted it by $\rho(G)$.

Clearly $\rho(G) = 1$ for graphs with just one automorphism.

Examples for the distinguishing cost





Examples for $\rho(G) = 2$, resp. 3.

Clearly $\rho(G) \leq |V(G)|/2$.

Remarks

Asymmetrization predates the introduction of the distinguishing number and there is a wealth of deep results.

In 1977 Babai proved that for $\alpha \geq 2$ and any two colors, there are $\max(2^\alpha, 2^\omega)$ pairwise inequivalent asymmetrizing 2-colorings of α-valent trees.

Between 1991 and 1996 Polat and Sabidussi wrote a series of papers on the asymmetrization of graphs of any cardinality. They are deep and interesting, but have almost no citations.

The concept of asymmetrization extends to groups of permutations on a set. Let me cite three results:

In 1983 Gluck showed that if the order of the group is odd, then it can be asymmetrized by two colors.

In 1984 Cameron, Neumann, and Saxl showed that all but finitely many primitive permutation groups other than A_n, S_n can be asymmetrized by two colors; in 1997 Seress classified the remaining ones.

1996 asymmetrization became popular under a new name

1996 Albertson and Collins introduced the distinguishing number: Suddenly asymmetrization came down to earth and became popular.

Their paper spawned a hundreds of publications. Most, but not all in the context of graphs.

For example also to distinguishing vector spaces over finite fields, automorphism groups of groups, maps.

Often motion plays a role. It is the minimum number of vertices moved by each automorphism. We have the "Motion Lemma" of Russell and Sundaram, 1998. It says:

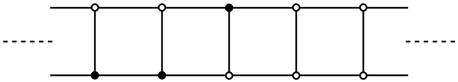
If G has motion m, then $D(G) = 2$ if

$$2^{m/2} \geq |Aut(G)|.$$

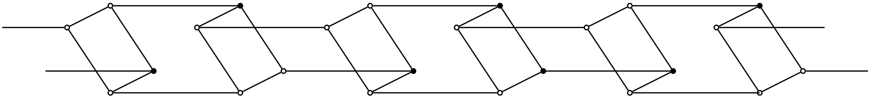
Note that the Motion Lemma appears in work of Laborde in 1985 and in a 1986 note of Cameron.

The 2-distinguishing density

Infinite graphs can have finite 2-distinguishing cost, e.g. the infinite ladder L.



If $D(G)$ is infinite, one has to color infinitely many vertices. In this case we wish to minimize their 2-distinguishing density $\delta(G)$.



The infinite chain of quadrangles K. It has distinguishing density $1/4$.

Questions

Are there any other v.t. cubic graphs with positive density?

Which densities are possible?

What is the structure of these graphs?

This talk deals with these questions and is based on a manuscript with Gundelinde Wiegel, Thomas Lachmann and Thomas Tucker:

K is member of an interesting class of cubic graphs with positive density.

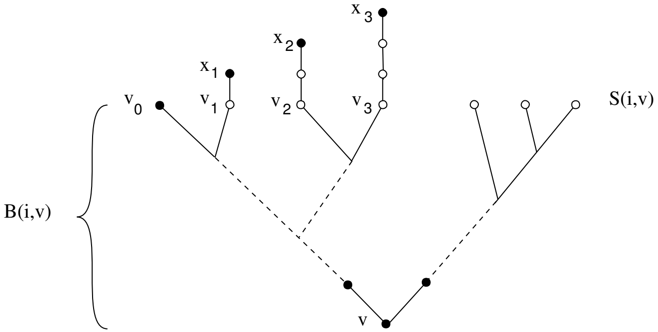
We determine the 2-distinguishing cost of large classes of cubic v.t. graphs.

On the way we provide a new characterization of Split Praeger-Xu graphs.

But we still have more questions than answers.

Example for density 0

Distinguishing with a sparse set of black vertices in trees without endpoints.



T is 2-distinguishable and that the black vertices form a very sparse set S.

We shall later see that S must be infinite when $\mathrm{Aut}(T)$ is uncountable.

The distinguishing density – definition

$$B(v, n) := \{w \in G : d(v, w) \leq n\}$$

denotes the ball of radius n with center v. Given $S \subseteq V(G)$ we call

$$\delta_v(S) := \limsup_{n\to\infty} \frac{\mid B(v, n) \cap S \mid}{\mid B(v, n) \mid}$$

the density of S at v.

If $\delta_v(S)$ exists for all vertices, then the density of S is

$$\delta(S) = \sup\{\delta_v(S) : v \in V(G)\}.$$

The infimum of $\delta(S)$ over all asymmetrizing sets S is then the distinguishing density $\delta(G)$ of G. (S.M. Smith, F. Lehner, W.I., 2020.)

When is density 0 well defined?

If there is $w \in V(G)$ and a constant c such that

$$|B(w, n+1)| < c \cdot |B(w, n)|,$$

and if G has distinguishing density 0 at one vertex v, then it has distinguishing density 0 at all vertices, and thus $\delta(G) = 0$.

