Highly symmetric polytopes with prescribed local combinatorics

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National Autonomous University of Mexico

Algebraic Graph Theory International Webminar
March 2021

First, a Caley graph...

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The idea...

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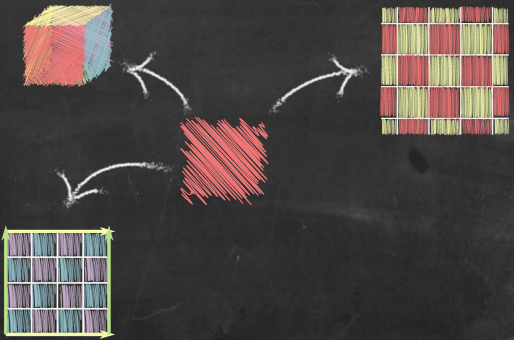

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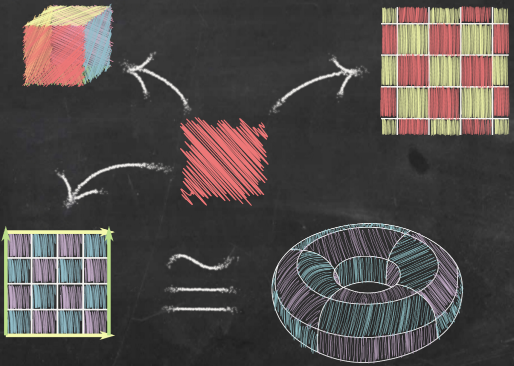

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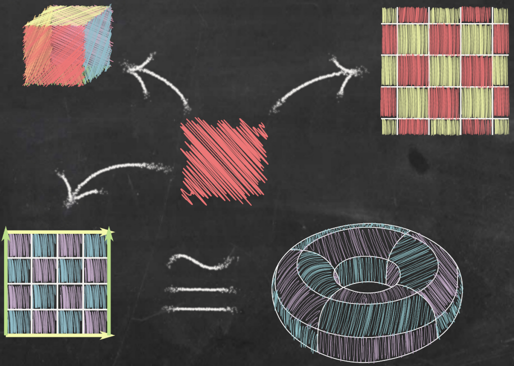

Given an abstract n-polytope $\mathcal{K}$...

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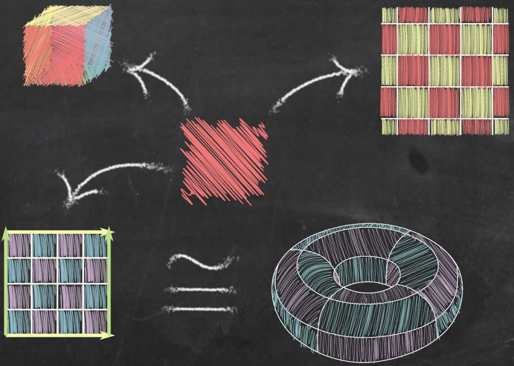

Given an abstract n-polytope $\mathcal{K}$... what are the possibilities for an $(n+1)$-polytope $\mathcal{P}$ with all the facets isomorphic to $\mathcal{K}$.

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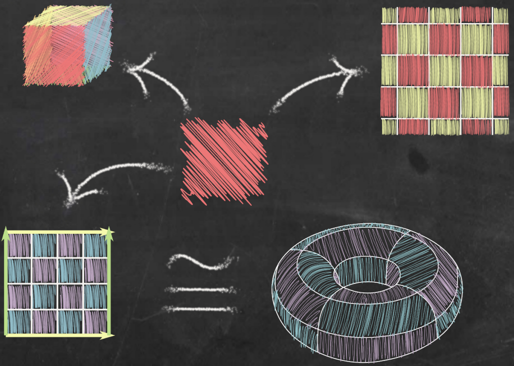

Existence?

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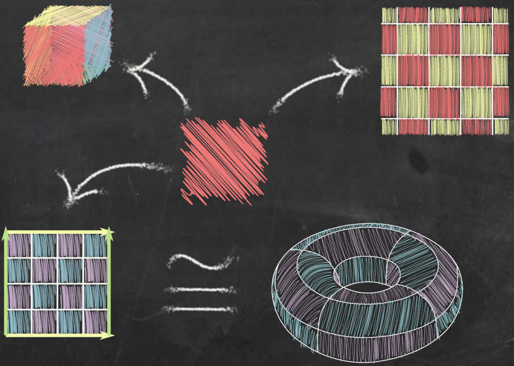

Finiteness?

