Equivalence of discrete groups

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Introduction

Here, a discrete group will be a finite group group acting on a compact connected surface. We will be mostly restricted to orientation-preserving case.

First instances of the problem appeared long time ago:

- Isometries of the sphere, *spherical groups*, as abstract groups these are subgroups of the dihedral groups and of the groups of symmetries of Platonic solids,
- Isometries of the Euclidean plane, these are subgroups of regular tesselations of E_2,
- Discrete groups for the projective plane, torus and Klein bottle can be obtained as quotients of the above families.
- In modern mathematics the problem was considered in the frame of investigation of symmetries of Riemann surfaces, algebraic curves,. . .

Classifications of D.G. of small genera

$g = 2, 3$	S. A. Broughton, 1990, J. Pure and Abstract Algebra
$g = 4$	O. V. Bogopolskij, 1996, Voprosy Algebry i Logiky,
$g = 4$	H. Kimura, 2003, J. Algebra,
$g = 5$	H.Kimura and Kuribayshi 1990, J.Algebra.
$g \leq 101$	**"Large groups"** $\lvert G\rvert > 4(g-1)$ M. Conder, 2006

Note: Large groups have restricted signatures $(0; m_1, m_2, m_2)$ or $(0; m_1, m_2, m_3, m_3)$.

Discrete groups

Warning: To classify discrete groups of given genus it is not enough to do it with respect to group isomorphism! But what is the proper equivalence relation? And how to compute the equivalence classes, at least for small genera?

Definition (topological): Denote by S an orientable surface of genus g,

$\mathrm{Hom}^+(S)$ its group of orientation-preserving homeomorphisms.

A finite group G **acts on** S if there is monomorphism $\varepsilon\colon G \to \mathrm{Hom}^+(S)$.

Two actions of G on S defined by embeddings $\varepsilon, \varepsilon'$ are **equivalent**, $\varepsilon' \approx \varepsilon$, iff there exists $\omega \in \mathrm{Aut}(G)$ and $h \in \mathrm{Hom}^+(S)$ such that

$$\varepsilon'(g) = h\epsilon(\omega(g))h^{-1}.$$

Problem

Given genus $g > 1$, classify discrete groups of genus g up to equivalence.

Fuchsian groups and discrete actions

Theorem (Riemann existence theorem)

Every action of G on S can be constructed by means of a pair of **Fuchsian groups** $K \lhd \Gamma < \mathrm{PSL}(2, \mathbb{R})$ *acting discontinuosly on the upperhalf plane U and by an* **order preserving epimorphism** $\eta \colon \Gamma \to G$ *with kernel K, where K is a surface group.*

In this representation of the action (G, S_g) we identify $G = \Gamma/K$ and $S_g = U/K$.

Advantage of such a representation is that we have replaced a topological surface by geometric! In particular, $\varepsilon(G)$ becomes a group of geometric transformations.

Transition to group theory

The group Γ have the presentation

$$\langle x_1, x_2, \ldots, x_r, a_1, b_1, \ldots, a_\gamma, b_\gamma \mid x_1^{m_1} = \cdots = x_r^{m_r} = 1, \prod_{i=1}^{\gamma}[a_i, b_i] \prod_{j=1}^{r} x_j = 1\rangle,$$

where $1 < m_1 \leq m_2 \leq \ldots m_r$ are integers and $\gamma \leq g$ is the genus of the quotient surface $S/\varepsilon(G)$ and m_i are the branch indices of the (regular) branched covering $S \to S/\varepsilon(G) \cong S_\gamma$.

Riemann-Hurwitz equation

All the numbers are related by the Riemann-Hurwitz equation

$$2 - 2g = |G| \left(2 - 2\gamma - \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right); m_i \geq 2, m_i \mid |G|.$$

Corollary (Hurwitz bound)

$|G| \leq 84(g-1)$.

An algorithm generating all discrete groups of given genus

Observation

The signature of Γ: The signature $(\gamma; m_1, m_2, \ldots, m_r)$ determines Γ.

Input: The genus, $g > 1$,
Output: A list of actions given by $\Gamma \to G$.

Step 1: Find all numerical solutions of Riemann-Hurwitz equation with respect to $\langle(\gamma; m_1, m_2, \ldots, m_r), |G|\rangle$.

Step 2: Generate all order-preserving epimorphisms $\Gamma \to G$. The group G is chosen from Small Group Library, if possible. Otherwise, one can use coset enumeration to find normal subgroups of Γ, of index $|G|$.

