On the Interplay Between Global and Local Symmetries

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Asymmetric graphs

A graph G is called *asymmetric* if it does not have a non-trivial automorphism.

Asymmetric graphs

Theorem (Erdös, Rényi, 1963)

Almost all finite graphs are asymmetric.

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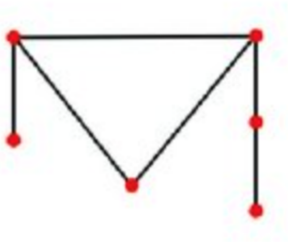

Figure: The smallest asymmetric graph

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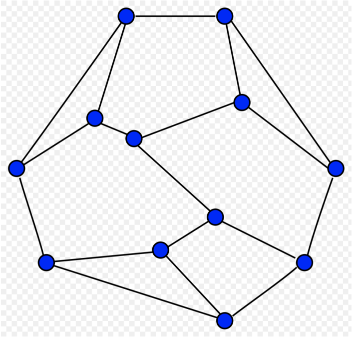

Figure: The Frucht graph, one of the five smallest asymmetric cubic graphs.

Measure of asymmetricity

How many vertices of a graph do we need to remove to get a symmetric graph?

Minimal asymmetric graphs

An undirected graph G on at least two vertices is *minimal asymmetric* if G is asymmetric and no proper induced subgraph of G on at least two vertices is asymmetric.

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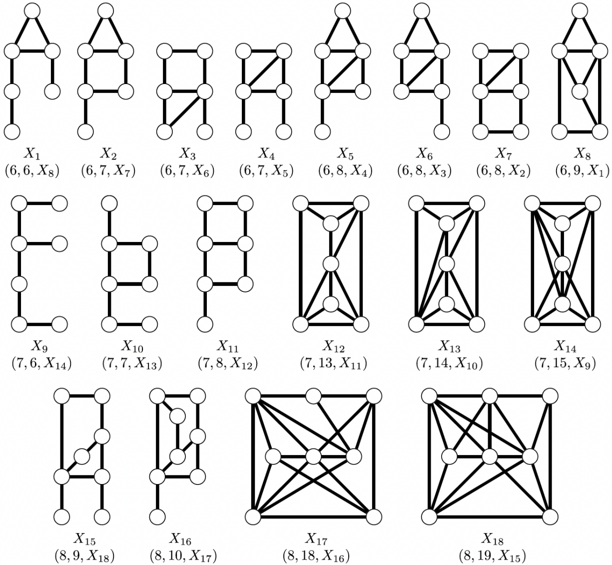
Nešetřil's conjecture: There are exactly 18 minimal asymmetric graphs (coming in 9 complementary pairs).

Nešetřil and Sabidussi earlier established a close connection between minimal asymmetric graphs and minimal involution-free graphs.

Minimal asymmetric graphs



On the other hand...

We like structures with rich automorphism groups.

Automorphism Groups of Graphs

Theorem (Frucht 1939)

For any finite group G there exists a graph Γ such that $Aut(\Gamma) \cong G$.

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Note: We do not specify the type of action required.

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Theorem (Cayley)

Every group G acts regularly on itself via (left) multiplications, i.e., G is isomorphic to the group $G_L = \{\sigma_g \mid g \in G\}$ of (left) translations:

$$\sigma_g(h) = g \cdot h, \quad \text{for all } h \in G$$

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Note:

- Every regular action of G on a set V can be viewed as the action of G_L on G.

Graphical Regular Representations (GRR)

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Theorem (Watkins, Imrich, Godsil, ...)

Let G be a finite group that does not have a GRR, i.e., a finite group that does not admit a regular representation as the full automorphism group of a graph. Then G is an abelian group of exponent greater than 2 or G is a generalized dicyclic group or G is isomorphic to one of the 13 groups : $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_5$, $\mathcal{A}_4$, $\mathcal{Q} \times \mathbb{Z}_3$, $\mathcal{Q} \times \mathbb{Z}_4$,
$\langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ abc = bca = cab \rangle$,
$\langle a, b \mid a^8 = b^2 = 1,\ b^{-1}ab = a^5 \rangle$,
$\langle a, b, c \mid a^3 = b^3 = c^2 = 1,\ ab = ba,\ (ac)^2 = (bc)^2 = 1 \rangle$,
$\langle a, b, c \mid a^3 = b^3 = c^3 = 1,\ ac = ca,\ bc = cb,\ b^{-1}ab = ac \rangle$.

