(Closed) distance magic circulants

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Circulant graphs

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Closed distance magic circulant graphs

Notation, definitions and examples

Graphs and labelings

Graph

- Γ - finite, simple graph (no loops, no multiple edges)
- V - vertex set of Γ
- $n = |V|$
- for $x \in V$ we denote by $\Gamma(x)$ the set of neighbours of x
- for $x \in V$ we let $\Gamma[x] = \{x\} \cup \Gamma(x)$

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Labeling

A labeling of Γ is a map $\ell : V \mapsto \mathbb{R}$.

Graphs and labelings

Weight of a vertex

Let ℓ be a labeling of Γ. For $x \in V$ we define

$$w(x) = w_\ell(x) = \sum_{y \in \Gamma(x)} \ell(y).$$

and

$$\overline{w}(x) = \overline{w}_\ell(x) = \sum_{y \in \Gamma[x]} \ell(y).$$

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We refer to $w(x)$ and $\overline{w}(x)$ as weight and closed weight of vertex x, respectively.

Distance Magic Graphs

Distance Magic Graphs

Graph Γ is said to be distance magic, if there exist a bijective labeling $\ell : V \mapsto \{1, 2, \ldots, n\}$ of Γ and a constant r, such that $w(x) = r$ for every $x \in V$.

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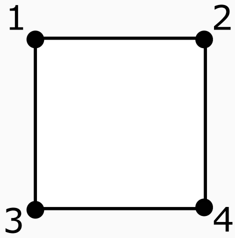
Graph Γ is said to be distance magic, if there exist a bijective labeling $\ell : V \mapsto \{1, 2, \ldots, n\}$ of Γ and a constant r, such that $w(x) = r$ for every $x \in V$.

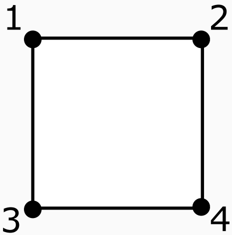
In this case:

- ℓ - distance magic labeling of Γ
- r - magic constant of Γ

Distance Magic Graphs - examples

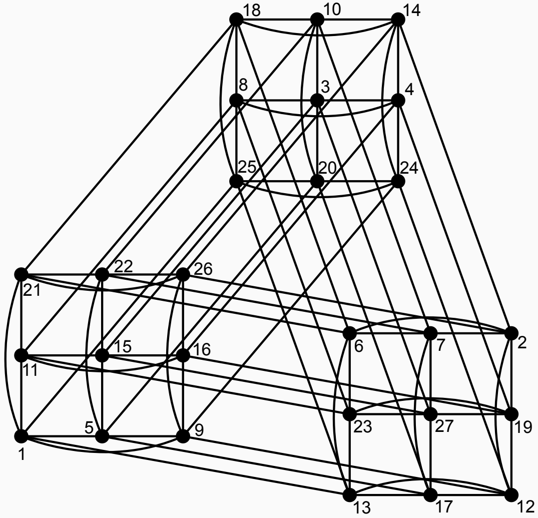


Distance Magic Graphs - examples



More general, hypercubes Q_D with $D \equiv 2 \pmod 4$ are distance-magic.

Distance Magic Graphs - examples



Distance Magic Graphs - nonexamples

- Complete graphs K_n for $n \geq 2$
- Cycles C_n for $n \geq 5$
- Hypercubes Q_D with $D \not\equiv 2 \pmod 4$
- ...

Motivation and related concepts

Distance Magic Graphs - couple of comments

- Application - tournaments

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- Application - tournaments
- Related concepts (closed distance magic graphs, d-distance magic graphs, anti distance magic graphs, group distance magic graphs, ...)

Regular distance magic graphs

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Regular Distance Magic Graphs

Therefore

$$r = \frac{k(n+1)}{2}$$

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In particular, k is even.