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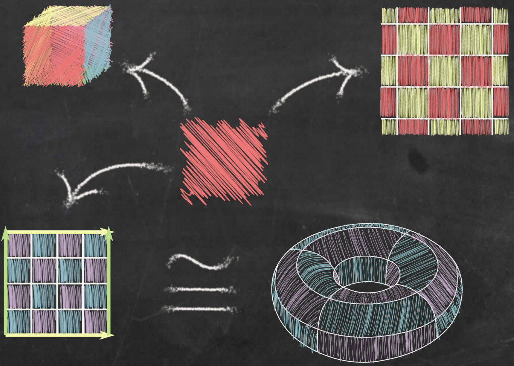

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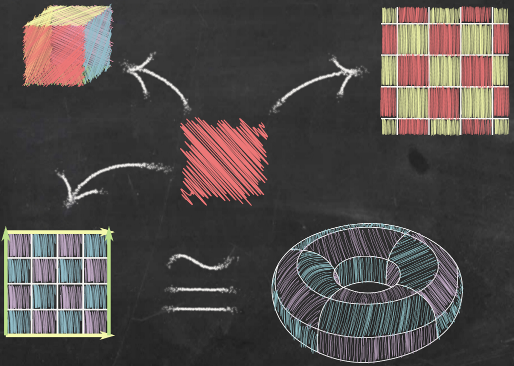

Universality?

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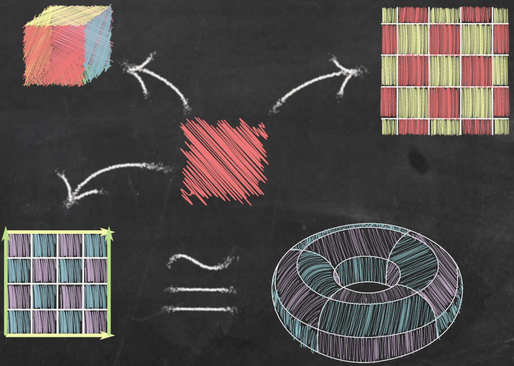

Symmetry?

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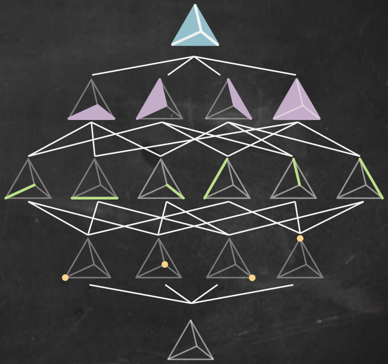

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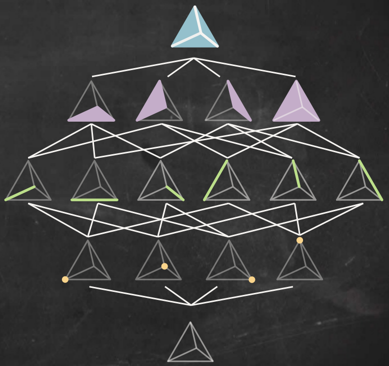

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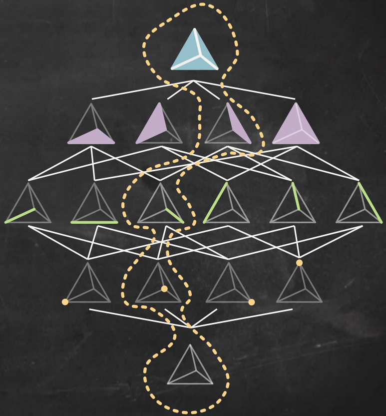

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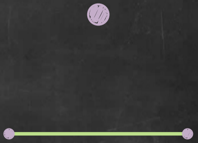

