Recognizing vertex-transitive digraphs which are wreath products, double coset digraphs, and generalized wreath products

Ted Dobson

University of Primorska

March 30, 2021

This is joint work with Rachel Barber of Mississippi State University

Definition

Let G be a group and $S \subset G$ such that $S = S^{-1}$ and $1_G \notin S$. Define a **Cayley graph of** *G, denoted $\mathrm{Cay}(G, S)$, to be the graph with $V(\mathrm{Cay}(G, S)) = G$ and $E(\mathrm{Cay}(G, S)) = \{\{g, gs\} : g \in G, s \in S\}$. We call S the* **connection set of** *$\mathrm{Cay}(G, S)$.*

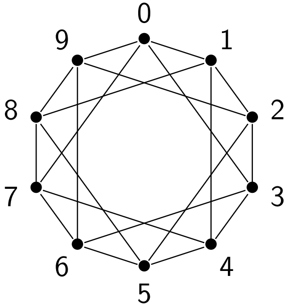


Figure: The Cayley graph $\mathrm{Cay}(\mathbb{Z}_{10}, \{1, 3, 7, 9\})$.

While I will only talk about graphs and Cayley graphs, all of the results hold for digraphs and double coset digraphs (with obvious modifications). Also, most hold similarly for bicoset graphs and Haar graphs.

Definition

Let Γ_1 *and* Γ_2 *be graphs. The* **wreath product of** Γ_1 **and** Γ_2*, denoted* $\Gamma_1 \wr \Gamma_2$*,*

Definition

Let Γ_1 *and* Γ_2 *be graphs. The* **wreath product of** Γ_1 **and** Γ_2*, denoted* $\Gamma_1 \wr \Gamma_2$*, is the graph with vertex set* $V(\Gamma_1) \times V(\Gamma_2)$ *and edge set*

$$\{(u,v)(u,v') : u \in V(\Gamma_1) \text{ and } vv' \in E(\Gamma_2)\}$$

$$\cup\{(u,v)(u',v') : uu' \in E(\Gamma_1) \text{ and } v,v' \in V(\Gamma_2)\}.$$

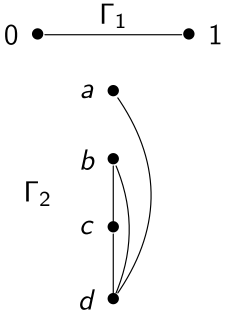
Intuitively, $\Gamma_1 \wr \Gamma_2$ is constructed as follows. First, we have $|V(\Gamma_1)|$ copies of the graph Γ_2, with these $|V(\Gamma_1)|$ copies indexed by elements of $V(\Gamma_1)$. Next, between corresponding copies of Γ_2 we place every possible edge from one copy to another if in Γ_1 there is an edge between the indexing labels of the copies of Γ_2, and no edges otherwise.

To find the wreath product of any two graphs Γ_1 and Γ_2:

1. First corresponding to each vertex of Γ_1, put a copy of Γ_2.

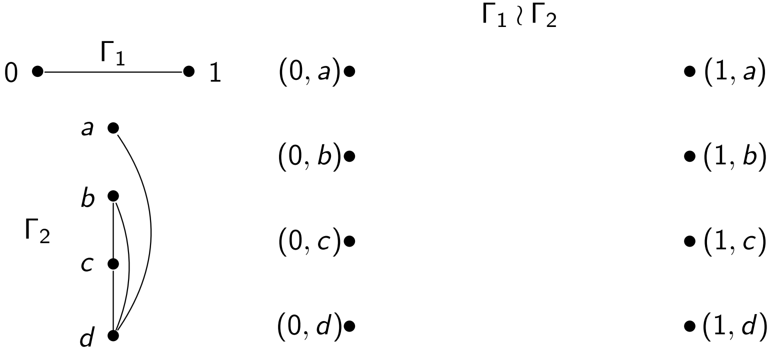
To find the wreath product of any two graphs Γ_1 and Γ_2:

1. First corresponding to each vertex of Γ_1, put a copy of Γ_2.
2.



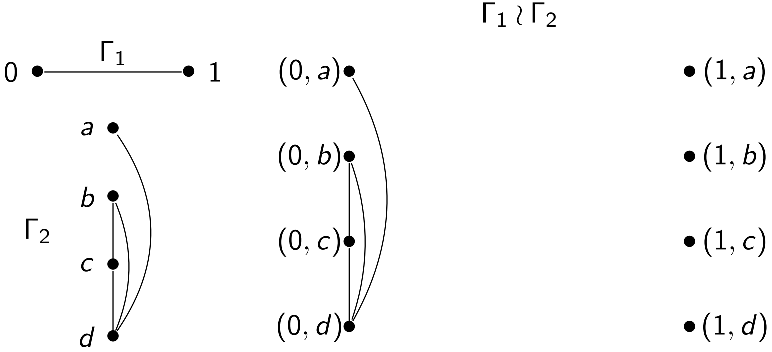
To find the wreath product of any two graphs Γ_1 and Γ_2:

1. First corresponding to each vertex of Γ_1, put a copy of Γ_2.
2.



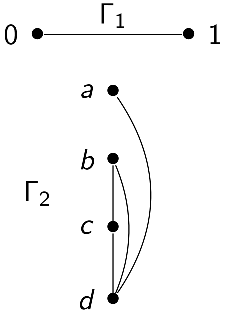
To find the wreath product of any two graphs Γ_1 and Γ_2:

1. First corresponding to each vertex of Γ_1, put a copy of Γ_2.
2.

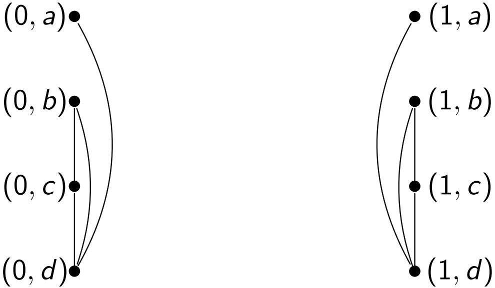


To find the wreath product of any two graphs Γ_1 and Γ_2:

