Orientably-regular maps with no non-trivial exponents

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What we want to proof

Theorem 1.1

For every hyperbolic pair (k, m) there exists infinitely many orientably-regular maps of type $\{m, k\}$ with no non-trivial exponents.

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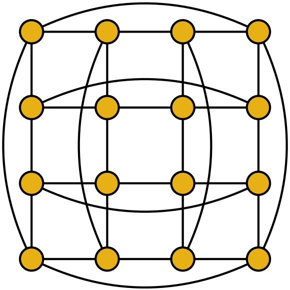
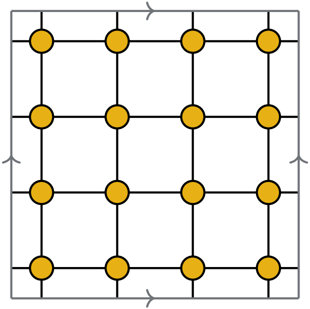
- Orientable maps
- Orientably-regular maps
- Maps of type $\{m, k\}$
- Exponents of orientable maps
- How we proof this

Part 1: Orientable maps



An **orientable map** is a 2-cell embedding of a graph on an orientable surface.



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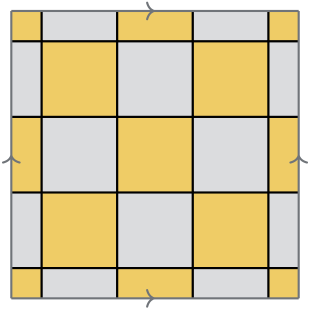

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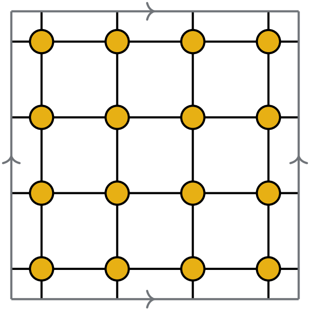

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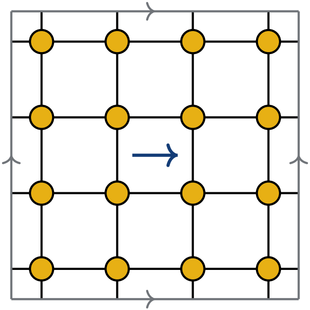

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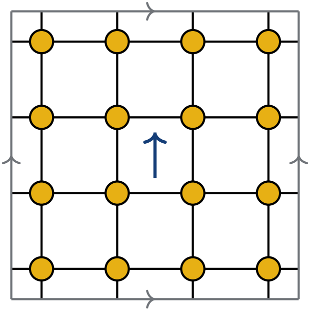

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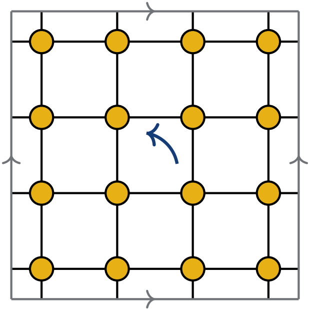

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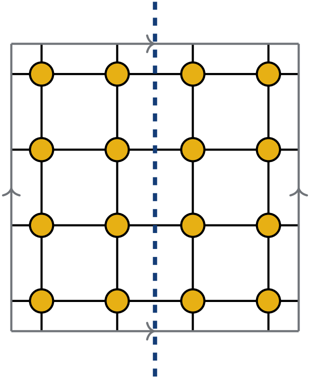

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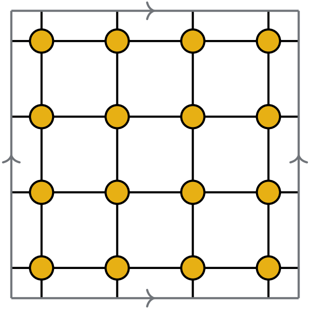

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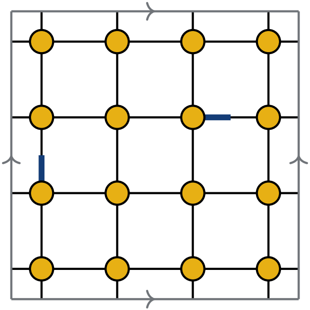
Orientation-preserving automorphisms form a group $\mathrm{Aut}^+(\mathcal{M})$.