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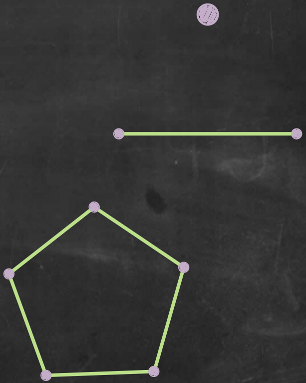

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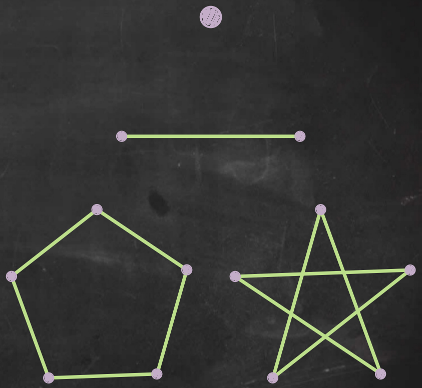

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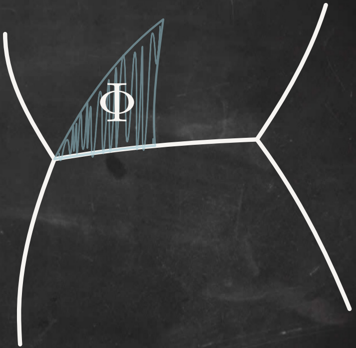

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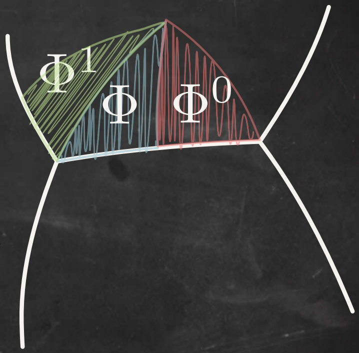

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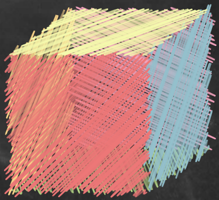

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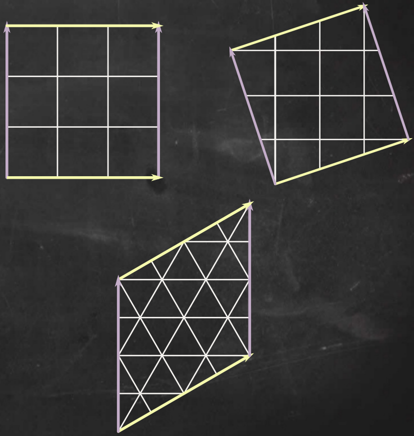

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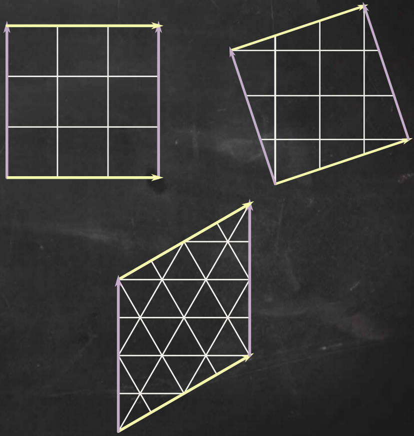

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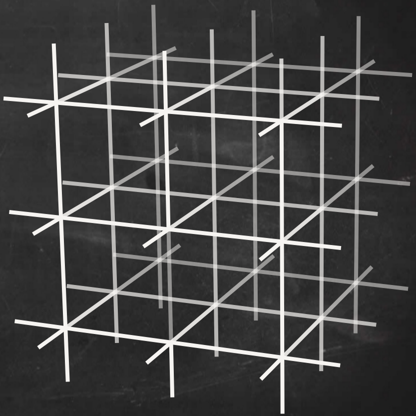

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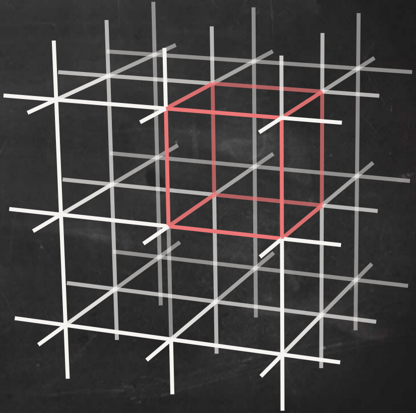