From vertex-transitive to asymmetric

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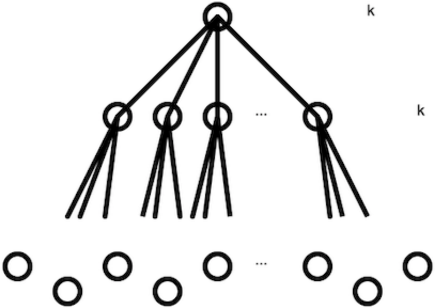

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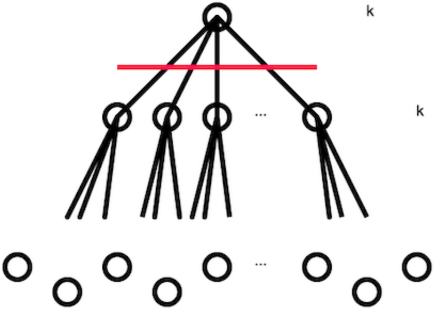

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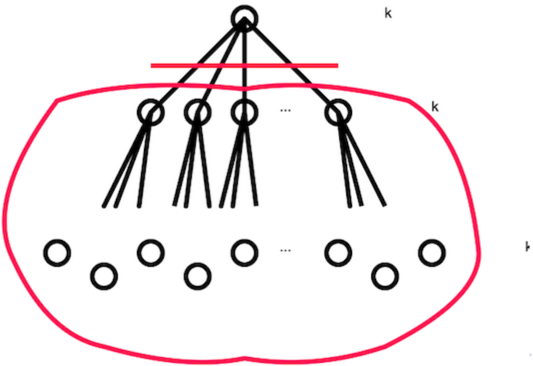

Partial graph automorphisms

Let $\Gamma = (V, \mathcal{E})$ be a finite graph

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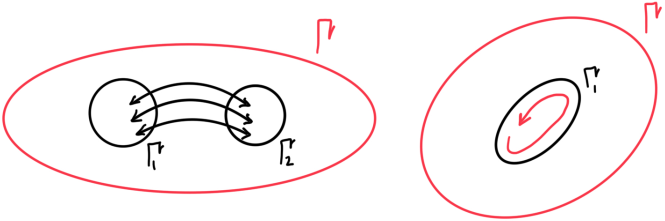

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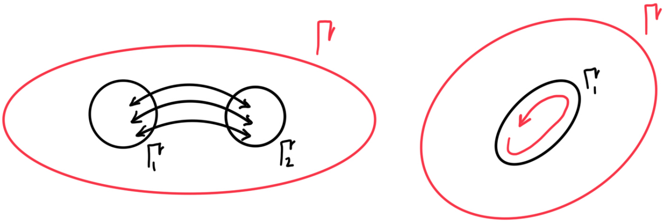

The set of all partial automorphisms, denoted $\mathrm{PAut}(\Gamma)$ with the composition and partial inverse of partial maps forms an inverse monoid.

$\mathrm{PAut}(\Gamma) \leq \mathrm{PSym}(V)$

Inverse Monoid

A set together with an associative binary operation is called a *semigroup*

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A monoid M is called **inverse**

- if for every $a \in M$ there exists a unique element a^{-1} s.t.

$$a \cdot a^{-1} \cdot a = a$$

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"Archetypal" inverse semigroup PSym(X)

PSym(X) - set of all partial permutations of X
$=$ bijections between subsets of X (including $\emptyset$).

$$\varphi : Y \to Z \qquad Y, Z \subseteq X$$

Y - domain $\quad dom\varphi$
Z - range $\quad ran\varphi$

$|dom\varphi| = |ran\varphi|$ - rank of φ

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The cycle notation of classical permutations generalizes by the addition of a notion called a path, which (unlike a cycle) ends when it reaches the "undefined" element.

$$\mathrm{dom}(x_1, x_2 \ldots x_k] = \{x_1, x_2, \ldots, x_{k-1}\}$$

$$\mathrm{ran}(x_1, x_2 \ldots x_k] = \{x_2, x_3, \ldots, x_k\}$$

PSym(X)

- Operation on PSym(X) - is composition of partial maps: Let α and β be partial permutations of a set X; α and β can be composed (from left to right) on the largest domain upon which it "makes sense" to compose them (may be the $\emptyset$)

$$\mathrm{dom}\, \alpha\beta = [\mathrm{im}\, \alpha \cap \mathrm{dom}\, \beta]\alpha^{-1}$$

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- Local Identities: $A \subset X$ - id_A, idempotents
- Zero - PSym(X) has also zero element - empty map - $id_\emptyset$

Wagner-Preston representation

While groups can be represented as symmetries:

Theorem (Cayley)

Every (finite) group can be represented as a group of permutations of a (finite) set.