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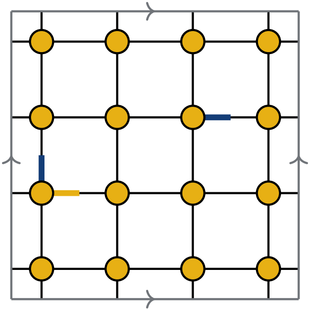

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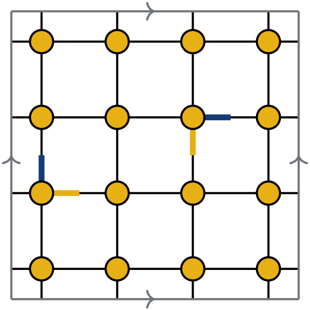

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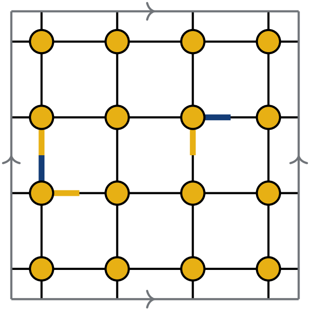

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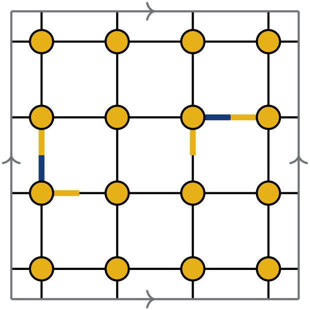

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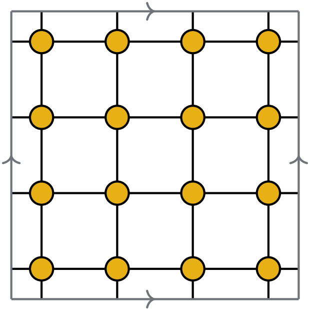

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Hence $|\mathrm{Aut}^+(\mathcal{M})| \leq |\mathcal{D}(\mathcal{M})|$.
If $|\mathrm{Aut}^+(\mathcal{M})| = |\mathcal{D}(\mathcal{M})|$ we say that $\mathcal{M}$ is **orientably-regular**.

The **valency** k of $\mathcal{M}$ is the least common multiple of valencies of vertices.
The **covalency** m of $\mathcal{M}$ is the least common multiple of valencies of faces.
We say that $\mathcal{M}$ has **type** $\{m, k\}$ and corresponds with pair (k, m).

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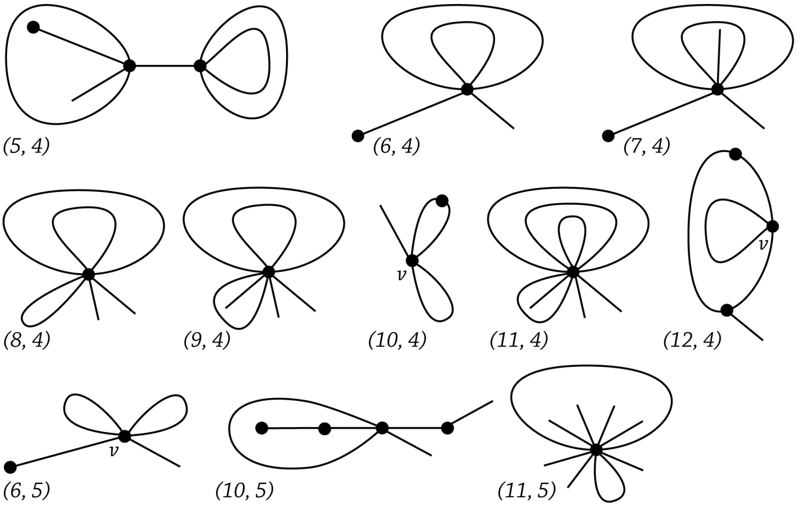