Remark

The epimorphisms are encoded as vectors of images of generators of Γ:

$$\eta = (\bar{a}_1, \bar{b}_1, \ldots, \bar{a}_\gamma, \bar{b}_\gamma, \bar{x}_1, \ldots, \bar{x}_r).$$

Example

We have an epimorphism $\varepsilon \colon \Gamma(1;2,2) \to (\mathbb{Z}_2,+)$, encoded by a vector $(1,0;1,1)$, i.e. $a \mapsto 1, b \mapsto 0, x_1 \mapsto 1, x_2 \mapsto 1$. Since this is a legitimate order-preserving epimorphism, there is an action of $G \cong \mathbb{Z}_2$ on the surface of genus two with signature $(1;2,2)$.

All epimorphisms are

$$\mathrm{Epi}_0((\Gamma(1;2,2),\mathbb{Z}_2) = \{(0,0;1,1),(0,1;1,1),(1,0;1,1),(1,1;1,1)\}.$$

- How many equivalence classes are there?
- By Broughton's list of actions of genera 2, there is **only one** class.
- Since $\mathrm{Aut}(G)$ is trivial, there must be a nontrivial (transitive) action of $\mathrm{Aut}^+(\Gamma)$ on $\mathrm{Epi}_0(\Gamma,\mathbb{Z}_2)\ldots$

Equivalence of actions in group-theoretical setting

We have seen that given $g > 1$ to generate actions is equivalent to a finite problem of group-theoretical nature.

Observation

If two actions G_1 and G_2 on S are equivalent, then both $G_1 \cong G_2$ and the associated Fuchsian groups $\Gamma_1 \cong \Gamma_2$.

Problem

Given a (finite) set of vectors $\mathrm{Epi}_0(\Gamma, G)$, determine the equivalence classes.

Translation to group theory (Lloyd 1972): Two order preserving epimorphisms $\eta_1, \eta_2 \in \mathrm{Epi}_0(\Gamma, G)$ are equivalent if and only if there exists $a \in \mathrm{Aut}(G)$ and $\alpha \in \mathrm{Aut}^+(\Gamma)$ such that $\eta_2 = a\eta_1\alpha$.

Here is the difficulty: To identify the equivalence classes, we have to understand the action of an infinite group $\mathrm{Aut}^+(\Gamma)$ on $\mathrm{Epi}_0(\Gamma, G)$. In fact, the essential part of the problem is to understand the action of $\mathrm{Out}(\Gamma) = \mathrm{Aut}^+(\Gamma)/\mathrm{Inn}(\Gamma)$, known in literature as the **mapping class group**.

A parallel with topological graph theory

A dictionary Surfaces versus Graphs:

- discrete group $\sim$
  (semiregular) subgroup of the automorphism group,
- (branched) regular covering of surfaces $\sim$
  (branched) regular covering between graphs,
- Fuchsian group $\sim$
  free group of rank $\beta = e - v + 1$
- order preserving epimorphism $\eta\colon\ :\Gamma \to G$ $\sim$
  a T-reduced (ordinary) voltage assignment: $\nu\colon D(Y) \to G$.
- Equivalence of discrete groups: (LLoyd 1972) $\eta_1 \cong \eta_2$ iff there exist $a \in \mathrm{Aut}(G)$ and $\alpha \in \mathrm{Aut}^+(\Gamma)$ such that $\eta_2 = a\eta_1\alpha$ $\sim$
  Equivalence of voltage assignments (Škoviera 1986): $\eta_1 \cong \eta_2$ iff there exist $a \in \mathrm{Aut}(G)$ and renormalisation $\alpha\colon T_1 \to T_2$ such that $\eta_2 = a\eta_1\alpha$.

Note: Renormalisation $\sim$ automorphism of the free group F_β.

Generators of $A(\Gamma) = \mathrm{Aut}^+(\Gamma)$, the planar case

Planar Fuchsian groups:

$$\Gamma = \langle x_1, x_2, \dots x_r \mid x_i^{m_i} = 1, i = 1, \dots, r, x_1 x_2 \dots x_r = 1 \rangle$$

Denote by

$$F_r = \langle \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_r \rangle,$$

free group of rank r. There is a natural epimorphism $F_r \to \Gamma$ taking $\tilde{x}_i \to x_i$, $i = 1, \dots, r$.

Theorem (Zieshang 1966)

Every automorphism $\phi \in A(\Gamma) = \mathrm{Aut}^+(\Gamma)$ lifts to an automorphism of F_r.