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For $a, b \in \mathbb{R}$, $a \neq 0$, and $x \in V$ define

$$\ell'(x) = a\ell(x) + b$$

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Regular Distance Magic Graphs

In particular, if $a = 1$ and $b = -r/k = -(n+1)/2$, then $w'(x) = 0$ for every $x \in V$.

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Theorem

Assume Γ is a regular distance magic graph (with evan valency k, distance magic labeling ℓ and magic constant r). Let $V = \{x_1, x_2, \ldots, x_n\}$. For $x \in V$ we let $\ell'(x) = \ell(x) - (n+1)/2$. Then vector

$$(\ell'(x_1), \ell'(x_2), \ldots, \ell'(x_n))^T$$

is an eigenvector of Γ with eigenvalue 0.

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is an eigenvector of Γ with eigenvalue 0.

In particular, if 0 is not an eigenvalue of Γ, then Γ is not distance magic.

Regular Distance Magic Graphs

Theorem

Assume Γ is a regular graph (with evan valency k). Then Γ is distance magic if and only if 0 is an eigenvalue of Γ and there exists an eigenvector $\mathbf{w}$ for the eigenvalue 0 with the property that a certain permutation of its entries results in the arithmetic sequence

$$\frac{1-n}{2}, \frac{3-n}{2}, \frac{5-n}{2}, \ldots, \frac{n-1}{2}.$$

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Observe: such an eigenvector $\mathbf{w}$ exists if and only if there exists en eigenvector $\mathbf{w}_1$ for the eigenvalue 0 with the property that a certain permutation of its entries results in the arithmetic sequence $1-n, 3-n, 5-n, \ldots, n-1$.

Circulant graphs

Circulant Graphs

Circulant graphs

Let $\mathbb{Z}_n$ denote the cyclic group of order n and let $S \subseteq \mathbb{Z}_n$ be such that $0 \notin S$ and $S = -S$. Let $\mathrm{Circ}(\mathbb{Z}_n; S)$ be a graph with vertex set $\mathbb{Z}_n$, where $x, y \in \mathbb{Z}_n$ are adjacent if and only if $x - y \in S$.

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Observe that $\mathrm{Circ}(\mathbb{Z}_n; S)$ is regular with valency $|S|$ and is connected if and only if S generates $\mathbb{Z}_n$.

Theorem (Cichacz and Froncek, 2016)

Let $S = \{\pm 1, \pm b\} \subseteq \mathbb{Z}_n$, $b \neq n/2$ odd. Then $\mathrm{Circ}(\mathbb{Z}_n; S)$ is distance magic if and only if $b^2 - 1 = n(2t+1)$ for some nonnegative integer t, or $n = 2b + 2$.

Tetravalent Circulant Graphs

Theorem (Cichacz and Froncek, 2016)

Let $S = \{\pm 1, \pm b\} \subseteq \mathbb{Z}_n$, $b \neq n/2$ odd. Then $\mathrm{Circ}(\mathbb{Z}_n; S)$ is distance magic if and only if $b^2 - 1 = n(2t + 1)$ for some nonnegative integer t, or $n = 2b + 2$.

Open problem (Cichacz and Froncek, 2016)

Characterize distance magic circulant graphs $\mathrm{Circ}(\mathbb{Z}_n; S)$, where $S = \{\pm 1, \pm b\} \subseteq \mathbb{Z}_n$ with $b \neq n/2$ even.

Characters of cyclic groups

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Let $\mathbb{Z}_n$ denote the cyclic group of order n. A *character* of $\mathbb{Z}_n$ is a homomorphism from $\mathbb{Z}_n$ to the multiplicative group $\mathbb{C} \setminus \{0\}$.