1. First corresponding to each vertex of Γ_1, put a copy of Γ_2.
2.

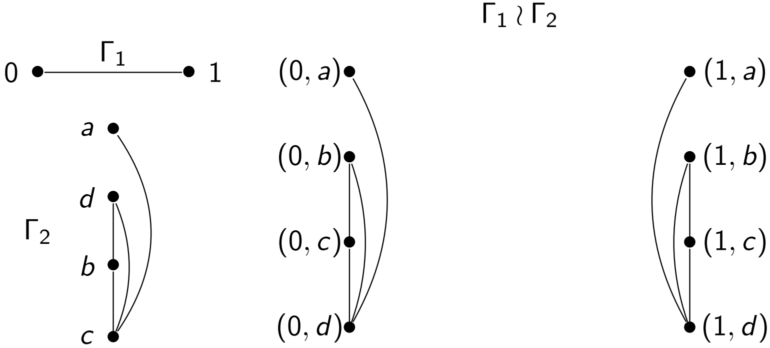


$\Gamma_1 \wr \Gamma_2$



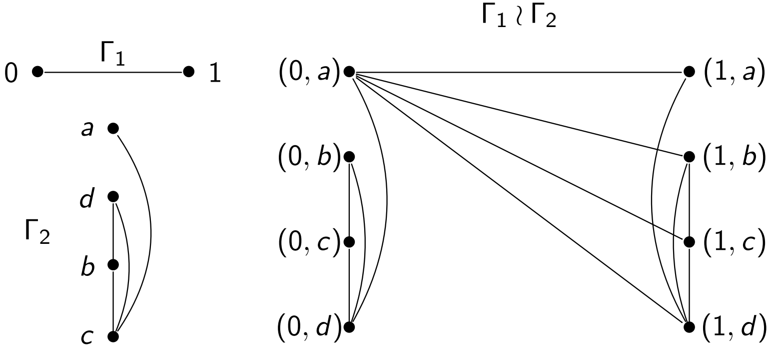
To find the wreath product of any two graphs Γ_1 and Γ_2:

1. First corresponding to each vertex of Γ_1, put a copy of Γ_2.
2. If v_1 and v_2 are adjacent in Γ_1, put every edge between corresponding copies of Γ_2.



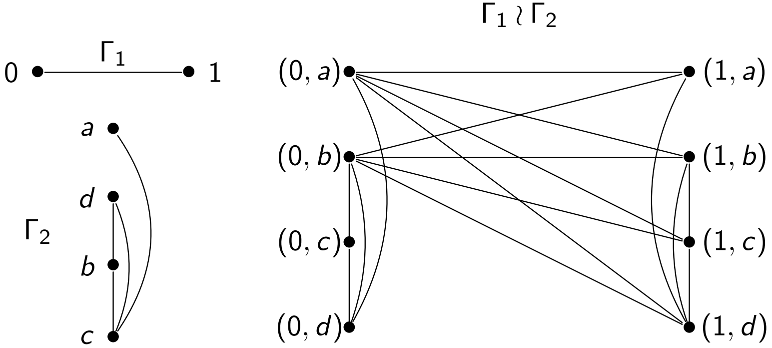
To find the wreath product of any two graphs Γ_1 and Γ_2:

1. First corresponding to each vertex of Γ_1, put a copy of Γ_2.
2. If v_1 and v_2 are adjacent in Γ_1, put every edge between corresponding copies of Γ_2.



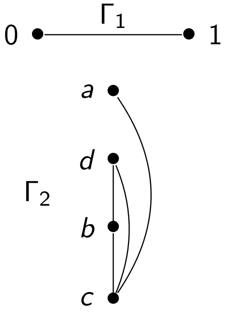
To find the wreath product of any two graphs Γ_1 and Γ_2:

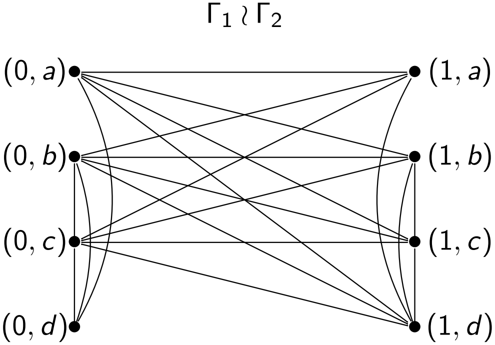
1. First corresponding to each vertex of Γ_1, put a copy of Γ_2.
2. If v_1 and v_2 are adjacent in Γ_1, put every edge between corresponding copies of Γ_2.



To find the wreath product of any two graphs Γ_1 and Γ_2:

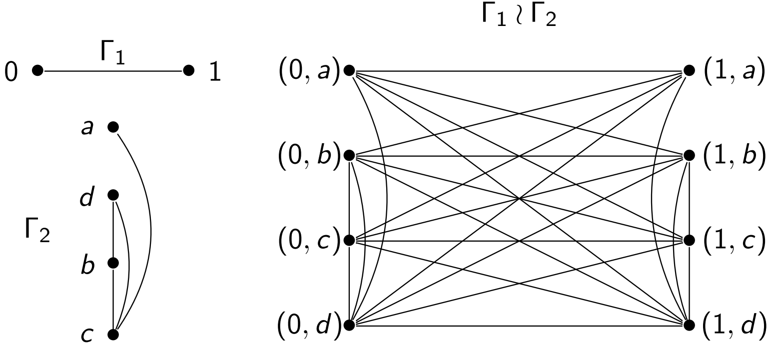
1. First corresponding to each vertex of Γ_1, put a copy of Γ_2.
2. If v_1 and v_2 are adjacent in Γ_1, put every edge between corresponding copies of Γ_2.





To find the wreath product of any two graphs Γ_1 and Γ_2:

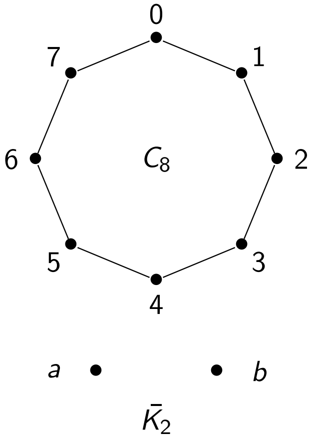
1. First corresponding to each vertex of Γ_1, put a copy of Γ_2.
2. If v_1 and v_2 are adjacent in Γ_1, put every edge between corresponding copies of Γ_2.