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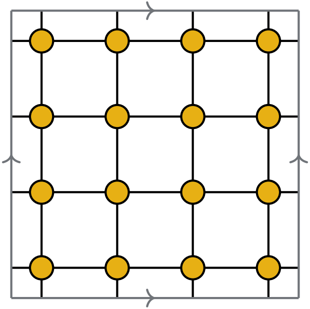

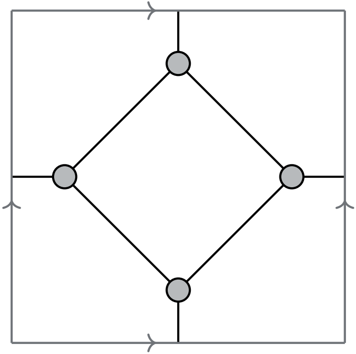

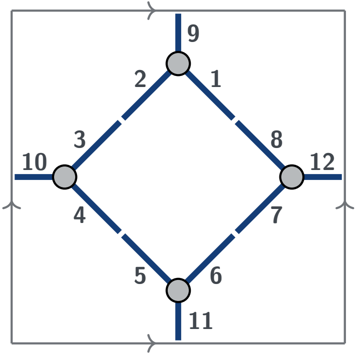
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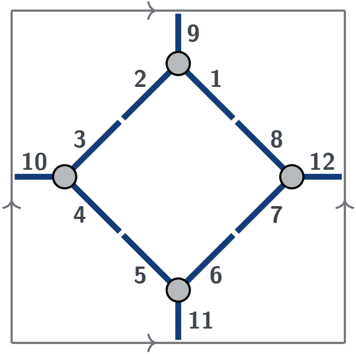

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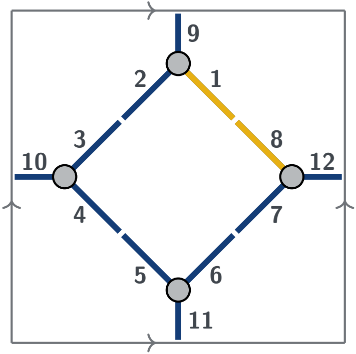

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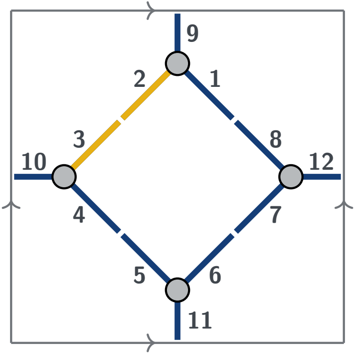

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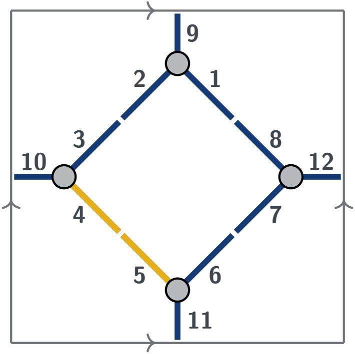

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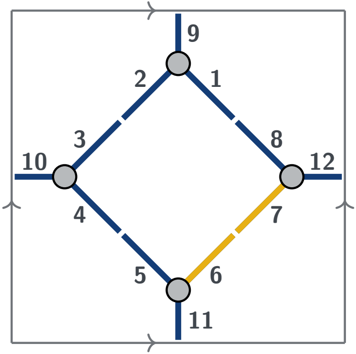

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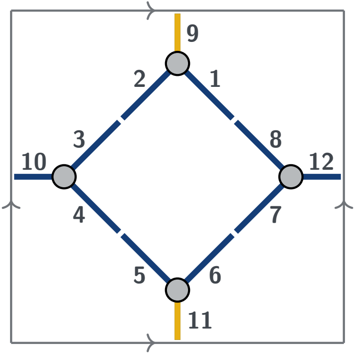