Inverse semigroups can be represented as partial symmetries:

Theorem (Wagner-Preston)

Every (finite) inverse semigroup can be represented as the inverse semigroup of partial bijections of a (finite) set.

Restrictions vs. extensions

It is clear that all restrictions of an automorphism of a graph are partial automorphisms.

But not all partial automorphisms extend to a (global) automorphism.

Homogeneous graphs

A graph Γ is *homogeneous* if any isomorphism between induced subgraphs extends to an automorphism of Γ.

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Theorem (Gardiner 1976)

A finite homogeneous graph is one of the following:

- *a disjoint union of complete graphs of the same size*
- *a regular complete multipartite graph*
- *the* 5*-cycle*
- *the line graph of* $K_{3,3}$

Hrushovski's extension theorem

Theorem (Hrushovski 1992)

Let X be a finite graph. Then there exists a finite graph Z containing X as an induced subgraph, such that every isomorphism between induced subgraphs of X extends to an automorphism of Z.

Extensions of partial automorphisms

Jaroslav Nešetřil, Matěj Konečný, ...

Extension property for partial automorphisms, EPPA

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Let A be a structure and let B be its (induced) substructure. A is an EPPA-witness for B if every partial automorphism of B extends to an automorphism of A.

A class $\mathcal{C}$ of **finite** structures has EPPA if for every $B \in \mathcal{C}$ there is $A \in \mathcal{C}$, which is an EPPA-witness for B.

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Theorem (Hrushovski 1992)

The class of all finite graphs has EPPA.

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Extension property for partial automorphisms, EPPA

- Class of all n-partite tournaments (orientations of complete n-partite graphs) has EPPA (Eurocomb2019)
- The question is still open for the class of *all* tournaments

Reconstruction

Definition

Let $\Gamma = (V, \mathcal{E})$ be a finite graph and $u \in V(\Gamma)$.
Then $\Gamma - u$ is called a *card*.
The collection $\mathcal{D}$ of the cards of a graph Γ is called the **deck** of Γ:
$\mathcal{D}$ is the multiset of all induced subgraphs $\Gamma - u$, $u \in V$.

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Pseudo-similarity

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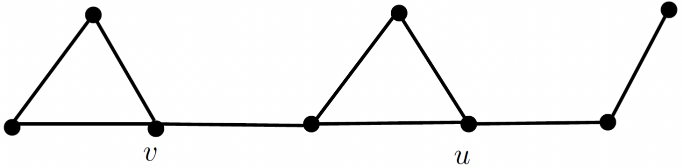

Figure: The Harary-Palmer Graph - the smallest graph containing a pair of pseudo-similar vertices

Reconstruction and symmetry

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Vertex-transitive graphs can have a large reconstruction number.

Theorem (Myrvold, Kurshunov, Müller, Bollobas)

If a graph Γ is asymmetric and all induced subgraphs on $n-3$ vertices are non-isomorphic, then any three cards of Γ reconstruct Γ uniquely.

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If a graph Γ is asymmetric and all induced subgraphs on $n-3$ vertices are non-isomorphic, then any three cards of Γ reconstruct Γ uniquely.

So, in this sense, it is easier to reconstruct asymmetric graphs then symmetric.

Many pseudo-similar vertices

Graphs that have several pseudo-similar vertices are interesting.

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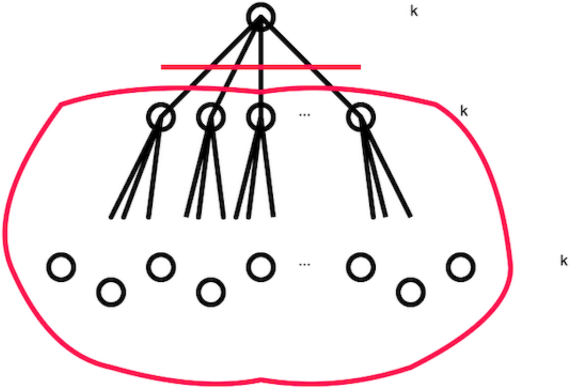
Open question:
What is the maximal number of mutually pseudo-similar vertices in a graph of order n?