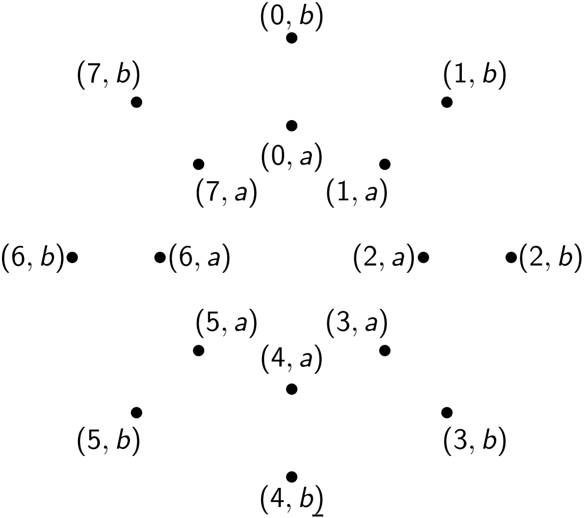


The wreath product of graphs has many names, The wreath product of graphs has many names, the lexicographic product, The wreath product of graphs has many names, the lexicographic product, graph composition, The wreath product of graphs has many names, the lexicographic product, graph composition, and the Γ_2-extension of Γ_1.

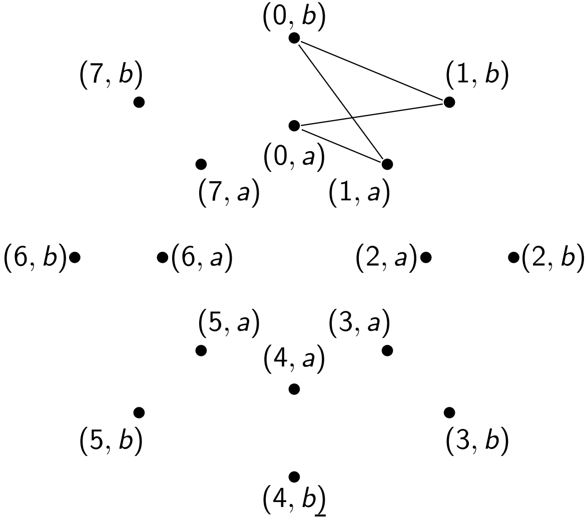
Let us consider the graph $C_8 \wr \bar{K}_2$.

Let us consider the graph $C_8 \wr \bar{K}_2$.

