On Cayley graphs of generalized dihedral groups

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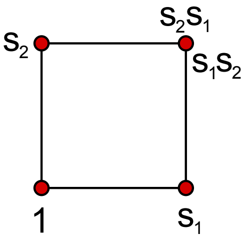
- Some (difficult) problems for Cayley graphs are “easy” when restricted to abelian groups. Why?
- Nice structure of $\Gamma = \mathrm{Cay}(A; S)$ for A abelian:
  - There are plenty of 4-cycles in such graphs (for $s_1, s_2 \in S$, $s_2 \neq s_1^{\pm 1}$ there is the 4-cycle $(1, s_1, s_1 s_2, s_2)$).



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- If there is $s \in S$ with $\langle S \setminus \{s, s^{-1}\} \rangle \neq A$ then Γ contains a spanning subgraph isomorphic to $\mathrm{Cay}(\langle S_0 \rangle; S_0) \Box P_m$, where $S_0 = S \setminus \{s, s^{-1}\}$ and $m = [A : \langle S_0 \rangle]$.

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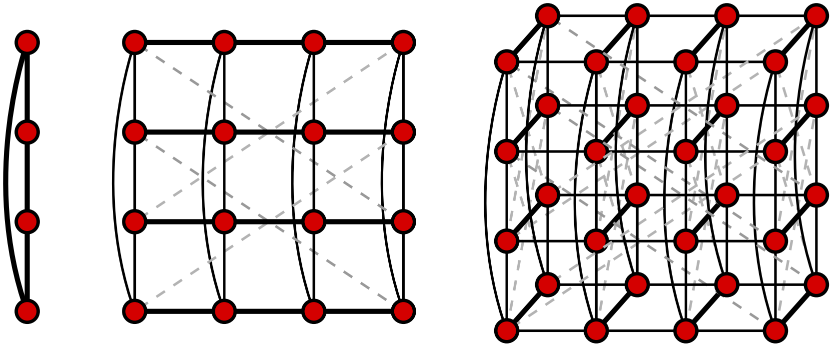
- If there is $s \in S$ with $|s| \neq |A|$ then “such” a spanning subgraph can be constructed inductively.
- ...
- These (and similar) observations are the most important ingredients of the proof of the remarkable Chen-Quimpo theorem.

An example:

$\mathrm{Cay}(\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2; \{\pm(2,1,0), \pm(1,0,1), \pm(2,1,1)\})$



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- If exists $a \in S \cap A$, then $\mathrm{Cay}(G; S)$ again has 4-cycles ($(1, tb, tba, a^{-1})$ for any $tb \in S$ with $b \in A$) and one can also consider the subgraph $\mathrm{Cay}(A; S \cap A)$ (unless of course $S \cap A = \emptyset$).

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- The corresponding results on Cayley graphs of abelian groups can (sometimes) be of help.

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- For cubic:

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#VT	3	4	3	4	5	7	3	11	5	6	10	10	5	12	5	12	10	7
#Cay	2	4	3	4	5	5	3	11	4	5	7	10	4	12	5	10	10	7
#GD	2	3	3	4	4	4	3	7	4	5	6	8	4	8	5	9	8	7
n	46	48	50	52	54	56	58	60	62	64	66	68	70	72	74	76	78	80
#VT	5	32	9	10	13	16	7	38	7	26	11	12	11	37	9	11	14	33
#Cay	5	32	8	9	13	13	6	30	7	26	11	11	11	36	8	11	14	29
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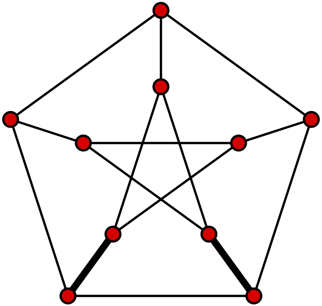
- Well-known graphs such as the Heawood graph and the Pappus graph.

Extendability of matchings

Extendability of matchings

- (Plummer, 1980): A graph Γ of even order at least $2\ell + 2$ is ℓ-extendable if it contains a matching of size ℓ and if every such matching is contained in a perfect matching of Γ.



Extendability of matchings

Theorem (Chan, Chen, Yu, 1995)

Let Γ be a connected Cayley graph of an abelian group of even order at least 6 and valency at least three. Then Γ is **not** *2-extendable if and only if it is isomorphic to one of* $\mathrm{Cay}(\mathbb{Z}_{2n}; \{\pm 1, \pm 2\})$, $(n \geq 3)$, $\mathrm{Cay}(\mathbb{Z}_{4n}; \{\pm 1, 2n\})$, $(n \geq 2)$, $\mathrm{Cay}(\mathbb{Z}_{4n+2}; \{\pm 2, 2n+1\})$ *and* $\mathrm{Cay}(\mathbb{Z}_{4n+2}; \{\pm 1, \pm 2n\})$.