The following graphs are 2-distinguishable with density zero:

- connected, locally finite, primitive graphs
- denumerable trees without leaves
- denumerable vertex-transitive graphs of connectivity 1
- the Cartesian product of any two connected denumerable graphs of infinite diameter.
- graphs of subquadratic growth and infinite motion.

When is positive distinguishing density well defined?

It is not hard to construct graphs with $\delta > 0$, but not when they are vertex transitive.

We shall thus focus attention on v.t. cubic graphs.

We still need a condition to assure that $\delta = a > 0$ for one vertex implies that $\delta = a$ for all vertices.

Suppose there exists a constant c such that

$$|B(v, n)| \leq |B(v, n+k)| \leq |B(v, n)| + kc$$

for all natural numbers k. Then $\delta_v(S) = a$ at some vertex v implies $\delta(G) = a$.

Properties of the infinite chain of quadrangles

$Aut(K)$ is uncountable. By the following theorem $\rho(K)$ is infinite.

Theorem [D. Boutin, W.I. 2017] Let G be a connected, locally finite graph with infinite automorphism group. Then $\rho(G)$ is finite if and only if $\text{Aut}(G)$ is countable.

One can say much more.

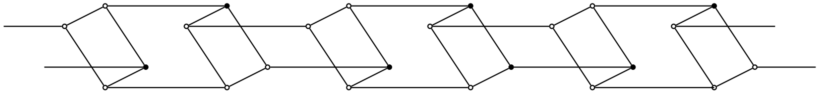
Let G be a connected, locally finite graph and $\aleph_0 \leq |\text{Aut}(G)| < 2^{\aleph_0}$ then:

- $\text{Aut}(G)$ is countable
- Each automorphism of G moves infinitely many vertices
- $D(G) = 2$
- $\rho(G) < \aleph_0$

Countability follows from Halin 1973, the next two assertions from Imrich, Smith, Tucker and Watkins 2015.

Arc orbits of K and v.t. cubic graphs in general

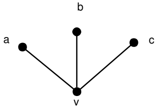
Another important property of K is that it has two arc orbits.



The orbit of an arc vw is

$$O(vw) = \{xy \,|\, x = \alpha(v), y = \alpha(w), \alpha \in \mathrm{Aut}(G)\}.$$

By vertex-transitivity every vertex has to be incident to at least one arc from every arc orbit, hence the number of arc orbits in a vertex-transitive cubic graph is 1,2 or 3.

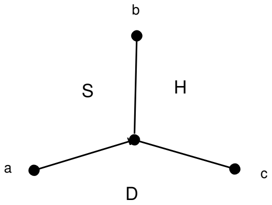


Three arc orbits

If G is v.t. cubic with three arc orbits, then $\rho(G) = 1$.

An example is the truncated icosidodecahedron, already described by Johannes Kepler.

Each vertex is in a square, a hexagon, and a decagon.



One arc orbit

These graphs are arc-transitive, in other words, to any two edges uv and xy there is an automorphism α such that $\alpha(u) = x$ and $\alpha(v) = y$.

Theorem Let G be an arc-transitive cubic graph different from K_4, $K_{3,3}$, the cube and the Petersen graph.

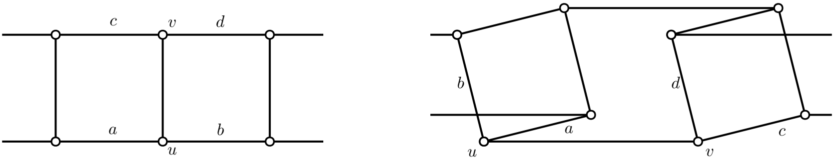
If G has finite girth, then $\rho(G) \leq 5$.

Otherwise $G = T_3$, $\rho(T_3) = \infty$, and $\delta(T_3) = 0$.

The proof mainly uses results of Tutte and Djokovic, who showed that arc-transitive cubic graphs of finite girth are s-arc regular, where $s \leq 5$.

Two arc orbits

There are two fundamentally different types of v.t. cubic graphs with two arc-orbits. Compare the edges uv in the two figures.



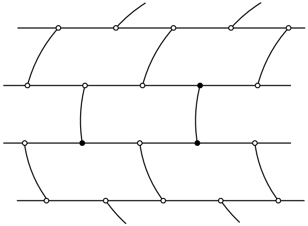
If one interchanges the edges a and b in the ladder, then c, d also have to be interchanged.

This is not the case in the figure on the right.

We call these graphs rigidly, resp. flexibly connected.

Rigidly connected graphs

Theorem Let G be a connected vertex-transitive cubic graph with two arc orbits that is rigidly connected. Then $\rho(G) \leq 3$.



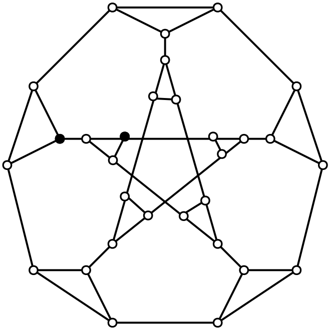
Flexibly connected graphs

If they have girth 3, they can easily be treated.
Let G be a connected, vertex-transitive cubic graph with an orbit of triangles and a matching orbit.