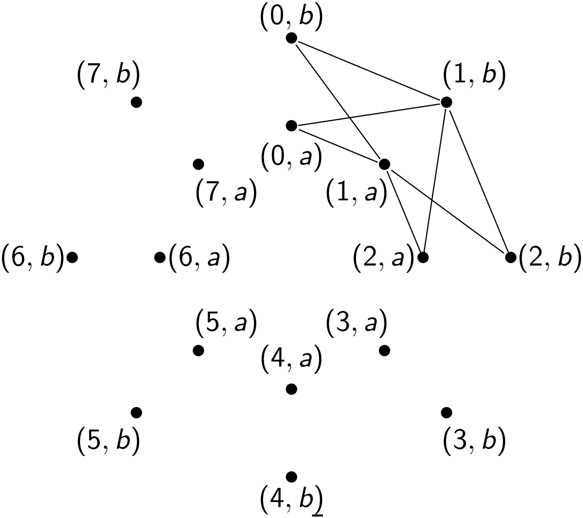




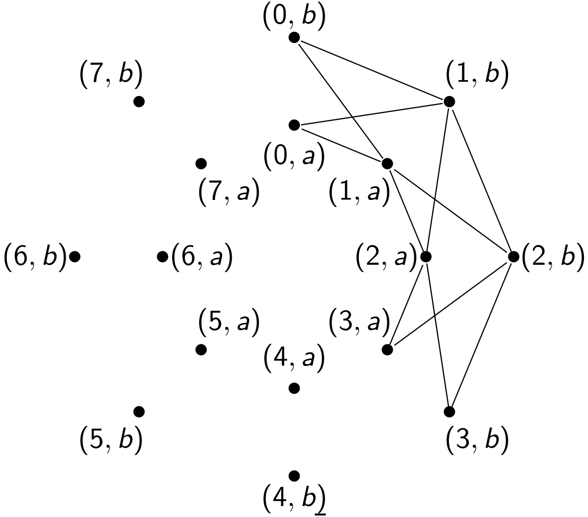
$C_8 \wr K_2$



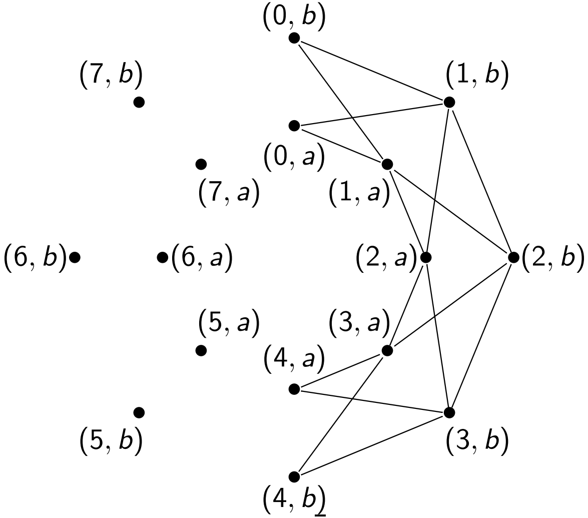
$C_8 \wr \bar{K}_2$



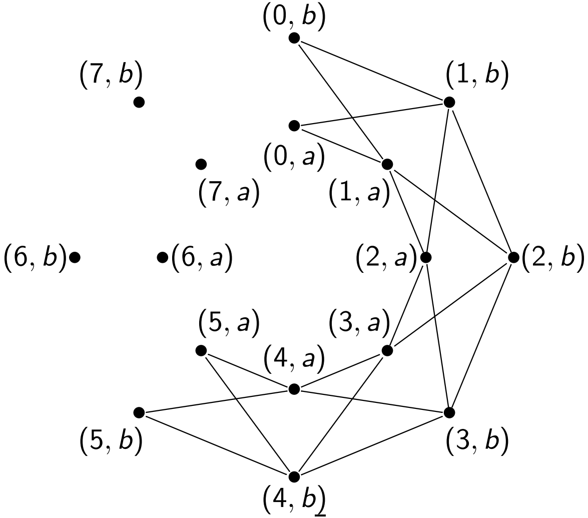
$C_8 \wr \bar{K}_2$



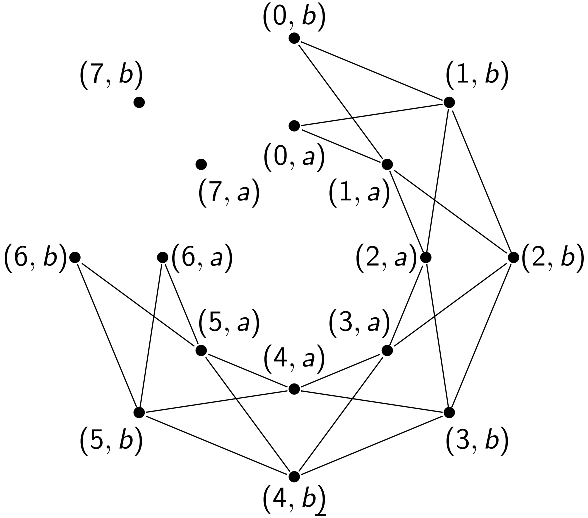
$C_8 \wr \bar{K}_2$



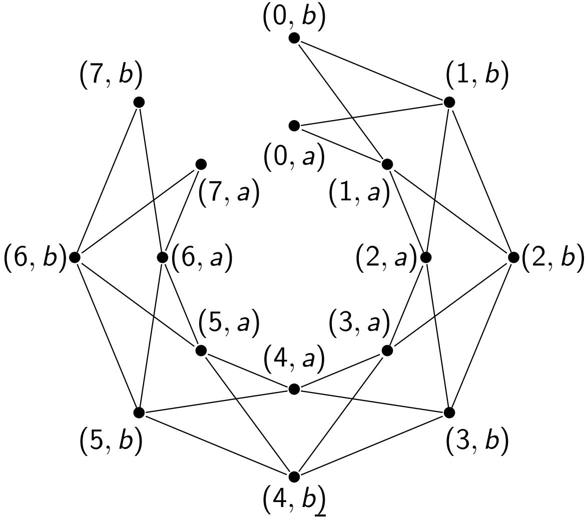
$C_8 \wr \bar{K}_2$



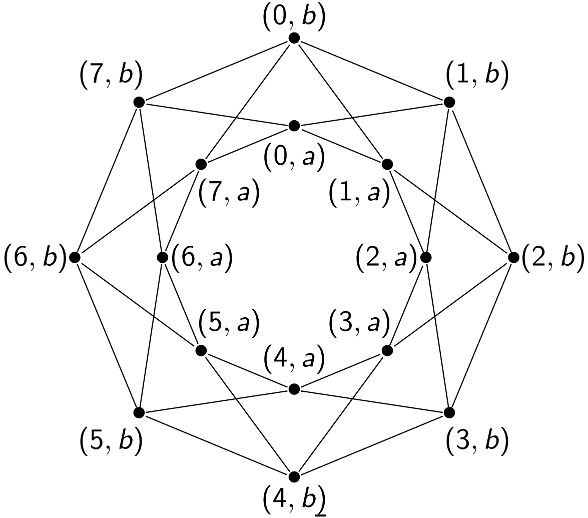
$C_8 \wr \bar{K}_2$



$C_8 \wr \bar{K}_2$



$C_8 \wr \bar{K}_2$



$C_8 \wr \bar{K}_2$

In the previous graph, think of the sets $\{(i,j) : j \in \mathbb{Z}_2\}$ as blocks.

In the previous graph, think of the sets $\{(i,j) : j \in \mathbb{Z}_2\}$ as blocks. Take any automorphism of C_8, and think of it as "permuting" the blocks.

In the previous graph, think of the sets $\{(i,j) : j \in \mathbb{Z}_2\}$ as blocks. Take any automorphism of C_8, and think of it as "permuting" the blocks. A block is mapped to a block by any automorphism of $\bar{K}_2$, and we can have different automorphisms of $\bar{K}_2$ for different blocks.

In the previous graph, think of the sets $\{(i,j) : j \in \mathbb{Z}_2\}$ as blocks. Take any automorphism of C_8, and think of it as "permuting" the blocks. A block is mapped to a block by any automorphism of $\bar{K}_2$, and we can have different automorphisms of $\bar{K}_2$ for different blocks. This is the group $\mathrm{Aut}(C_8) \wr \mathrm{Aut}(\bar{K}_2)$.

In the previous graph, think of the sets $\{(i, j) : j \in \mathbb{Z}_2\}$ as blocks. Take any automorphism of C_8, and think of it as "permuting" the blocks. A block is mapped to a block by any automorphism of $\bar{K}_2$, and we can have different automorphisms of $\bar{K}_2$ for different blocks. This is the group $\mathrm{Aut}(C_8) \wr \mathrm{Aut}(\bar{K}_2)$.

Definition

Let G be a permutation group on X and H a permutation group on Y.

In the previous graph, think of the sets $\{(i,j) : j \in \mathbb{Z}_2\}$ as blocks. Take any automorphism of C_8, and think of it as "permuting" the blocks. A block is mapped to a block by any automorphism of $\bar{K}_2$, and we can have different automorphisms of $\bar{K}_2$ for different blocks. This is the group $\mathrm{Aut}(C_8) \wr \mathrm{Aut}(\bar{K}_2)$.

Definition

Let G be a permutation group on X and H a permutation group on Y. Define the **wreath product of G and H**,

In the previous graph, think of the sets $\{(i,j) : j \in \mathbb{Z}_2\}$ as blocks. Take any automorphism of C_8, and think of it as "permuting" the blocks. A block is mapped to a block by any automorphism of $\bar{K}_2$, and we can have different automorphisms of $\bar{K}_2$ for different blocks. This is the group $\mathrm{Aut}(C_8) \wr \mathrm{Aut}(\bar{K}_2)$.

