On the eigenvalues of the graphs $D(5, q)$

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Introduction

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Let G be a d-regular connected graph and λ_2 be the second largest eigenvalue of its adjacency matrix. Then

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Theorem (Alon-Boppana, (1986, 1991))

Let $\{G_n\}$ be an infinite family of d-regular connected graphs with $|V(G_n)| \to \infty$ as $n \to \infty$. Then

$$\liminf_{n\to\infty} \lambda_2(G_n) \geq 2\sqrt{d-1}.$$

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- Marcus, Spielman, Srivastava (2015) showed there exist infinite families of bipartite Ramanujan graphs for any degree greater than 2.

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3. The girth of $D(k, q)$ (and therefore $CD(k, q)$) is *at least* $k + 4$ when k is even, and $k + 5$ when k is odd. (Lazebnik and Ustimenko 1995) For certain values of k, q this value of the girth is known to be exact. (Füredi, Lazebnik, Seress, Ustimenko, and Woldar 1995)

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4. The above gives asymptotically the best known general lower bound (except for $k=5$) for $\mathrm{ex}(n, \{C_{2k+1}, \ldots, C_4, C_3\})$ implying $ex(n, \{C_{2k+1}, \ldots, C_4, C_3\}) \geq n^{1+\frac{2}{3k-3+\epsilon}}$. where ϵ is 1 if k is even, and 0 if k is odd.

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Conjecture (Ustimenko)

The graphs $CD(k, q)$ are nearly Ramanujan. More precisely, $\lambda_2(CD(k, q)) \leq 2\sqrt{q}$ for any integer $k \geq 2$ and prime power q.

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We now define the graphs $D(k,q)$.

Definition (Algebraically Defined Graphs)

Let $P = L = \mathbb{F}_q^m$ be two copies of the m-dimensional vector space over $\mathbb{F}_q$ with $q = p^e$. Call the set P points and L lines, with the distinction in notation by $(a) \in P$ and $[a] \in L$. Define $\Gamma_q = \Gamma_q(f_2, f_3, \ldots, f_m)$ to be the bipartite graph with parts P and L and with edge relation defined between them as follows: If $(p) = (p_1, \ldots, p_m) \in P$ and $[\ell] = [\ell_1, \ldots, \ell_m]$, then $(p) \sim [l]$ if and only if

$$\begin{aligned} \ell_2 + p_2 &= f_2(\ell_1, p_1) \\ \ell_3 + p_3 &= f_3(\ell_1, p_1, \ell_2, p_2) \\ &\vdots \\ \ell_m + p_m &= f_m(\ell_1, p_1, \ldots, \ell_{m-1}, p_{m-1}) \end{aligned}$$

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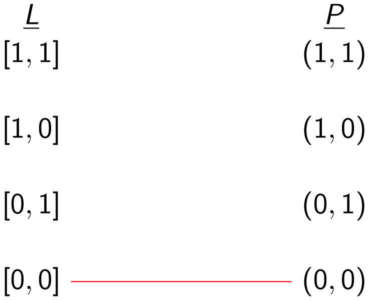

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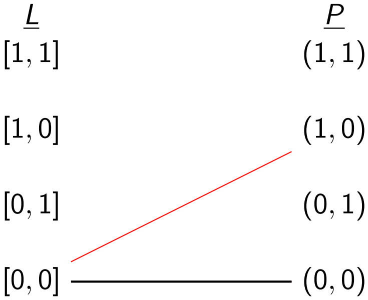

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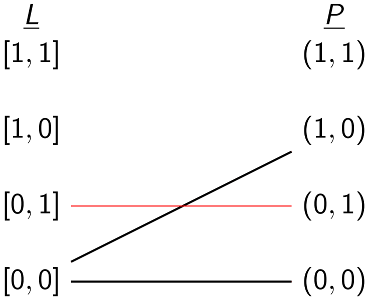

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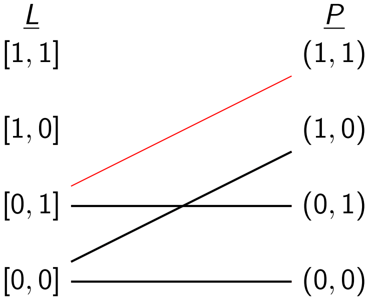