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Theorem

Let $\mathbf{i}$ denote the imaginary unit of $\mathbb{C}$. The characters of $\mathbb{Z}_n$ are precisely the homomorphisms

$$\chi_j \colon \mathbb{Z}_n \to \mathbb{C} \setminus \{0\} \qquad (0 \leq j \leq n-1),$$

where for each $x \in \mathbb{Z}_n$ we have

$$\chi_j(x) = \left(e^{\frac{2\pi\mathbf{i}}{n}}\right)^{jx} = \cos\left(\frac{2\pi jx}{n}\right) + \mathbf{i}\sin\left(\frac{2\pi jx}{n}\right).$$

Eigenvalues of circulant graphs

Theorem

The spectrum of $\mathrm{Circ}(\mathbb{Z}_n; S)$ is equal to

$$\{\chi_j(S) \mid 0 \leq j \leq n-1\},$$

where

$$\chi_j(S) = \sum_{s \in S} \chi_j(s).$$

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where

$$\chi_j(S) = \sum_{s \in S} \chi_j(s).$$

Moreover,

$$(\chi_j(0), \chi_j(1), \ldots, \chi_j(n-1))^T$$

is the eigenvector corresponding to the eigenvalue $\chi_j(S)$.

Distance magic circulant graphs with valency 4 and 6

Circulant graphs with valency 4

Let $\Gamma = \mathrm{Circ}(\mathbb{Z}_n; \{\pm a, \pm b\})$, where $1 \leq a < b < n/2$ and $\gcd(n, a, b) = 1$, be a connected tetravalent circulant. Pick $0 \leq j \leq n-1$. Then $\chi_j(S) = 0$ if and only if

$$\cos \frac{2\pi ja}{n} + \cos \frac{2\pi jb}{n} = 0$$

Circulant graphs with valency 4

Let $\Gamma = \mathrm{Circ}(\mathbb{Z}_n; \{\pm a, \pm b\})$, where $1 \leq a < b < n/2$ and $\gcd(n, a, b) = 1$, be a connected tetravalent circulant. Pick $0 \leq j \leq n-1$. Then $\chi_j(S) = 0$ if and only if

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if and only if

$$j = \frac{n(2k+1)}{2(b+a)} \in \{0, 1, \ldots, n-1\} \text{ for some } 0 \leq k \leq b+a-1, \text{ or}$$

$$j = \frac{n(2k+1)}{2(b-a)} \in \{0, 1, \ldots, n-1\} \text{ for some } 0 \leq k \leq b-a-1.$$

Circulant graphs with valency 4

Theorem (M. & Šparl, 2021)

Let $\Gamma = \mathrm{Circ}(\mathbb{Z}_n; \{\pm a, \pm b\})$, where $1 \leq a < b < n/2$ and $\gcd(n, a, b) = 1$, be a connected tetravalent circulant. Then Γ is distance magic if and only if n is even, at least one of a and b is coprime to n, and Γ is isomorphic to $\mathrm{Circ}(\mathbb{Z}_n; \{\pm 1, \pm c\})$ for some $1 < c < n/2$ such that the following holds:

- if c is even then $2(c^2 - 1)$ is an odd multiple of n;
- if c is odd then either $c^2 - 1$ is an odd multiple of n or $n = 2c + 2 \equiv 4 \pmod 8$.

Circulant graphs with valency 6

Let $\Gamma = \mathrm{Circ}(\mathbb{Z}_n; \{\pm a, \pm b, \pm c\})$, where $1 \leq a < b < c < n/2$ and $\gcd(n, a, b, c) = 1$, be a connected circulant with valency 6. Pick $0 \leq j \leq n-1$. Then $\chi_j(S) = 0$ if and only if

$$\cos\frac{2\pi ja}{n} + \cos\frac{2\pi jb}{n} + \cos\frac{2\pi jc}{n} = 0. \tag{1}$$

Problem (H. S. M. Coxeter, 1944)

Determine all rational solutions of the equation

$$\cos(r_1\pi) + \cos(r_2\pi) + \cos(r_3\pi) = 0, \quad 0 \le r_1 \le r_2 \le r_3 \le 1.$$