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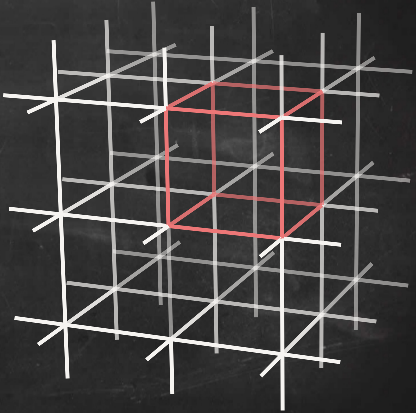

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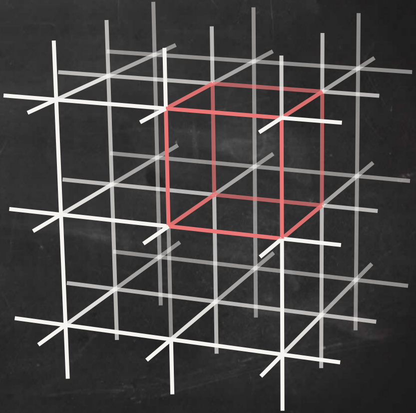

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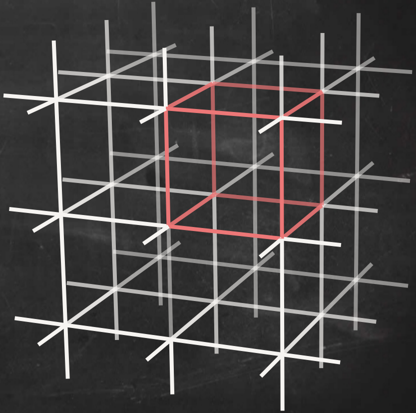

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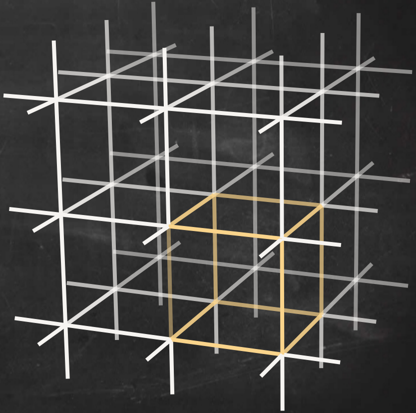

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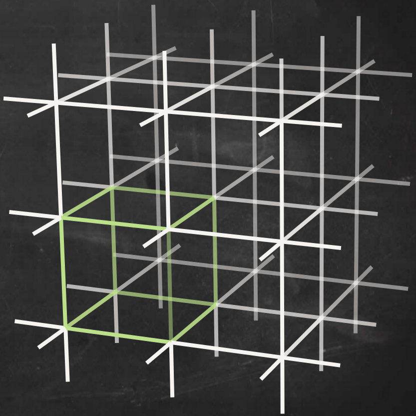

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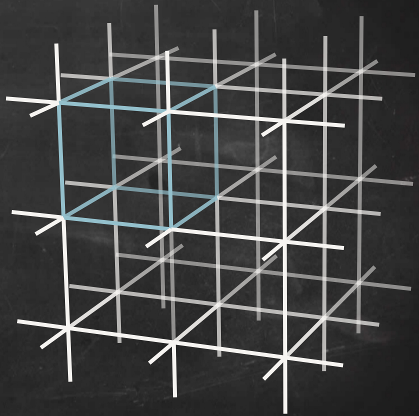

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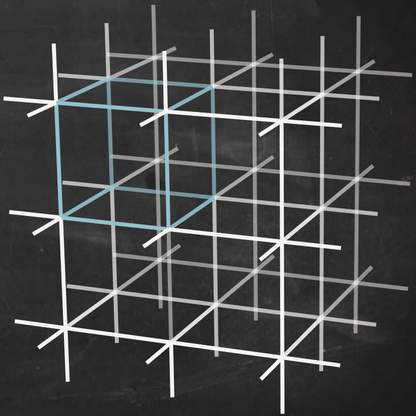