If automorphism $\tilde{\phi} \in \tilde{A}(\Gamma) < \mathrm{Aut}(F_r)$, then there exists a permutation $\mu \in \mathrm{Sym}(r)$ satisfying $|x_{\mu(i)}| = m_i = |x_i|$, for $i = 1, \dots, r$, and elements λ_i, λ of F_r such that

1. $\tilde{\phi}(\tilde{x}_i) = \lambda_i \tilde{x}_{\mu(i)} \lambda_i^{-1}$, $i = 1, 2, \dots, r$, *and*
2. $\tilde{\phi}(\tilde{x}_1 \tilde{x}_2 \dots \tilde{x}_r) = \lambda \tilde{x}_1 \tilde{x}_2 \dots \tilde{x}_r \lambda^{-1}$,

Moreover, given $\tilde{\phi} \in \tilde{A}(\Gamma)$ determined by μ, λ, and $\lambda_1, \dots, \lambda_r$, projects to a unique $\phi \in A(\Gamma)$.

Algebraic braid group

Let

$$B_r = \langle \sigma_1, \ldots, \sigma_{r-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2\rangle$$

Let $\nu : B_r \to \mathrm{Sym}(r)$ be the homorphism defined by $\nu(\sigma_i) = (i, i+1)$, in fact ν is an epimorphism.

Kernel $\ker(\nu)$ is called the **pure braid group** $P_r = \langle A_{i,j}\rangle$, where

$$A_{ij} = \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+1}^{-1}{\sigma_i}^2\sigma_{i+1}\ldots\sigma_{j-1},$$

for $1 \leq i < j \leq r$.

Braid group as a group of automorphisms F_r

Further let $\delta\colon B_r \to \mathrm{Aut}(F_r)$ be the embedding determined by $\sigma_i \mapsto c_i$, where $c_i : x_i \mapsto x_i x_{i+1} x_i^{-1}$, $x_{i+1} \mapsto x_i$ and $x_j \mapsto x_j$ for $j \notin \{i, i+1\}$. How to recognise elements of B_r in $\mathrm{Aut}(F_r)$?

Theorem (Birman 1974, Chow 1948)

An automorphism $\beta \in \delta(B_r)$ if and only if there exists a permutation $\mu \in \mathrm{Sym}(r)$ and elements λ_i such that

1. $\beta(\tilde{x}_i) = \lambda_i \tilde{x}_{\mu(i)} \lambda_i^{-1}$, $i = 1, 2, \ldots, r$, *and*
2. $\beta(\tilde{x}_1 \tilde{x}_2 \ldots \tilde{x}_r) = \tilde{x}_1 \tilde{x}_2 \ldots \tilde{x}_r$.

Characterisation of $\mathrm{Aut}^+(\Gamma)$

Conclusion of Zieshang and Birman-Chow theorems:

Every automorphism in $\mathrm{Aut}^+(\Gamma) = \mathrm{Inn}(\Gamma)B_r^*$, where B_r^* denotes the projection of the subgroup of $B_r \leq \mathrm{Aut}(F_r)$, is made of those elements whose projection into $\mathrm{Sym}(r)$ preserves
the partition $\mathcal{P}$ of $\{1, 2, \ldots, r\}$ defined by $i \approx j$ iff $m_i = m_j$.

Theorem

The group of inner automorphisms $\mathrm{Inn}(\Gamma)$ *lifts to a subgroup of the embedding* $\delta(P_r) \leq \delta(B_r) < \mathrm{Aut}(F_r)$ *of the pure braid group. In particular,* $\mathrm{Aut}^+(\Gamma) = B_r^*$.

Structure of $\mathrm{Aut}^+(\Gamma)$

Corollary

There is an epimorphism $\nu^* \colon \mathrm{Aut}^+(\Gamma) \to \mathrm{Sym}(r, \mathcal{P})$ *with kernel* P_r^**. In particular,* $\mathrm{Aut}^+(\Gamma)$ *is an extension of* P_r^* *by* $\mathrm{Sym}(r, \mathcal{P}) \cong \prod_{j=1}^{\ell} \mathrm{Sym}(r_{i_j} - r_{i_{j-1}})$*, where* $0 = r_0 < r_1 < \cdots < r_\ell = r$ *and* $m_{r_j} < m_{r_j+1}$ *for* $j = 1, \ldots, \ell - 1$.

The epimorphism is defined by $\nu^*(\phi) = \nu(\delta^{-1}(\tilde{\phi}))$.

Generators of $P_r^* \lhd \mathrm{Aut}^+(\Gamma)$

The generators of $\mathrm{Aut}^+(\Gamma)$ are the generators $\bar{A}_{s,t} = \mathcal{A}_{s,t}$ of the image P_r^* of pure braid group and the images $\bar{c}_i$ such that $i \approx i+1$ in the equivalence defined by $\mathcal{P}$.