Circulant graphs with valency 6

Solution (W. J. R. Crosby, 1946)

$$0 \le r_1 \le \frac{1}{2}, \quad r_2 = \frac{1}{2}, \quad r_3 = 1 - r_1, \tag{2}$$

$$0 \le r_1 \le \frac{1}{3}, \quad r_2 = \frac{2}{3} - r_1, \quad r_3 = \frac{2}{3} + r_1. \tag{3}$$

$$r_1 = \frac{1}{5},\ r_2 = \frac{3}{5},\ r_3 = \frac{2}{3} \quad \text{and} \quad r_1 = \frac{1}{3},\ r_2 = \frac{2}{5},\ r_3 = \frac{4}{5}. \tag{4}$$

Circulant graphs with valency 6

For a given integer $n \geq 7$ and a subset $S = \{\pm a, \pm b, \pm c\} \subset \mathbb{Z}_n$ of size 6, suppose that for $j \in \{0, 1, 2, \ldots, n-1\}$ we have $\chi_j(S) = 0$. Then we say that j (as well as the corresponding character χ_j) is of *type 1*, *type 2* or *type 3*, respectively, if the corresponding solution of Equation (1) is of type (2), (3) or (4), respectively.

Circulant Graphs with valency 6

With P. Šparl we were able to classify distance magic circulants $\mathrm{Circ}(n; S)$ with $S = \{\pm a, \pm b, \pm c\} \subset \mathbb{Z}_n$, for which all $j \in \{0, 1, 2, \ldots, n-1\}$ with $\chi_j(S) = 0$ are of the same type.

Circulant Graphs with valency 6

Theorem (M. & Šparl, 2021)

Let $n \geq 7$ be an integer and let $S = \{\pm a, \pm b, \pm c\} \subset \mathbb{Z}_n$ be such that $|S| = 6$ and $\langle S \rangle = \mathbb{Z}_n$. Suppose that all $j \in \{0, 1, 2, \ldots, n-1\}$ with $\chi_j(S) = 0$ are of type 2. Then $\Gamma = \mathrm{Circ}(n; S)$ is distance magic if and only if $n = 3n_0$ for some $n_0 \geq 3$, and either $\Gamma \cong C_{n_0}[3K_1]$, or the following both hold:

Circulant Graphs with valency 6

Theorem (M. & Šparl, 2021)

- $n_0 = dd'$ for coprime d and d' with $1 < d < d'$ both of which are coprime to 3;
- letting $\delta \in \{-1, 1\}$ be such that $n_0 \equiv \delta \pmod 3$ and letting $c' \in \{1, 2, \ldots, n-1\}$ be the unique solution of the system of congruences

$$\begin{aligned} c' &\equiv 0 \pmod 3 \\ c' &\equiv 1 \pmod d \\ c' &\equiv -1 \pmod{d'}, \end{aligned} \tag{5}$$

there exists a $q \in \mathbb{Z}_n^*$ such that $qS = \{\pm 1, \pm(n_0 + \delta), \pm c'\}$.

Closed distance magic circulant graphs

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Similarly as in distance magic case we see, that if Γ is a regular (with valency k) closed distance magic graph, then

$$r = \frac{(k+1)(n+1)}{2}.$$

Regular Closed Distance Magic Graphs

Theorem

Assume Γ is a regular graph. Then Γ is closed distance magic if and only if -1 is an eigenvalue of Γ and there exists an eigenvector $\mathbf{w}$ for the eigenvalue 0 with the property that a certain permutation of its entries results in the arithmetic sequence

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Observe: such an eigenvector $\mathbf{w}$ exists if and only if there exists en eigenvector $\mathbf{w}_1$ for the eigenvalue -1 with the property that a certain permutation of its entries results in the arithmetic sequence $1-n, 3-n, 5-n, \ldots, n-1$.

Closed Distance Magic Circulants - some known results

Theorem (Simanjuntak et al.)