Definition

Let G be a permutation group on X and H a permutation group on Y. Define the **wreath product of G and H**, *denoted $G \wr H$,*

In the previous graph, think of the sets $\{(i,j) : j \in \mathbb{Z}_2\}$ as blocks. Take any automorphism of C_8, and think of it as "permuting" the blocks. A block is mapped to a block by any automorphism of $\bar{K}_2$, and we can have different automorphisms of $\bar{K}_2$ for different blocks. This is the group $\mathrm{Aut}(C_8) \wr \mathrm{Aut}(\bar{K}_2)$.

Definition

Let G be a permutation group on X and H a permutation group on Y. Define the **wreath product of G and H**, *denoted $G \wr H$, to be the set of all permutations of $X \times Y$ of the form $(x,y) \to (g(x), h_x(y))$, where $g \in G$ and each $h_x \in H$.*

In the previous graph, think of the sets $\{(i,j) : j \in \mathbb{Z}_2\}$ as blocks. Take any automorphism of C_8, and think of it as "permuting" the blocks. A block is mapped to a block by any automorphism of $\bar{K}_2$, and we can have different automorphisms of $\bar{K}_2$ for different blocks. This is the group $\mathrm{Aut}(C_8) \wr \mathrm{Aut}(\bar{K}_2)$.

Definition

Let G be a permutation group on X and H a permutation group on Y. Define the **wreath product of G and H**, *denoted $G \wr H$, to be the set of all permutations of $X \times Y$ of the form $(x,y) \to (g(x), h_x(y))$, where $g \in G$ and each $h_x \in H$.*

It is easy to see that for graphs Γ and Δ, $\mathrm{Aut}(\Gamma) \wr \mathrm{Aut}(\Delta) \leq \mathrm{Aut}(\Gamma \wr \Delta)$.

Problem:

Problem: Find necessary and sufficient conditions on the connection set S of a Cayley graph of G for $\mathrm{Cay}(G,S) \cong \Gamma_1 \wr \Gamma_2$, $|V(\Gamma_1)|, |V(\Gamma_2)| \geq 2$,

Problem: Find necessary and sufficient conditions on the connection set S of a Cayley graph of G for $\mathrm{Cay}(G, S) \cong \Gamma_1 \wr \Gamma_2$, $|V(\Gamma_1)|, |V(\Gamma_2)| \geq 2$, and determine Γ_1 and Γ_2.

Problem: Find necessary and sufficient conditions on the connection set S of a Cayley graph of G for $\mathrm{Cay}(G, S) \cong \Gamma_1 \wr \Gamma_2$, $|V(\Gamma_1)|, |V(\Gamma_2)| \geq 2$, and determine Γ_1 and Γ_2.

What is known?

Problem: Find necessary and sufficient conditions on the connection set S of a Cayley graph of G for $\mathrm{Cay}(G, S) \cong \Gamma_1 \wr \Gamma_2$, $|V(\Gamma_1)|, |V(\Gamma_2)| \geq 2$, and determine Γ_1 and Γ_2.

What is known? It is known that a Cayley digraph $\mathrm{Cay}(A, S)$ of an abelian group is isomorphic to a wreath product of two smaller digraphs if and only if there exists $1 < B < A$ such that $S \setminus B$ is a union of cosets of B. This was shown explicitly for prime powers by Morris, Kovács and Servatius and mentioned without proof in Bhoumik, Dobson, and Morris.

Problem: Find necessary and sufficient conditions on the connection set S of a Cayley graph of G for $\mathrm{Cay}(G, S) \cong \Gamma_1 \wr \Gamma_2$, $|V(\Gamma_1)|, |V(\Gamma_2)| \geq 2$, and determine Γ_1 and Γ_2.

What is known? It is known that a Cayley digraph $\mathrm{Cay}(A, S)$ of an abelian group is isomorphic to a wreath product of two smaller digraphs if and only if there exists $1 < B < A$ such that $S \setminus B$ is a union of cosets of B. This was shown explicitly for prime powers by Morris, Kovács and Servatius and mentioned without proof in Bhoumik, Dobson, and Morris.

It is easy to construct Cayley graphs of nonabelian group G whose connection sets are unions of left or right cosets of a subgroup of G that is not isomorphic to a wreath product.

Problem: Find necessary and sufficient conditions on the connection set S of a Cayley graph of G for $\mathrm{Cay}(G, S) \cong \Gamma_1 \wr \Gamma_2$, $|V(\Gamma_1)|, |V(\Gamma_2)| \geq 2$, and determine Γ_1 and Γ_2.

What is known? It is known that a Cayley digraph $\mathrm{Cay}(A, S)$ of an abelian group is isomorphic to a wreath product of two smaller digraphs if and only if there exists $1 < B < A$ such that $S \setminus B$ is a union of cosets of B. This was shown explicitly for prime powers by Morris, Kovács and Servatius and mentioned without proof in Bhoumik, Dobson, and Morris.

It is easy to construct Cayley graphs of nonabelian group G whose connection sets are unions of left or right cosets of a subgroup of G that is not isomorphic to a wreath product. An obvious next guess is unions of left *and* right cosets of a subgroup of G.

Definition

Let G *be a group,* $H \leq G$*, and* $s \in G$*. The set* $HsH = \{h_1sh_2 : h_1, h_2 \in H\}$ *is a* **double coset of** H **in** G*. Similarly, for* $S \subseteq G$*, the set* $HSH = \{h_1sh_2 : h_1, h_2 \in H, s \in S\}$ *is a union of double cosets of* H *in* G*.*

Definition

Let G be a group, $H \leq G$, and $s \in G$. The set $HsH = \{h_1sh_2 : h_1, h_2 \in H\}$ is a **double coset of H in G**. *Similarly, for $S \subseteq G$, the set $HSH = \{h_1sh_2 : h_1, h_2 \in H, s \in S\}$ is a union of double cosets of H in G.*

The set of double cosets of H in G are a partition of G.

Definition

Let G be a group, $H \leq G$, and $s \in G$. The set $HsH = \{h_1sh_2 : h_1, h_2 \in H\}$ is a **double coset of H in G**. *Similarly, for $S \subseteq G$, the set $HSH = \{h_1sh_2 : h_1, h_2 \in H, s \in S\}$ is a union of double cosets of H in G.*

The set of double cosets of H in G are a partition of G.