Suppose there is only one edge between adjacent triangles. If we contract the triangles to single vertices, then we obtain an arc transitive graph.

One can then show, either directly or with the result about arc-transitive graphs, that $\rho(G) \leq 5$, or $\delta(G) = 0$.

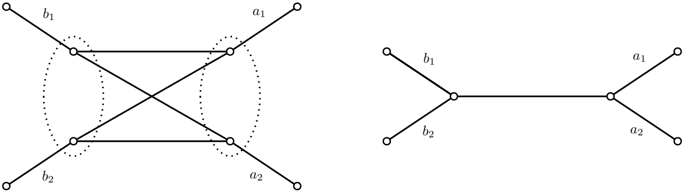
Example



Here G is contracted to the Petersen graph, but $\rho(G) = 2$.

Girth 4, two arc orbits, flexible connection

We define a folding operation for the quadrangles



Folding of quadrangles

With this operation the chain of quadrangles is folded to a chain of single and double edges

.

Defolding

It is easily seen that the process of defolding is unique.

Furthermore, distinguishing colorings of a graph obtained by folding can be extended to the original graph.

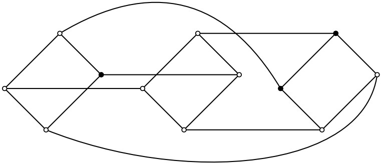
One can thus characterize the cost and density of graphs that can be folded to a finite or infinite chain of single and double edges.
Graphs that can be folded to a ring of m single and double edges by n foldings will be denoted by $P(n, m)$.

As the processes of defolding yields a unique graph, up to isomorphisms, $P(n, m)$ is uniquely defined.

$P(n, m)$ graphs

The $P(n, m)$ graphs with two arc-orbits that are flexibly connected are those with the parameters $m \geq 3$ and $1 \leq n \leq m - 1$.

These are the interesting ones and many have large distinguishing cost.



P(1,3)

Surprisingly these graphs are the Split Praeger–Xu graphs, SPX–graphs for short.

SPX– **graphs**

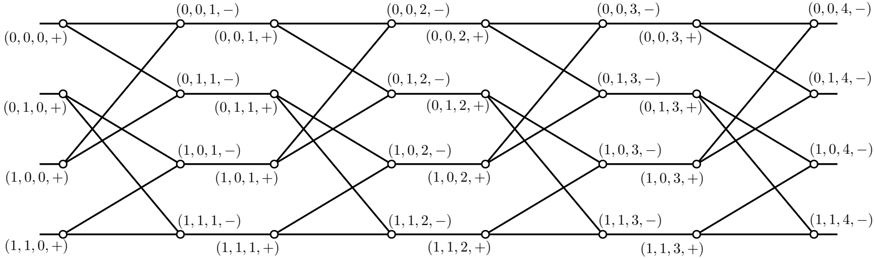
Let n, m be positive integers with $m \geq 3$ and $1 \leq n \leq m-1$.

$\mathrm{SPX}(2,n,m)$ has vertex-set $\mathbb{Z}_2^n \times \mathbb{Z}_m \times \{+,-\}$ and edge-set

$$\{\{(i_0, i_1, \ldots, i_{n-1}, x, +), (i_1, i_2, \ldots, i_n, x+1, -)\} \mid i_j \in \mathbb{Z}_2, x \in \mathbb{Z}_m\}$$
$$\cup \{\{(i_0, i_1, \ldots, i_{n-1}, x, +), (i_0, i_1, \ldots, i_{n-1}, x, -)\} \mid i_j \in \mathbb{Z}_2, x \in \mathbb{Z}_m\}.$$

These are cubic, bipartite graphs.

Figure



Part of SPX$(2, 2, m)$ for large m

The distinguishing cost of $\mathrm{SPX}(2, 2, m)$

Let $m \geq 5$ and $1 \leq n \leq m - 1$.

Then $\rho(SPX(2, n, m)) = \lceil \frac{m}{n} \rceil$, unless

$\lceil \frac{m}{n} \rceil = 2$. Then $\rho = 3$.

If $m = 3, 4$, then $\rho = 3$.

Infinite Split Praeger–Xu graphs

If we replace $\mathbb{Z}_m$ in the Definition of $\mathrm{SPX}(2, n, m)$ by $\mathbb{Z}$ we obtain an infinite graph, say $\mathrm{SPX}(2, n)$. It is

- cubic
- vertex transitive
- bipartite
- 2-distinguishable
- has two arc orbits
- is flexibly connected
- satisfies the conditions needed to be able to define positive density

The 2-distinguishing density of $\mathrm{SPX}(2, n)$ is $\frac{1}{n2^{n+1}}$.

We have complete answers about cost and density for v.t. cubic graphs with one or three arc orbits.

For v.t. cubic graphs with two arc orbits we have complete answers if they are rigidly connected.

For flexibly connected v.t. cubic graphs we have complete answers for girth 3.

For girth 4 we have partial answers, but many examples of graphs with positive density.

But even in this case we do not know which densities are possible and whether there are such graphs with nonlinear growth.

THANK YOU FOR YOUR ATTENTION