For a positive integer k, the circulant graph $\mathrm{Circ}(n; \{1, 2, \ldots, k-1, k+1, \ldots, \lfloor n/2 \rfloor\})$ is closed distance magic if and only if $n = 4k$.

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Theorem (Simanjuntak et al.)

For $n \geq 2k+2$, the circulant graph $\mathrm{Circ}(n; \{1, 2, \ldots, k\})$ is not closed distance magic.

Closed Distance Magic Circulants - some known results

Theorem (Anholzer, Cichacz, Peterin)

For a positive integers k, c, the circulant graph $\mathrm{Circ}(n; \{c, 2c, \ldots, kc\})$ is closed distance magic if and only if either $n = 2kc$, or $n = (2k+1)c$ and c is odd.

Closed Distance Magic Circulants - valency 3 or 4

It is easy to see (but it also follows from the above Theorem by Simanjuntak et al.) that the cycle C_n is closed distance magic if and only if $n = 3$.

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Theorem (Fernández, M., Maleki, Sarobidy)

Let Γ be a connected circulant graph with valency 3 or 4. Then Γ is closed distance magic if and only if Γ is isomorphic to K_4 or K_5.

sketch of the proof - valency 4

- Let $\Gamma = \mathrm{Circ}(n; \{\pm a, \pm b\})$ for $1 \leq a < b < n/2$.

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$$\chi_j(\{\pm a, \pm b\}) = 2\cos\left(\frac{2\pi ja}{n}\right) + 2\cos\left(\frac{2\pi jb}{n}\right) = -1,$$

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$$\chi_j(\{\pm a, \pm b\}) = 2\cos\left(\frac{2\pi ja}{n}\right) + 2\cos\left(\frac{2\pi jb}{n}\right) = -1,$$

- which is equivalent to

$$\cos\left(\frac{2\pi ja}{n}\right) + \cos\left(\frac{2\pi jb}{n}\right) + \cos\frac{\pi}{3} = 0.$$

sketch of the proof - valency 4

Therefore, by the above solution of Crosby, one of the following holds for some integers k_1, k_2:

1. $$\left\{\frac{2\pi ja}{n}, \frac{2\pi jb}{n}\right\} = \left\{\frac{\pi}{2} + k_1\pi, \pm\frac{2\pi}{3} + 2k_2\pi\right\},$$

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3. $$\left\{\frac{2\pi ja}{n}, \frac{2\pi jb}{n}\right\} = \left\{\pm\frac{2\pi}{5} + 2k_1\pi, \pm\frac{4\pi}{5} + 2k_2\pi\right\}.$$

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- As $a < b < n/2$ we have $a = n/5$ and $b = 2n/5$.
- By connectedness, $n = 5$, $a = 1$ and $b = 2$.

Closed Distance Magic Circulants - valency 5

Theorem (Fernández, M., Maleki, Sarobidy)

Let Γ be a connected circulant graph with valency 5. Then Γ is closed distance magic if and only if Γ is isomorphic to $\mathrm{Circ}(\mathbb{Z}_n; \{\pm 1, \pm c, n/2\})$ with n even and $1 < c < n/2$, and one of the following (i)–(iv) holds:

(i) $c = n/2 - 1$;

(ii) $n \equiv 2 \pmod 4$, c even, and $2(c^2 - 1)$ is an odd multiple of n;

(iii) $n = 3 \cdot 2^t(6k + (-1)^t)$ and $c = 2^{t-1}(6k + (-1)^t) - 1$ for some integer $t \geq 2$ and some integer $k \geq 0$ such that $c \geq 2$;

(iv) $n = 3 \cdot 2^t(6k - (-1)^t)$ and $c = 2^{t-1}(6k - (-1)^t) + 1$ for some integer $t \geq 2$ and some integer $k \geq 0$ such that $c \geq 2$.

Thank you!