Definition

Let G be a group, $H \leq G$ and $S \subset G$. Define a graph $\mathrm{Cos}(G, H, S)$ by $V(\mathrm{Cos}(G, H, S))$ is the set of left cosets of H in G and

$$E(\mathrm{Cos}(G, H, S)) = \{\{gH, gaH\} : g \in G, a \in HSH\}.$$

The graph $\mathrm{Cos}(G, H, S)$ is the **double coset graph of G with stabilizer H and connection set S**.

Definition

Let G be a group, $H \leq G$, and $s \in G$. The set $HsH = \{h_1sh_2 : h_1, h_2 \in H\}$ is a **double coset of H in G**. *Similarly, for $S \subseteq G$, the set $HSH = \{h_1sh_2 : h_1, h_2 \in H, s \in S\}$ is a union of double cosets of H in G.*

The set of double cosets of H in G are a partition of G.

Definition

Let G be a group, $H \leq G$ and $S \subset G$. Define a graph $\mathrm{Cos}(G, H, S)$ by $V(\mathrm{Cos}(G, H, S))$ is the set of left cosets of H in G and

$$E(\mathrm{Cos}(G, H, S)) = \{\{gH, gaH\} : g \in G, a \in HSH\}.$$

The graph $\mathrm{Cos}(G, H, S)$ is the **double coset graph of G with stabilizer H and connection set S**.

A Cayley graph of G is then just a double coset graph of G with $H = \{1\}$.

There is a convention with double coset graphs to insist that H contains no normal nontrivial subgroup of G. This convention gives that the action of G on G/H (the set of left cosets of H in G) by left multiplication ($g \cdot kH = (gk)H$) is faithful. This convention is useful if one is constructing a double coset graph. We will NOT follow this convention (it makes life harder for us and some results simply won't be true).

It may also be useful for this talk to think of double cosets a little differently than normal. Normally, G is chosen, then H is chosen, and S is chosen so that $S = HSH$. Here it will be better to think of choosing G, then choosing S, and then the possible choices for H are determined. The advantage is obvious - I get to choose any connection set I like. The disadvantage is that I don't even know the order of my double coset graph as that is determined by the choice of H.

Definition

Let Γ *be graph and* $\mathcal{P}$ *a partition of* $V(\Gamma)$. *Define the* **quotient graph of** Γ **with respect to** $\mathcal{P}$, *denoted* $\Gamma/\mathcal{P}$, *to be the graph with vertex set* $\mathcal{P}$ *and edge set* $\{\{P,P'\} : P,P' \in \mathcal{P}, P \neq P', \text{ and } uv \in E(\Gamma) \text{ for some } u \in P \text{ and } v \in P'\}$.

Definition

Let Γ *be graph and* $\mathcal{P}$ *a partition of* $V(\Gamma)$*. Define the* **quotient graph of** Γ **with respect to** $\mathcal{P}$, *denoted* $\Gamma/\mathcal{P}$, *to be the graph with vertex set* $\mathcal{P}$ *and edge set* $\{\{P, P'\} : P, P' \in \mathcal{P}, P \neq P', \text{ and } uv \in E(\Gamma) \text{ for some } u \in P \text{ and } v \in P'\}$.

So this quotient maps edges to edges and doesn't care at all about non-edges.

Definition

Let Γ be graph and $\mathcal{P}$ a partition of $V(\Gamma)$. Define the **quotient graph of Γ with respect to $\mathcal{P}$**, *denoted $\Gamma/\mathcal{P}$, to be the graph with vertex set $\mathcal{P}$ and edge set $\{\{P,P'\} : P,P' \in \mathcal{P}, P \neq P', \text{ and } uv \in E(\Gamma) \text{ for some } u \in P \text{ and } v \in P'\}$.*

So this quotient maps edges to edges and doesn't care at all about non-edges.

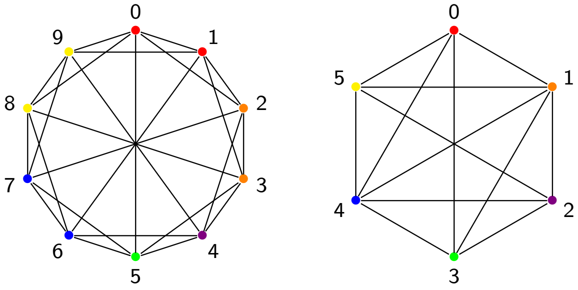
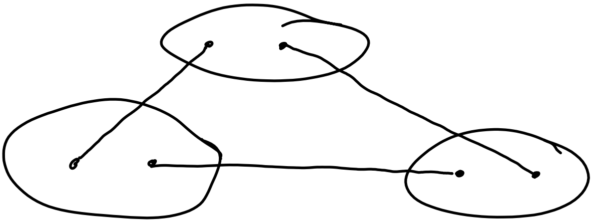


Figure: The Cayley graph $\mathrm{Cay}(\mathbb{Z}_{10}, \{1,2,5,8,9\})$ and its quotient graph.

This quotient also does not “preserve” graph theoretic information.

This quotient also does not “preserve” graph theoretic information. For example, here is a graph whose quotient is a complete graph and the quotient of its complement is the same complete graph:



Theorem

A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ *of a group* G *is isomorphic to a wreath product of two vertex-transitive graphs of smaller order if and only if there exists* $1 < H < G$ *such that* $S \setminus H$ *is a union of double cosets of* H *in* G*. If such a* $1 < H < G$ *exists and* $\mathcal{B}$ *is the partition of* G *that consists of the left cosets of* H*, then*

$$\mathrm{Cay}(G, S) \cong \Gamma/\mathcal{B} \wr \Gamma[H] \cong \mathrm{Cos}(G, H, S) \wr \mathrm{Cay}(H, S \cap H).$$

Additionally, if Γ *is not complete nor the complement of a complete graph and* H *is chosen to be maximal in* G *with the above properties, then*

$$\mathrm{Aut}(\mathrm{Cay}(G, S)) \cong \mathrm{Aut}(\mathrm{Cos}(G, H, S)) \wr \mathrm{Aut}(\mathrm{Cay}(H, S \cap H)).$$

Sabidussi showed in 1964 (Theorem 2 of his paper) that every vertex-transitive graph Γ is isomorphic to a double coset digraph of G, where $G \leq \mathrm{Aut}(\Gamma)$ is transitive.