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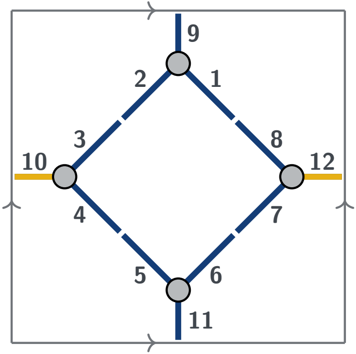

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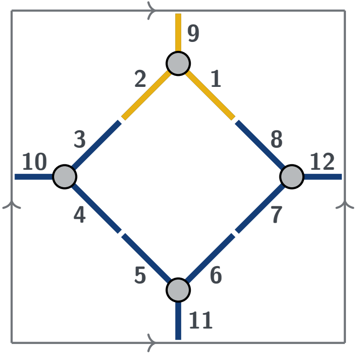

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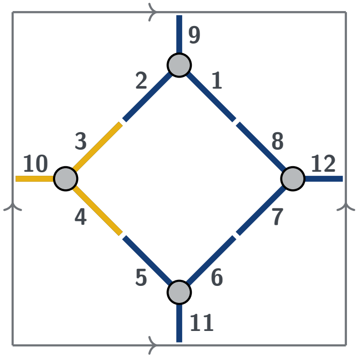
Orientable maps



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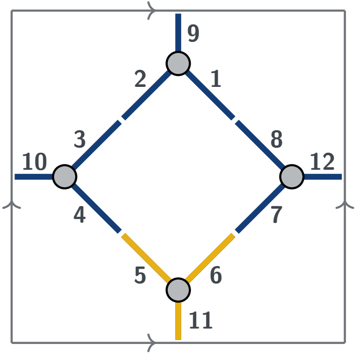

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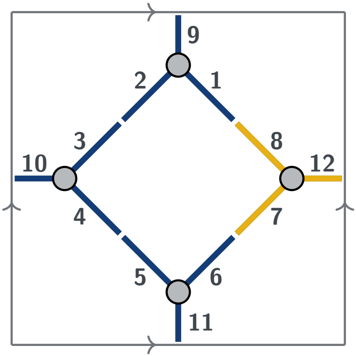

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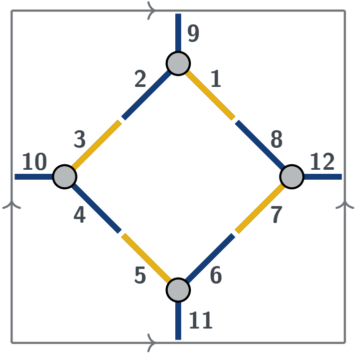

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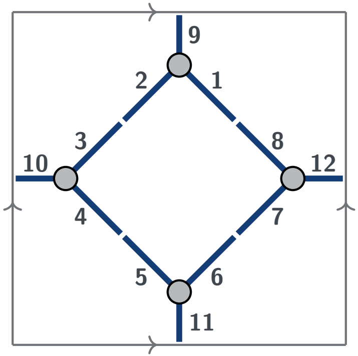

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Map $\mathcal{M} = (\mathcal{D}, \mathcal{L}, \mathcal{R})$ has type $\{\mathrm{ord}(\mathcal{L}\mathcal{R}), \mathrm{ord}(\mathcal{R})\}$.

Map $\mathcal{M}$ is orientably-regular if and only if $|G| = |\mathcal{D}|$, where $G = \langle \mathcal{R}, \mathcal{L} \rangle$.