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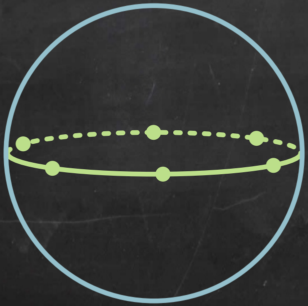

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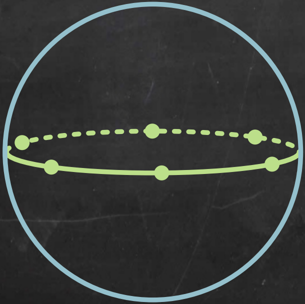

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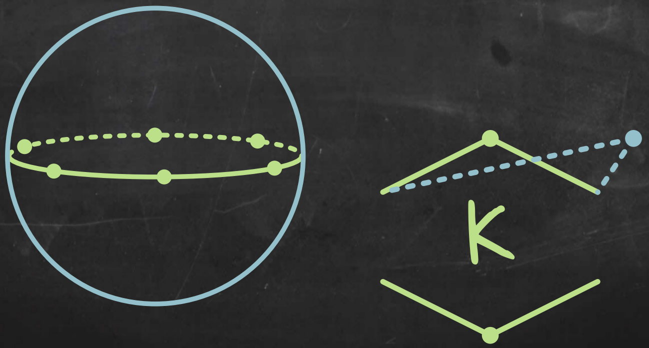

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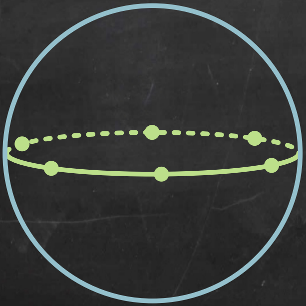

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* If this action is also transitive, we say that $\mathcal{P}$ is regular.

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* Facets of regular polytopes are regular.

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* If $\mathcal{K}$ is a n-polytope, does $\mathcal{K}$ admit a non-degenerate regular extension?

Symmetries of RP

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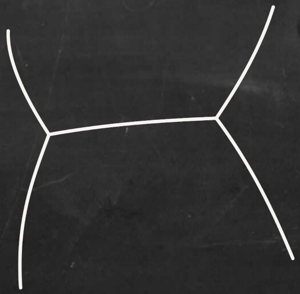

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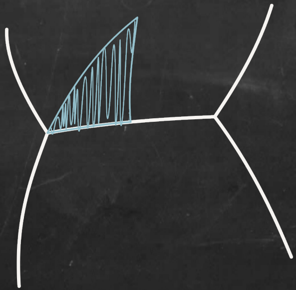

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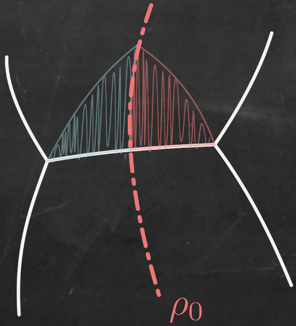

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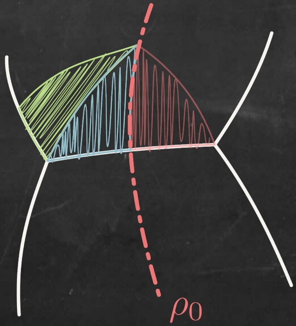

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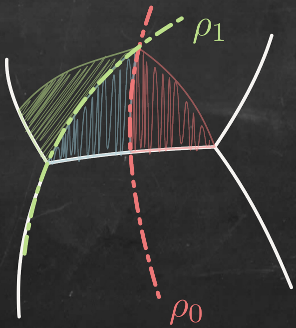

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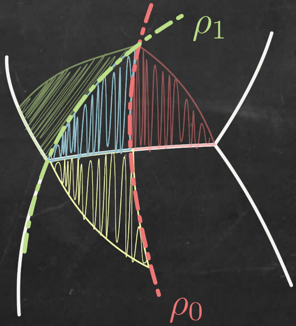

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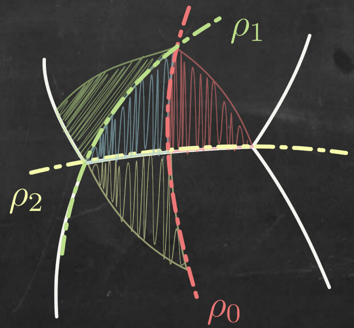