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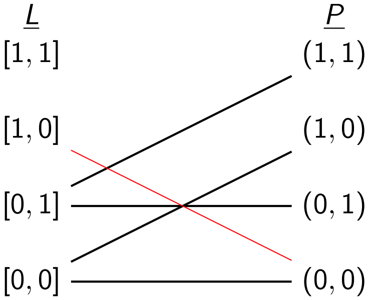

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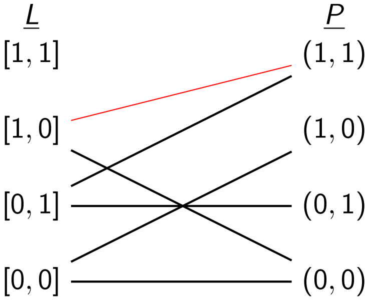

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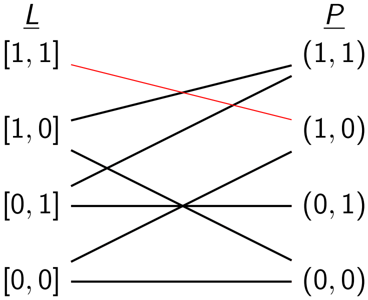

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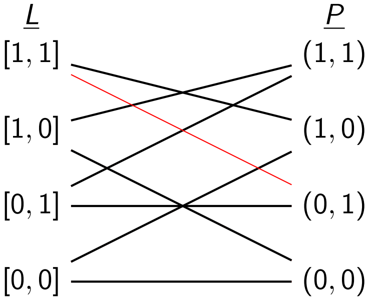

The graphs $D(k, q)$

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$$p_j + \ell_j = \begin{cases} p_{j-2}\ell_1 & \text{if } j \equiv 0,1 \pmod 4 \\ p_1\ell_{j-2} & \text{if } j \equiv 2,3 \pmod 4 \end{cases}$$

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So $D(5, q)$ in particular is given by:

$$p_2 + \ell_2 = p_1\ell_1$$
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$$p_4 + \ell_4 = p_2\ell_1$$
$$p_5 + \ell_5 = p_3\ell_1$$

$D(5, q)$ and its point graph

Definition

The point graph $P_{D(5,q)}$ of $D(5, q)$ has vertex set $P = \mathbb{F}_q^5$ and two points are adjacent in $P_{D(5,q)}$ if they are at distance 2 in $D(5, q)$.

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Note that λ is an eigenvalue of $D(5, q)$ if and only if $\lambda^2 - q$ is an eigenvalue of $P_{D(5,q)}$.

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Definition

Let G be a group with identity 0, and $S \subset G$ be an inverse closed subset of G with $0 \notin S$. Then the undirected cayley graph $Cay(G, S)$ is the graph with vertex set G and two vertices $g, h \in G$ are adjacent if there exists an $s \in S$ such that $gs = h$.

$D(5, q)$ and its point graph

Lemma

Let q be an odd prime power. Then $P_{D(5,q)}$ is isomorphic to the Cayley graph $Cay(G, S)$, where

- $G = (\mathbb{F}_q^5, \oplus)$
- $S = \{(x, ax, ax^2, a^2x, a^2x^2) : a, x \in \mathbb{F}_q, x \neq 0\}$,
- $\oplus$ *is defined as*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 + 2x_1y_2 \\ x_5 + y_5 + 2x_1y_3 \end{pmatrix}$$

Representation Theory

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Example: Consider $\rho : \mathbb{Z}_3 \to GL(3, \mathbb{C})$, defined by

$$\rho(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho(1) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \rho(2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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Let $e_i \in \mathbb{C}^3$ be the vector with 1 in the i^{th} coordinate, and 0 elsewhere. Observe that $\rho(j)e_i = e_{i+j \pmod 3}$. So in this example, the representations encode the group operation in their action on the standard basis $\{e_1, e_2, e_3\}$.

Representation Theory

Let C_3 be the graph of a 3-cycle. C_3 can be though of as the Cayley graph $Cay(\mathbb{Z}_3, S)$, where $S = \{1, 2\}$. Let ρ be the representation of $\mathbb{Z}_3$ given above.

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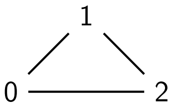

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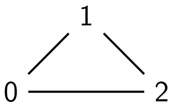

Observe that the adjacency matrix of C_3, is

$$A_{C_3} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\rho(1)} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{\rho(2)} = \sum_{s \in S} \rho(s).$$

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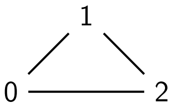

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This is no coincidence, we can generalize this idea to any Cayley graph over any group G.