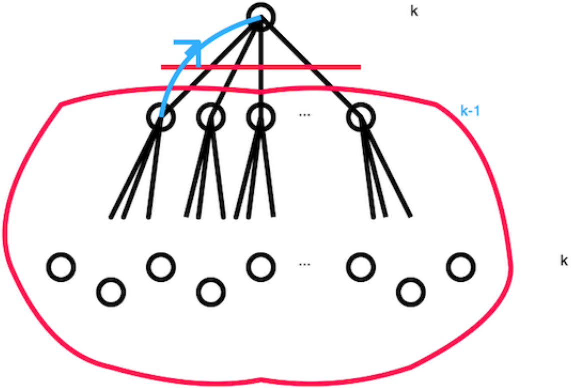
Example with many pairs of pseudo-similar vertices

Let G be a group of *odd* order. Let graph Γ be a GRR of a group G, and let $u \in V(\Gamma)$.



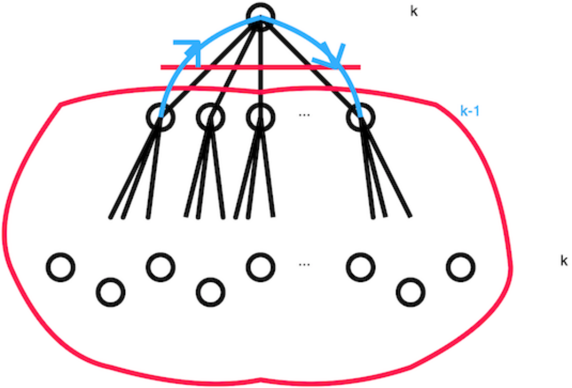
Example with many pairs of pseudo-similar vertices

Let $x_1 \in V(\Gamma - u)$. There is an automorphism α of Γ such that $\alpha(x_1) = u$



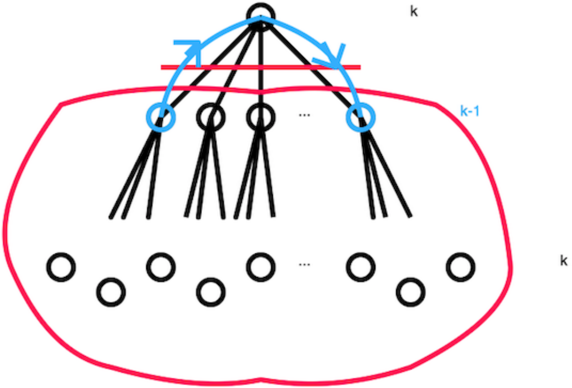
Example with many pairs of pseudo-similar vertices

and $\alpha(u) = x_2$. Then $x_1 \neq x_2$ since G has odd order.



Example with many pairs of pseudo-similar vertices

$\Gamma - x_1 - u \cong \Gamma - u - x_2$, and therefore x_1 has a pseudo-similar mate x_2.



Cages

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A k-regular graph Γ of girth g is called a (k, g)-cage if Γ is of smallest possible order among all k-regular graphs of girth g.

Open problem: Does there exist a $(57, 5)$-graph of order 3250?

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Open problem: Does there exist a $(57,5)$-graph of order 3250?

We do know that if the graph exists, it is not vertex-transitive, but for any two vertices u, v of such graph, there would exist a partial automorphism mapping u to v whose domain would constitute a significant part of the graph.

Main Questions:

0. Understand and describe the structure of $\mathrm{PAut}(\Gamma)$.

1. Classify finite inverse monoids that are *isomorphic* to inverse monoids of partial automorphisms of a graph

Analogue of Frucht's theorem for groups.

2. For a specific class of representations of finite inverse semigroups (e.g., those given by Wagner-Preston theorem) classify finite inverse semigroups that admit a combinatorial structure for which the inverse semigroup of partial automorphisms is *equal to* the partial bijections from the representation.

Analogue of GRR's for groups.

Analogue of Frucht's theorem

Note: The inverse semigroup of partial automorphisms of a *graph* $\Gamma = (V, \mathcal{E})$ with more than one vertex is never trivial:

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$u \longleftrightarrow v$ $\qquad\qquad$ $u \longleftrightarrow v$

Analogue of Frucht's theorem

Note: The inverse semigroup of partial automorphisms of a *graph* $\Gamma = (V, \mathcal{E})$ with more than one vertex is never trivial:

Corollary

Not every finite inverse monoid is the inverse monoid of partial automorphisms of a graph.