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For $\mathcal{P}$ a regular n-polytope

* $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$

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* $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$

* If $\mathcal{P}$ is of type $\{p_1, \ldots, p_{n-1}\}$,

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Given a regular n polytope $\mathcal{K}$, with $\Gamma(\mathcal{K}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ and a group $\Gamma = \langle \tilde{\rho}_0, \ldots, \tilde{\rho}_n \rangle$

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The regular polytope $\mathcal{P}(\Gamma)$ is a regular extension of $\mathcal{K}$ of type $\{\mathcal{K}, q\}$ where $q = o(\tilde{\rho}_{n-1}\tilde{\rho}_n)$.

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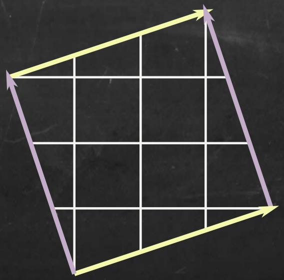

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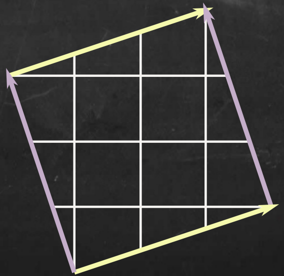

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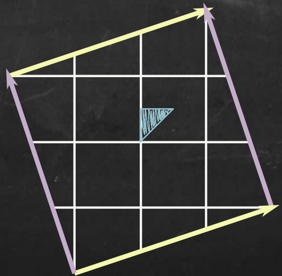

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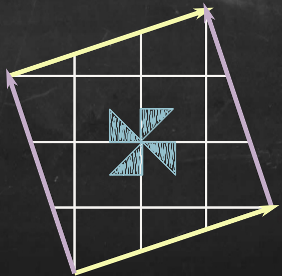

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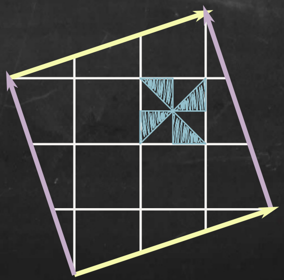

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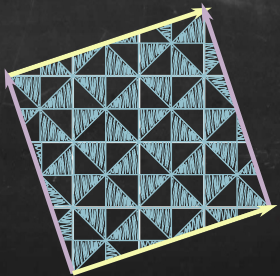

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  - (McMullen-Schulte, 96): No chiral n-polytopes from Euclidean tilings ($n \geqslant 4$).

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Proposition

Let $\mathcal{P}$ be a chiral n-polytope:

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* All the facets of $\mathcal{P}$ are either orientably regular or chiral but all the $(n-2)$-faces are regular.

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Theorem (M., 2021 $^{+}$)

* If $\mathcal{K}$ is a finite dually bipartite chiral polytope with regular facets, there are infinitely many numbers s such that $\mathcal{K}$ has a chiral extension of type $\{\mathcal{K}, 2s\}$.

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* (M.-Pellicer-Toledo, 2021 $^{+}$): If n is even, almost every regular n-toroid admits a chiral extension.

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- (Mochán, 2021 $^+$): Some of those maniplexes are in fact polytopes.

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* The extension problem makes sense and has two possibilities.

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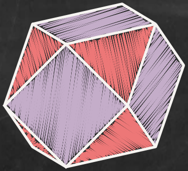

Figure: $\{{}^{4}_{3}, 2\}$

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* (Schulte-Monson, 2012) Characterise the automorphism groups of the alternating polytopes.

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* Problem Characterise the triplets $(\mathcal{P}, \mathcal{Q}, k)$ for which there exists a finite alternating polytope of type $\{^{\mathcal{P}}_{\mathcal{Q}}, k\}$.

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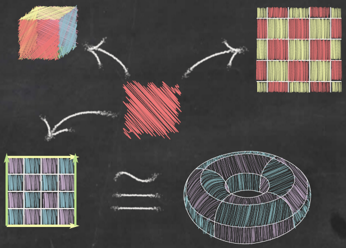

Thanks!