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Definition

Let G be a group of order n. Then $\rho : G \to GL(|G|, \mathbb{C})$, defined by

$$\rho(g)e_h = e_{gh}, \ \forall h \in G$$

is called the **regular representation of G over $\mathbb{C}$**.

Representation Theory

Denote $\mathbb{C}^G$ to be the vector space of dimension $|G|$ over $\mathbb{C}$ with vectors indexed by $g \in G$. The standard basis is $\{e_g : g \in G\}$.

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Lemma

Let G be a group and $S \subset G$ so that $Cay(G,S)$ is an undirected Cayley graph. Let ρ be the regular representation of G over $\mathbb{C}$. Then the adjacency matrix of $Cay(G,S)$ can be written as

$$A_{Cay(G,S)} = \sum_{s \in S} \rho(s).$$

Representation Theory

Example: Let ρ be the regular representation of $\mathbb{Z}_3$ as before and $\zeta = e^{2\pi i/3}$. Let $T = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{bmatrix}$.

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$$T\rho(0)T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; T\rho(1)T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix}, \; T\rho(2)T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta \end{bmatrix}$$

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Applying this transformation to A_{C_3}, we get

$$TA_{C_3}T^{-1} = T\rho(1)T^{-1} + T\rho(2)T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

So the eigenvalues of C_3 are $\{-1, -1, 2\}$.

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4. Repeat on each "subrepresentation" until they cannot be reduced further.

Representation Theory

In essence, it is possible to break down $\rho(G)$ uniquely into a direct sum of *irreducible representations* (i.e. ones that do not act on any non-trivial subspace).

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One dimensional representations are inherently irreducible.

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Theorem

Let ρ and π be two irreducible representations of a finite group G. Then

$$\frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)\overline{\chi_\pi(g)} = \begin{cases} 1 & \text{if } \rho = \pi \\ 0 & \text{otherwise} \end{cases}$$

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The above theorem is often referred to as the orthogonality relation and can be useful in finding character, and hence finding representations.

Lifting Representations

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Let G be a finite group and N a normal subgroup. Suppose that $\pi : G \to GL(n, \mathbb{C})$ is a representation of G/N. Let $\alpha : G \to G/N$ be the canonical homomorphism. Then $\rho = \pi \circ \alpha : G \to GL(n, \mathbb{C})$ is a representation of G.

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I would recommend the book by James and Liebeck, "Representations and Characters of Groups"

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- If G is a semidirect product of subgroups H and N, where N is normal and abelian. Then *every* irreducible representation of G can be obtained via a tensor product involving particular representations of H and N.
- If G is a p-group, then each irreducible representation of G is induced by a one dimensional representation from a subgroup of G.

Irreducible Representations of $P_{D(5,q)}$

Recall that the point graph of $D(5,q)$ is isomorphic to the Cayley graph $Cay(G,S)$, where

- $G = (\mathbb{F}_q^5, \oplus)$
- $S = \{(x, ax, ax^2, a^2x, a^2x^2) : a, x \in \mathbb{F}_q, x \neq 0\}$,
- $\oplus$ is defined as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 + 2x_1y_2 \\ x_5 + y_5 + 2x_1y_3 \end{pmatrix}$$

Irreducible Representations of $P_{D(5,q)}$

Let $q = p^e$, and $\zeta = e^{2\pi i/p}$. For each $\alpha, \beta, \gamma \in \mathbb{F}_q$, the linear characters of G are given by

$$\chi_{\alpha,\beta,\gamma}(X) := \zeta^{tr(\alpha x_1 + \beta x_2 + \gamma x_3)}.$$

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$$M_{\alpha,\beta,\gamma}(X) := [\zeta^{tr\left(\left(x_2 + \frac{\beta}{\alpha} x_3\right) j + \alpha x_4 + \beta x_5 + \gamma x_3\right)} \delta_{2x_1\alpha + j, k}]_{j,k \in \mathbb{F}_q}.$$

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The regular representation of G is the direct sum of all these irreducible representations. So the adjacency matrix A of $P_{D(5,q)}$ is similar to a block diagonal matrix with block sizes as $q \times q$ or 1×1.

Future Work

- Find eigenvectors and eigenvalues of $D(7, q)$.
- Find eigenvectors and eigenvalues of $D(5, q)$ for even q.
- Find a way to generalize these ideas to tackle the case for general k.

Questions?

Thanks!