Classification of inverse semigroups of partial automorphisms of combinatorial structures

Theorem (Nemirovskaya 1997)

If S is a finite inverse semigroup, then there exists a weighted graph Γ such that $S \cong PAut_{\omega}(\Gamma)$.

Theorem (Sieben 2008)

The inverse semigroup of partial automorphisms of the **Cayley color graph** *of an inverse semigroup is isomorphic to the original inverse semigroup.*

Structure of inverse monoids

$e \in S$ is an idempotent, if $e^2 = e$.

$E(S)$ - set of all idempotents of S.

$\forall s \in S$, $ss^{-1}, s^{-1}s \in E(S)$ (generally different)

- in inverse monoids, idempotents commute
- they form a subsemilattice
- the partial order induced by this semilattice extends naturally to the whole inverse semigroup:

  $$s \leq t \Leftrightarrow \exists \text{ an idempotent } e \text{ such that } s = te$$

  This is called the *natural partial order*

Structure of inverse monoids

In $PSym(X)$,

- idempotents are the partial identical maps,
- the natural partial order is defined by restriction of domains.

$\mathrm{PAut}(\Gamma)$ of a graph Γ is a *full* submonoid (= contains all idempotents) of $\mathrm{PSym}(V)$.

Structure of inverse monoids

Green's relations:

$s, t \in S$. We define $\mathcal{L}$ and $\mathcal{R}$:

- $s \,\mathcal{L}\, t \Leftrightarrow \exists\, x, y \in S$ s.t. $xs = t$ & $yt = s$,
- $s \,\mathcal{R}\, t \Leftrightarrow \exists\, x, y \in S$ s.t. $sx = t$ & $ty = s$.

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In $PSym(X)$,

$$\varphi_1 \, \mathcal{L} \, \varphi_2 \Leftrightarrow \operatorname{dom} \varphi_1 = \operatorname{dom} \varphi_2,$$
$$\varphi_1 \, \mathcal{R} \, \varphi_2 \Leftrightarrow \operatorname{ran} \varphi_1 = \operatorname{ran} \varphi_2.$$

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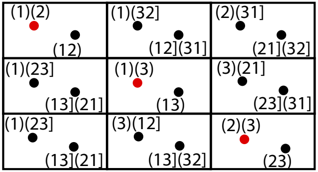

Figure: The $\mathcal{D}$-class of rank 2 partial one-to-one maps of PSym$(\{1, 2, 3\})$

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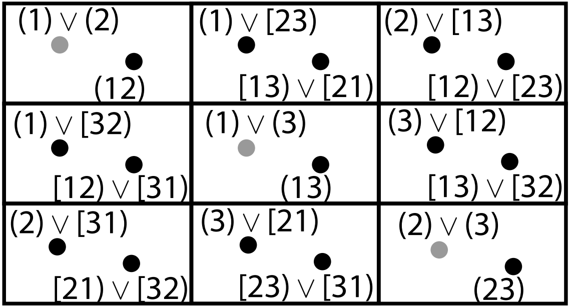

Figure: The $\mathcal{D}$-class of rank 2 partial one-to-one maps of PSym($\{1, 2, 3\}$)

Structure of $\mathrm{PAut}(\Gamma)$ for graph Γ

Proposition (R.Jajcay, T.J., N. Szakács, M. Szendrei 2021)

For any graph Γ*, the* $\mathcal{D}$*-classes of* $\mathrm{PAut}(\Gamma)$ *correspond to the isomorphism classes of induced subgraphs of* Γ*, that is, two elements are* $\mathcal{D}$*-related if and only if the subgraphs induced by their respective domains (or images) are isomorphic.*

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Partial order for $\mathcal{D}$-classes: "subgraph" relation