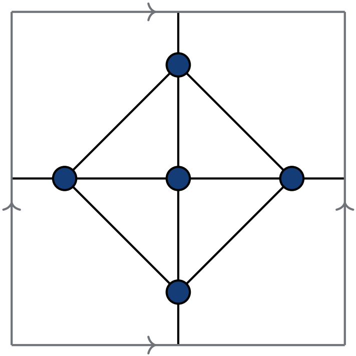
$\mathcal{E} \in \mathrm{Sym}(\mathcal{D})$ is an orientation-preserving automorphism of $\mathcal{M} = (\mathcal{D}, \mathcal{L}, \mathcal{R})$ if and only if $\mathcal{R}\mathcal{E} = \mathcal{E}\mathcal{R}$ and $\mathcal{L}\mathcal{E} = \mathcal{E}\mathcal{L}$.

There is an o-p isomorphism between $\mathcal{M}_1 = (\mathcal{D}, \mathcal{L}_1, \mathcal{R}_1)$ and $\mathcal{M}_2 = (\mathcal{D}, \mathcal{L}_2, \mathcal{R}_2)$ if and only if there exists $\mathcal{E} \in \mathrm{Sym}(\mathcal{D})$ such that $\mathcal{R}_1\mathcal{E} = \mathcal{E}\mathcal{R}_2$ and $\mathcal{L}_1\mathcal{E} = \mathcal{E}\mathcal{L}_2$.

Group $\mathrm{Aut}^+(\mathcal{M})$ is the centraliser of G in $\mathrm{Sym}(\mathcal{D})$.

Part 2: Exponents of orientable maps

Can we form (possibly) new orientable maps from a given map $\mathcal{M} = (\mathcal{D}, \mathcal{L}, \mathcal{R})$?



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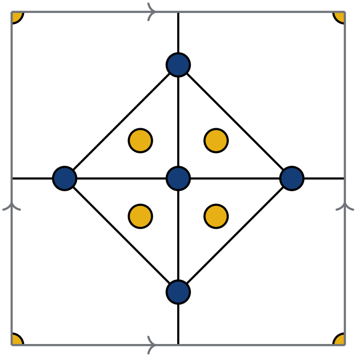

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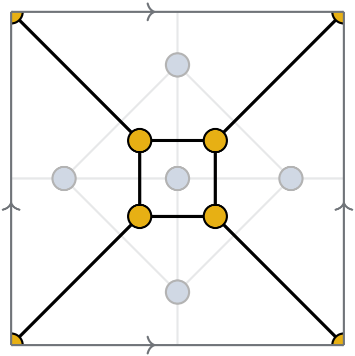

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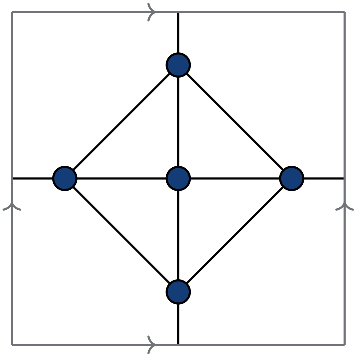

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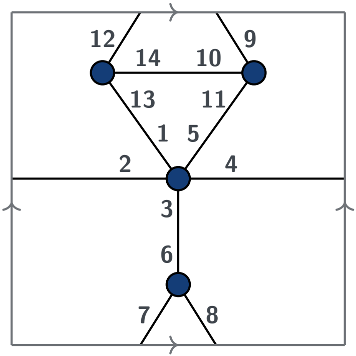
The e th **rotational power** of $\mathcal{M}$ (where $\gcd(k, e) = 1$) is $\mathcal{M}^e = (\mathcal{D}, \mathcal{L}, \mathcal{R}^e)$.



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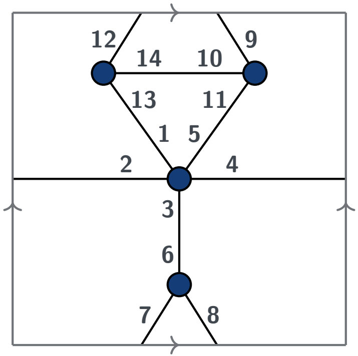

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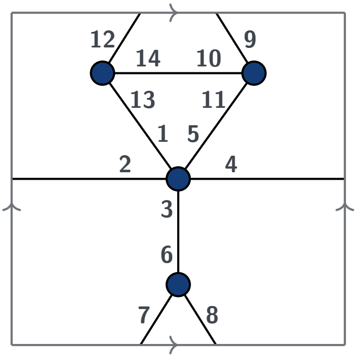