Sabidussi showed in 1964 (Theorem 2 of his paper) that every vertex-transitive graph Γ is isomorphic to a double coset digraph of G, where $G \leq \mathrm{Aut}(\Gamma)$ is transitive. His definition of a "double coset graph" is different from the one presented - he showed that there was a Cayley graph $\mathrm{Cay}(G, S)$ and subgroup $H \leq G$ such that $\mathrm{Cay}(G, S)/(G/H) \cong \Gamma$ (with the quotient defined as above).

Sabidussi showed in 1964 (Theorem 2 of his paper) that every vertex-transitive graph Γ is isomorphic to a double coset digraph of G, where $G \leq \mathrm{Aut}(\Gamma)$ is transitive. His definition of a "double coset graph" is different from the one presented - he showed that there was a Cayley graph $\mathrm{Cay}(G, S)$ and subgroup $H \leq G$ such that $\mathrm{Cay}(G, S)/(G/H) \cong \Gamma$ (with the quotient defined as above).
In Theorem 4 of his paper he shows that a "multiple" of Γ is isomorphic to a Cayley graph of G (again $G \leq \mathrm{Aut}(\Gamma)$ is transitive, and let the stabilizer of a point be H).

Sabidussi showed in 1964 (Theorem 2 of his paper) that every vertex-transitive graph Γ is isomorphic to a double coset digraph of G, where $G \leq \mathrm{Aut}(\Gamma)$ is transitive. His definition of a "double coset graph" is different from the one presented - he showed that there was a Cayley graph $\mathrm{Cay}(G, S)$ and subgroup $H \leq G$ such that $\mathrm{Cay}(G, S)/(G/H) \cong \Gamma$ (with the quotient defined as above).
In Theorem 4 of his paper he shows that a "multiple" of Γ is isomorphic to a Cayley graph of G (again $G \leq \mathrm{Aut}(\Gamma)$ is transitive, and let the stabilizer of a point be H). This multiple is isomorphic to $\Gamma \wr \bar{K}_{|H|}$.

Sabidussi showed in 1964 (Theorem 2 of his paper) that every vertex-transitive graph Γ is isomorphic to a double coset digraph of G, where $G \leq \mathrm{Aut}(\Gamma)$ is transitive. His definition of a "double coset graph" is different from the one presented - he showed that there was a Cayley graph $\mathrm{Cay}(G, S)$ and subgroup $H \leq G$ such that $\mathrm{Cay}(G, S)/(G/H) \cong \Gamma$ (with the quotient defined as above).

In Theorem 4 of his paper he shows that a "multiple" of Γ is isomorphic to a Cayley graph of G (again $G \leq \mathrm{Aut}(\Gamma)$ is transitive, and let the stabilizer of a point be H). This multiple is isomorphic to $\Gamma \wr \bar{K}_{|H|}$. If one inspects the proofs of Theorems 2 and 4 (but NOT the statements), it turns out that the two Cayley graphs constructed in Theorems 2 and 4 are isomorphic! Sabidussi does not mention this outside of the proofs nor discuss its consequences.

Suppose $S \cap H = \emptyset$. Then S is a union of double cosets of H in G, and

$$\mathrm{Cay}(G, S) \cong \mathrm{Cos}(G, H, S) \wr \bar{K}_h$$

where $|H| = h$.

Suppose $S \cap H = \emptyset$. Then S is a union of double cosets of H in G, and

$$\mathrm{Cay}(G, S) \cong \mathrm{Cos}(G, H, S) \wr \bar{K}_h$$

where $|H| = h$. This means that we don't have to use the quotient we saw before! We can quotient by sending edges to edges and nonedges to nonedges (the strong quotient) as $\mathrm{Cay}(G, S)$ has, between any two left cosets of H in G, either every edge or no edges. Much information about $\mathrm{Cay}(G, S)$ "projects" to $\mathrm{Cos}(G, H, S)$ and much information about $\mathrm{Cos}(G, H, S)$ "lifts" to $\mathrm{Cay}(G, S)$.
We also obtain an intuitively easy to understand definition of a double coset graph of a group G -

Suppose $S \cap H = \emptyset$. Then S is a union of double cosets of H in G, and

$$\mathrm{Cay}(G, S) \cong \mathrm{Cos}(G, H, S) \wr \bar{K}_h$$

where $|H| = h$. This means that we don't have to use the quotient we saw before! We can quotient by sending edges to edges and nonedges to nonedges (the strong quotient) as $\mathrm{Cay}(G, S)$ has, between any two left cosets of H in G, either every edge or no edges. Much information about $\mathrm{Cay}(G, S)$ "projects" to $\mathrm{Cos}(G, H, S)$ and much information about $\mathrm{Cos}(G, H, S)$ "lifts" to $\mathrm{Cay}(G, S)$.

We also obtain an intuitively easy to understand definition of a double coset graph of a group G - it is simply a natural quotient of an unworthy Cayley graph.

Suppose $S \cap H = \emptyset$. Then S is a union of double cosets of H in G, and

$$\mathrm{Cay}(G, S) \cong \mathrm{Cos}(G, H, S) \wr \bar{K}_h$$

where $|H| = h$. This means that we don't have to use the quotient we saw before! We can quotient by sending edges to edges and nonedges to nonedges (the strong quotient) as $\mathrm{Cay}(G, S)$ has, between any two left cosets of H in G, either every edge or no edges. Much information about $\mathrm{Cay}(G, S)$ "projects" to $\mathrm{Cos}(G, H, S)$ and much information about $\mathrm{Cos}(G, H, S)$ "lifts" to $\mathrm{Cay}(G, S)$.