Example

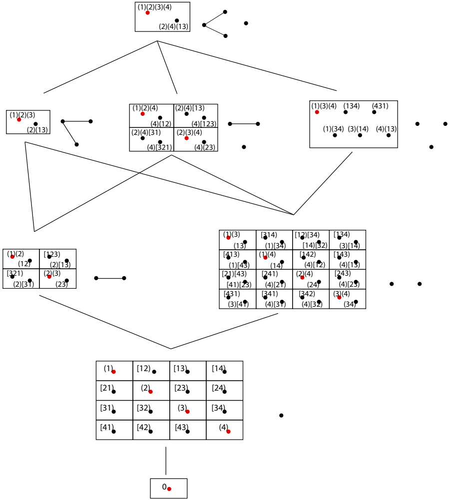
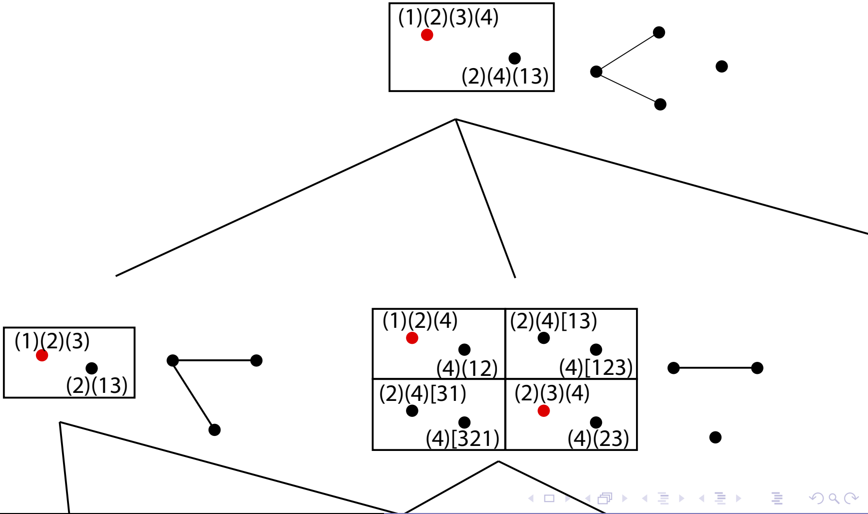


Figure: The Green-class structure of partial graph automorphisms

Example



Structure of $\mathrm{PAut}(\Gamma)$ for graph Γ

Lemma (R.Jajcay, T.J., N. Szakács, M. Szendrei 2021)

Let $\Gamma = (X, E)$ be a graph, and let $\varphi \in \mathrm{PSym}(X)$ be a partial permutation of rank at least 2. Then $\varphi \in \mathrm{PAut}(\Gamma)$ if and only if $\varphi|_Y \in \mathrm{PAut}(\Gamma)$ for any 2-element subset Y of $\operatorname{dom}\varphi$.

Proposition

The partial automorphism monoid $\mathcal{S} = \mathrm{PAut}(\Gamma)$ of any edge-colored digraph Γ has the following property:

(U) *For any compatible subset $A \subseteq \mathcal{S}$ of partial permutations of rank 1, if $\mathcal{S}$ contains the join of any two elements of A, then $\mathcal{S}$ contains the join of the set A.*

Proposition

If $\mathcal{S}, \mathcal{T}$ are full inverse submonoids of $\mathrm{PSym}(X)$ which coincide on their elements of rank at most 2 and satisfy condition (U), then $\mathcal{S} = \mathcal{T}$.

When is an inverse monoid of partial permutations the partial automorphism monoid of a graph?

Theorem (R.Jajcay, T.J., N. Szakács, M. Szendrei 2021)

Given an inverse submonoid $S \leq \mathrm{PSym}(X)$, where X is a finite set, there exists a graph with vertex set X whose partial automorphism monoid is S if and only if the following conditions hold:

1. *S is a full inverse submonoid of $\mathrm{PSym}(X)$,*
2. *for any compatible subset $A \subseteq S$ of rank 1 partial permutations, if S contains the join of any two elements of A, then S contains the join of the set A,*
3. *the rank 2 elements of S form at most two $\mathcal{D}$-classes,*
4. *the $\mathcal{H}$-classes of rank 2 elements are nontrivial.*

When is an (abstract) inverse monoid *isomorphic* to the partial automorphism monoid of a graph?

Theorem (R.Jajcay, T.J., N. Szakács, M. Szendrei 2021)

Given a finite inverse monoid S, there exists a finite graph whose partial automorphism monoid is isomorphic to S if and only if the following conditions hold:

1. *S is Boolean,*
2. *S is fundamental,*
3. *for any subset $A \subseteq S$ of compatible 0-minimal elements, if all 2-element subsets of A have a join in S, then the set A has a join in S,*
4. *S has at most two $\mathcal{D}$-classes of height 2,*
5. *the $\mathcal{H}$-classes of the height 2 $\mathcal{D}$-classes of S are nontrivial.*

Further questions

- Find a class of graphs, that will have an interesting (recognizable) class of inverse monoids as their PAut(Γ)
- Extensions of graphs vs. extensions of monoids (Hrushovski type of questions)
- PAut($\mathcal{C}$) for other combinatorial structures

$\vdots$

Thank you!

Thank you!

Greetings from Bratislava
Happy Holidays!