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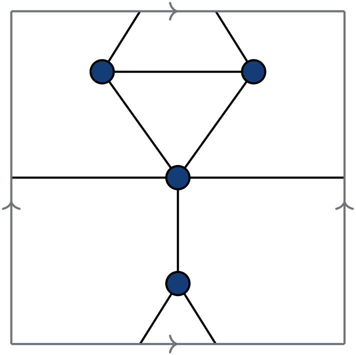
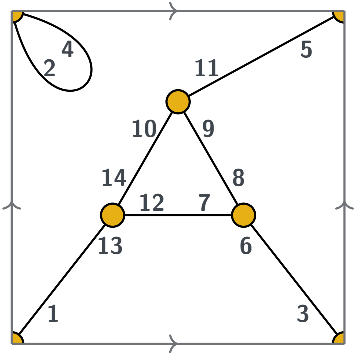
Exponents of orientable maps





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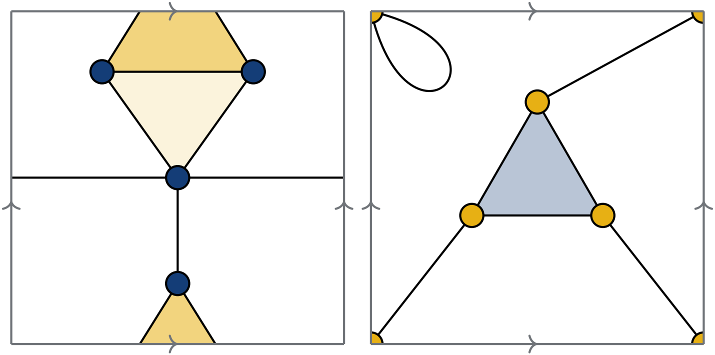

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Underlying graphs of $\mathcal{M}$ and $\mathcal{M}^2$ are the same.

$\mathcal{L}\mathcal{R}^2 = (1,12,6,5,10,13,3,8,11,2)(4)(7,14,9)$
$v - e + f = 4 - 7 + 3 = 0 = 2 - 2g \quad \Rightarrow$ the carrier surface of $\mathcal{M}^2$ is torus

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HYPERBOLIC $(1/k + 1/m < 1/2)$:

Bachratá, B '22

For every hyperbolic pair (k, m) there exists infinitely many orientably-regular maps of type $\{m, k\}$ with no non-trivial exponents.

Part 3: Idea of the proof

Instead of constructing orientably-regular maps for every hyperbolic type $\{m, k\}$, we construct a single orientable map $\mathcal{M} = (\mathcal{D}, \mathcal{L}, \mathcal{R})$ such that:

- ◇ $\mathcal{M}$ has type $\{m, k\}$
- ◇ $\mathcal{M}$ has at least 7 darts
- ◇ $\langle \mathcal{L}, \mathcal{R} \rangle = \mathrm{Alt}(\mathcal{D})$ or $\mathrm{Sym}(\mathcal{D})$
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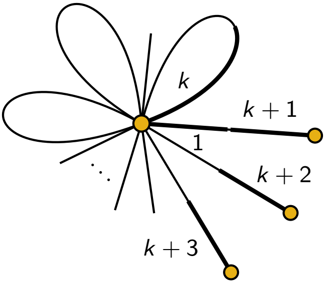
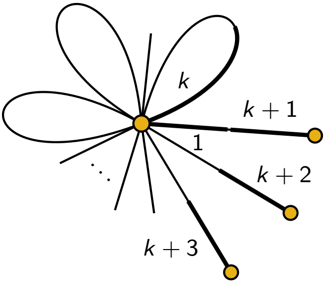
Then we take the **canonical regular cover** $\mathcal{M}' = (\langle \mathcal{L}, \mathcal{R} \rangle, \mathcal{L}, \mathcal{R})$ of $\mathcal{M}$.