We also obtain an intuitively easy to understand definition of a double coset graph of a group G - it is simply a natural quotient of an unworthy Cayley graph. Here, unworthy is in the Steve Wilson sense of unworthy - there are two vertices that have the same set of neighbors

Suppose $S \cap H = \emptyset$. Then S is a union of double cosets of H in G, and

$$\mathrm{Cay}(G, S) \cong \mathrm{Cos}(G, H, S) \wr \bar{K}_h$$

where $|H| = h$. This means that we don't have to use the quotient we saw before! We can quotient by sending edges to edges and nonedges to nonedges (the strong quotient) as $\mathrm{Cay}(G, S)$ has, between any two left cosets of H in G, either every edge or no edges. Much information about $\mathrm{Cay}(G, S)$ "projects" to $\mathrm{Cos}(G, H, S)$ and much information about $\mathrm{Cos}(G, H, S)$ "lifts" to $\mathrm{Cay}(G, S)$.
We also obtain an intuitively easy to understand definition of a double coset graph of a group G - it is simply a natural quotient of an unworthy Cayley graph. Here, unworthy is in the Steve Wilson sense of unworthy - there are two vertices that have the same set of neighbors (this also gives a graph theoretic test of for which S a double coset graph is well-defined).

The net effect of this is that some problems about Cayley graphs of G and double coset graphs of G are equivalent!

The net effect of this is that some problems about Cayley graphs of G and double coset graphs of G are equivalent! Rachel and I discovered or rediscovered that the strong quotient could be used.

The net effect of this is that some problems about Cayley graphs of G and double coset graphs of G are equivalent! Rachel and I discovered or rediscovered that the strong quotient could be used.

Here are some problems where enough information projects and lifts to show the problem for Cayley graphs is equivalent to the problem for double coset graphs:

The net effect of this is that some problems about Cayley graphs of G and double coset graphs of G are equivalent! Rachel and I discovered or rediscovered that the strong quotient could be used.

Here are some problems where enough information projects and lifts to show the problem for Cayley graphs is equivalent to the problem for double coset graphs:

1. Chromatic number

The net effect of this is that some problems about Cayley graphs of G and double coset graphs of G are equivalent! Rachel and I discovered or rediscovered that the strong quotient could be used.

Here are some problems where enough information projects and lifts to show the problem for Cayley graphs is equivalent to the problem for double coset graphs:

1. Chromatic number
2. Independence number

The net effect of this is that some problems about Cayley graphs of G and double coset graphs of G are equivalent! Rachel and I discovered or rediscovered that the strong quotient could be used.

Here are some problems where enough information projects and lifts to show the problem for Cayley graphs is equivalent to the problem for double coset graphs:

1. Chromatic number
2. Independence number
3. Automorphism groups!

The net effect of this is that some problems about Cayley graphs of G and double coset graphs of G are equivalent! Rachel and I discovered or rediscovered that the strong quotient could be used.

Here are some problems where enough information projects and lifts to show the problem for Cayley graphs is equivalent to the problem for double coset graphs:

1. Chromatic number
2. Independence number
3. Automorphism groups!
4. Isomorphisms for a fixed G

The net effect of this is that some problems about Cayley graphs of G and double coset graphs of G are equivalent! Rachel and I discovered or rediscovered that the strong quotient could be used.

Here are some problems where enough information projects and lifts to show the problem for Cayley graphs is equivalent to the problem for double coset graphs:

1. Chromatic number
2. Independence number
3. Automorphism groups!
4. Isomorphisms for a fixed G
5. Isomorphisms for all coset graphs provided one has also solved the problem of determining when a Cayley graph of G is isomorphic to a Cayley graph of other groups!

The net effect of this is that some problems about Cayley graphs of G and double coset graphs of G are equivalent! Rachel and I discovered or rediscovered that the strong quotient could be used.

Here are some problems where enough information projects and lifts to show the problem for Cayley graphs is equivalent to the problem for double coset graphs:

1. Chromatic number
2. Independence number
3. Automorphism groups!
4. Isomorphisms for a fixed G
5. Isomorphisms for all coset graphs provided one has also solved the problem of determining when a Cayley graph of G is isomorphic to a Cayley graph of other groups!

Moral of the Talk:

The net effect of this is that some problems about Cayley graphs of G and double coset graphs of G are equivalent! Rachel and I discovered or rediscovered that the strong quotient could be used.

Here are some problems where enough information projects and lifts to show the problem for Cayley graphs is equivalent to the problem for double coset graphs:

1. Chromatic number
2. Independence number
3. Automorphism groups!
4. Isomorphisms for a fixed G
5. Isomorphisms for all coset graphs provided one has also solved the problem of determining when a Cayley graph of G is isomorphic to a Cayley graph of other groups!

Moral of the Talk: If you have solved a problem for Cayley graphs of G or double coset digraphs of G,

The net effect of this is that some problems about Cayley graphs of G and double coset graphs of G are equivalent! Rachel and I discovered or rediscovered that the strong quotient could be used.

Here are some problems where enough information projects and lifts to show the problem for Cayley graphs is equivalent to the problem for double coset graphs:

1. Chromatic number
2. Independence number
3. Automorphism groups!
4. Isomorphisms for a fixed G
5. Isomorphisms for all coset graphs provided one has also solved the problem of determining when a Cayley graph of G is isomorphic to a Cayley graph of other groups!

Moral of the Talk: If you have solved a problem for Cayley graphs of G or double coset digraphs of G, check to see if your property can be "projected" or "lifted" to the other class.

The net effect of this is that some problems about Cayley graphs of G and double coset graphs of G are equivalent! Rachel and I discovered or rediscovered that the strong quotient could be used.

Here are some problems where enough information projects and lifts to show the problem for Cayley graphs is equivalent to the problem for double coset graphs:

1. Chromatic number
2. Independence number
3. Automorphism groups!
4. Isomorphisms for a fixed G
5. Isomorphisms for all coset graphs provided one has also solved the problem of determining when a Cayley graph of G is isomorphic to a Cayley graph of other groups!

Moral of the Talk: If you have solved a problem for Cayley graphs of G or double coset digraphs of G, check to see if your property can be "projected" or "lifted" to the other class.

DANGER:

DANGER: The Cayley graphs have much larger order and many more edges than their corresponding double coset graphs!

DANGER: The Cayley graphs have much larger order and many more edges than their corresponding double coset graphs! So in practice, one would not solve a problem by lifting the double coset graph to its corresponding Cayley graph, solve the problem, and project back.