Recall, $c_i : x_i \mapsto x_i x_{i+1} x_i^{-1}$, $x_{i+1} \mapsto x_i$ and $x_j \mapsto x_j$ for $j \notin \{i, i+1\}$.
Explicit definition of $\mathcal{A}_{s,t}$ follows

$$\mathcal{A}_{s,t}(x_i) = \begin{cases} x_i, & t < i \text{ or } i < s, \\ x_i^{x_s}, & t = i, \\ x_i^{x_s x_t}, & s = i, \\ x_i^{[x_s, x_t]}, & s < i < t \end{cases}$$

for $1 \leq s < t \leq r$

Action of $\mathrm{Aut}^+(\Gamma)$ on $\mathrm{Epi}_0(\Gamma, G)$

Recall that the elements of $\mathrm{Epi}_0(\Gamma, G)$ are r-dimensional vectors with entries in G.

The **vertical action** is the action of the image of the pure braid group. It is generated by the images of the "pure braid group" generators $\langle \bar{A}_{s,t} \rangle$, $1 \leq s < t \leq r$ and the action on X is defined by

$$(y_1, \ldots, y_r) \mapsto (\mathcal{A}_{s,t}(y_1), \ldots, \mathcal{A}_{s,t}(y_r)),$$

$1 \leq s < t \leq r$.

The **horisontal action** of $\mathrm{Aut}^+(\Gamma)$ on X is generated by the $\mathcal{P}$-invariant generators $\bar{c}_i$ $(i \approx i+1)$,

$$(y_1, \ldots, y_i, y_{i+1}, \ldots, y_r) \mapsto (y_1, \ldots, y_i y_{i+1} y_i^{-1}, y_i, \ldots, y_r).$$

Special cases

Corollary

- *If G is abelian, the equivalence is determined by the action of $\mathrm{Sym}(r, \mathcal{P})$ on the indices.*
- *If $m_1 = m_2 = \cdots = m_r$, then $\mathcal{P}$ has just one class and the equivalence is determined by the "horisontal action".*
- *If $m_1 < m_2 < \cdots < m_r$, then $\mathcal{P}$ is "$=$" and the equivalence is determined by the vertical action of P_r^*.*
- *If $r = 3$, then $P^* \cong \mathrm{Inn}(\Gamma)$. In particular, the classification of such actions with respect to the equivalence coincides with the classification of oriented hypermaps of given genus up to duality. (see Conder's lists up to genus 301).*

An algorithm computing equivalence classes

STEP 1 Construct a graph with vertex-set $\mathrm{Epi}_0(\Gamma, G)$, where u is adjacent to v iff there is a generator $\alpha \in \mathrm{Aut}^+(\Gamma)$ and $a \in \mathrm{Aut}(G)$ such that $u = a(\alpha(v))$. Based on the previous explanation, it is not difficult to find the neighbours of each vertex.

STEP 2 Find the connectivity components using one of the standard algorithms.

An output for genera $g = 2, 3, 4, 5, 6$ can be find in the Appendix. We expect to obtain a complete classification up to genus 10.

Two obstacles

- the action graph for higher genera may be huge
- with increasing g the amount of admissible groups G increases, in particular if $g - 1$ is a power of two.

Future projects

Problem 1: Derive an algorithm computing the equivalence classes of discrete groups. In particular, if G acts with trivial stabilisers, i.e. Γ is a surface group.

Problem 2: Enumeration of actions. Derive a formula for $|\mathrm{Epi}_0(\Gamma, G)|$, $|\mathrm{Sequiv}(\Gamma, G)|$ and $|\mathrm{Equiv}(\Gamma, G)|$ if G belongs to a particular class of groups. $\mathrm{Sequiv}(\Gamma, G)$ denotes the set of classes of epimorphisms in $\mathrm{Epi}_0(\Gamma, G)$ with respect to equivalence $\eta_2 \sim \eta_1$ iff there exists $a \in \mathrm{Aut}(G)$ s.t. $\eta_2 = a(\eta_1)$.

Notes

- A formula counting $|\mathrm{Epi}_0(\Gamma, \mathbb{Z}_n)|$ was derived by Mednykh and N. (2006).
- It is a multivariable arithmetic function and we have derived its aditive form.
- A multiplicative form was derived by Liskovets (2009).
- It has applications in map enumeration, and consequently, in 2-dimensional gravity models in theoretical physics.

Thank you!