- ⋄ $\mathcal{M}$ and $\mathcal{M}'$ have the same type
- ⋄ $\mathcal{M}'$ is always orientably-regular
- ⋄ if $|\mathcal{D}| \geq 7$, $\langle \mathcal{L}, \mathcal{R} \rangle = \mathrm{Alt}(\mathcal{D})$ or $\mathrm{Sym}(\mathcal{D})$, and $\mathcal{M}$ has no non-trivial exponents, then $\mathcal{M}'$ has no non-trivial exponenets



Base map of type $\{k, k\}$ for $k \geq 10$.

$\mathcal{L} = (1, k+1)(2, k+2)(3, k+3)(k-6, k-5)(k-4, k-3)(k-1, k)$
$\mathcal{R} = (1, 2, \ldots, k)$



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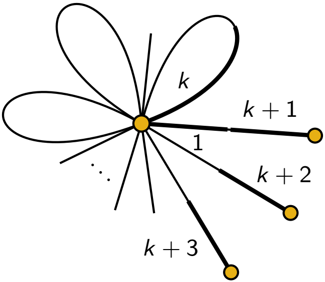

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$\mathcal{R} = (1, 2, \ldots, k)$
$\mathcal{LR} = (1, k+1, 2, k+2, 3, k+3, 4, 5, \ldots, k-6, k-4, k-2, k-1)$

Jones '14

Let G be a ~~primitive~~ 2-transitive permutation group of degree n that contains a cycle of length not exceeding $n-3$. Then G is isomorphic to the symmetric or the alternating group of degree n.

We constructed 7 infinite families for type(s):

- ⋄ $\{k, k\}$ for $k \geq 10$
- ⋄ $\{k+1, k\}$ and $\{k, k+1\}$ for $k \geq 8$
- ⋄ $\{m, k\}$ and $\{k, m\}$ for $k \geq 6$ and $k+2 \leq m \leq 2k-4$
- ⋄ $\{m, k\}$ and $\{k, m\}$ for $k \geq 5$ and $\max(9, 2k-3) \leq m \leq 4k-11$
- ⋄ $\{m, k\}$ and $\{k, m\}$ for $k \geq 5$ and $m \geq 3k-3$
- ⋄ $\{m, 4\}$ and $\{4, m\}$ for $m \geq 13$
- ⋄ $\{3, k\}$ and $\{k, 3\}$ for $k = 13$ and $k \geq 15$

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This leaves 51 hyperbolic pairs, for which we have found suitable permutations $\mathcal{R}$ and $\mathcal{L}$.

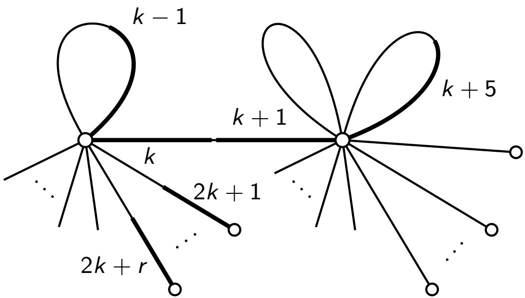

Figure: Base maps of types $\{m, k\}$ and $\{k, m\}$ for $k \geq 5$ and $\max(9, 2k-3) \leq m \leq 4k-11$

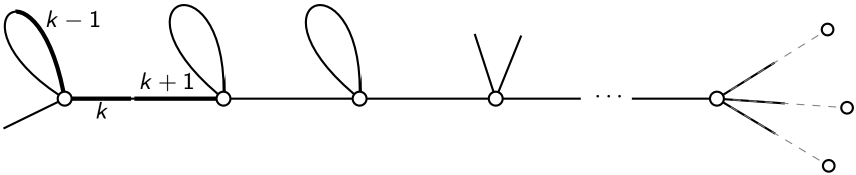

Figure: Base maps of types $\{m, 4\}$ and $\{4, m\}$ for $m \geq 13$

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