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Theorem (Miklavič, Š, 2009)

Precisely the same for the Cayley graphs of generalized dihedral groups.

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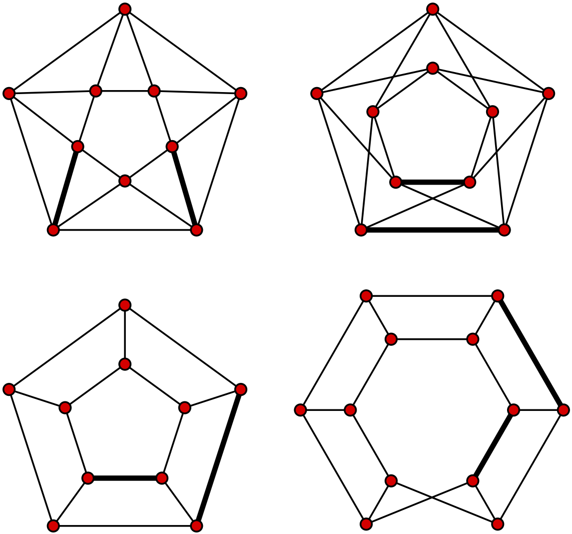
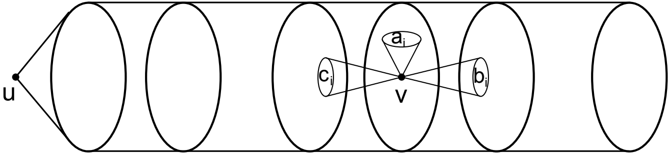


Distance regular graphs

Distance regular graphs

- A regular graph Γ is distance regular if for any integer i with $1 \leq i \leq \mathrm{diam}(\Gamma)$ and any pair of vertices u, v at distance i the numbers
  - $c_i = |\Gamma_{i-1}(u) \cap \Gamma(v)|$,
  - $a_i = |\Gamma_i(u) \cap \Gamma(v)|$ and
  - $b_i = |\Gamma_{i+1}(u) \cap \Gamma(v)|$

  do not depend on the particular pair $\{u, v\}$ but only on the distance i between these two vertices.



Distance regular graphs

- Hamming graphs (and thus hypercubes), odd graphs, Kneser graphs $K(n, 2)$, Grassmann graphs, $K_{n,n}$, $K_{n,n,n}$, the Heawood graph, the Pappus graph, the Desargues graph, Tutte's 12-cage, etc.

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- Classifying all distance regular Cayley graphs much too difficult.
- For some restricted types:
  - cyclic (Miklavič, Potočnik, 2003)
  - dihedral* (Miklavič, Potočnik, 2007)
  - very small valency* (van Dam, Jazeari, 2019)
  - SRG on $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$ (Leifman, Muzychuk, 2005)
  - ...

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- There exists $s \in S$ such that $\mathrm{Cay}(\langle S_0 \rangle; S_0)$ is not connected, where $S_0 = S \setminus \{s^{\pm 1}\}$.
- Well, at least for the abelian and generalized dihedral groups...

Distance regular graphs

- For the abelian groups A with respect to a minimal connection set S:

Theorem (Miklavič, Š, 2014)

The Cayley graph $\mathrm{Cay}(A; S)$ is distance-regular if and only if it is isomorphic to one of the following graphs:

(i) *The complete bipartite graph $K_{3,3}$.*

(ii) *The complete tripartite graph $K_{2,2,2}$.*

(iii) *The complete bipartite graph minus a 1-factor $K_{6,6} - 6K_2$.*

(iv) *The cycle C_n for $n \geq 3$.*

(v) *The Hamming graph $H(d, n)$, where $d \geq 1$ and $n \in \{2, 3, 4\}$.*

(vi) *The Doob(s) graph $D(n, m)$ where $n, m \geq 1$.*

(vii) *The antipodal quotient of the Hamming graph $H(d, 2)$, where $d \geq 2$.*

Distance regular graphs

- For the generalized dihedral groups G with respect to a minimal connection set S:

Theorem (Miklavič, Š, 2020)

The Cayley graph $\mathrm{Cay}(G; S)$ is distance-regular if and only if it is isomorphic to one of the following graphs:

(i) *The complete bipartite graph minus a 1-factor $K_{6,6} - 6K_2$.*

(ii) *The Pappus graph.*

(iii) *The cycle C_n for $n \geq 4$ even.*

(v) *The Hamming graph $H(d, n)$, where $d \geq 2$ and $n \in \{2, 4\}$.*

(vi) *The Doob(s) graph $D(n, m)$ where $n, m \geq 1$.*

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Distance regular graphs

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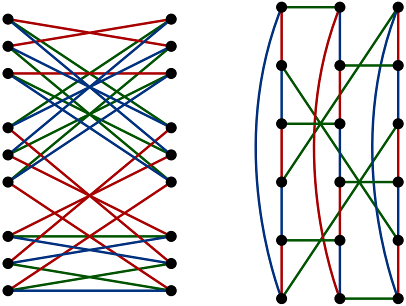
- The Pappus graph is a (minimal) Cayley graph of the generalized dihedral group over $\mathbb{Z}_3 \times \mathbb{Z}_3$.
- This makes it $\mathrm{HTG}(3, 6, 3)$.



Efficient domination

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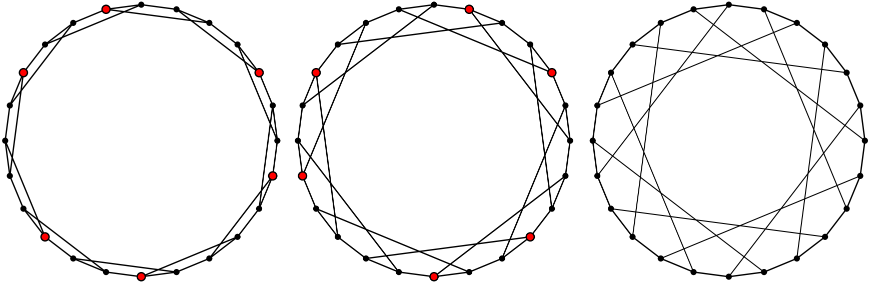
- Also known as perfect 1-codes (Biggs, 1973).

Efficient domination



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- Done for order 2^k (Knor, Potočnik, 2012).
- Easy for cubic Cayley graphs of abelian groups, more difficult for the quartic ones (Caliskan, Miklavič, Özkan, 2019).
- Not super easy for cubic Cayley graphs of generalized dihedral groups (Caliskan, Miklavič, Özkan, Š, 2020).
- We had to investigate the *Honeycomb toroidal graphs*.

Efficient domination

Theorem (Miklavič, Š, 2020)

Let $n \geq 3$ be an integer, $G = \langle t, a \mid t^2, a^n, tat = a^{-1} \rangle = D_n$ and let $b = a^k$ for some $1 < k \leq n-1$. Write $n = 2^r \ell$, for ℓ odd. Then $\mathrm{Cay}(G; \{t, ta, tb\})$ admits an e.d.s. if and only if $r \geq 2$ and $2k - 1 \not\equiv \pm 1 \pmod{2^{r+1}}$.

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Theorem (Miklavič, Š, 2020)

Let $G = \langle t, a, b \rangle$ with $A = \langle a, b \rangle$ abelian, $t \notin A$ and $tat = a^{-1}$, $tbt = b^{-1}$. If none of a, b and ba^{-1} generates A then $\mathrm{Cay}(G; \{t, ta, tb\})$ admits an e.d.s. if and only if at least one of a, b and ba^{-1} generates a subgroup of A which is of even order and of even index.

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  - m, n positive integers, where $n \geq 4$ is even.
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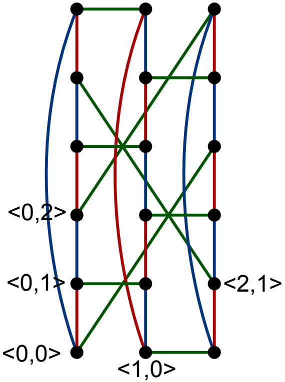
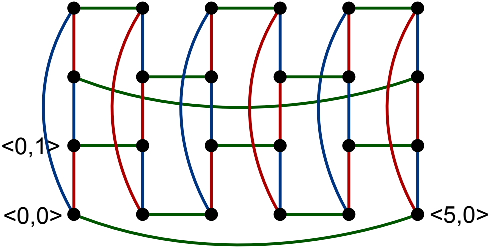
- The *Honeycomb toroidal graphs:*
  - m, n positive integers, where $n \geq 4$ is even.
  - ℓ an integer with $0 \leq \ell \leq n-1$ of the same parity as m.
  - The honeycomb toroidal graph $\mathrm{HTG}(m, n, \ell)$ has vertex-set $\{\langle i, j \rangle : i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$ and the following adjacencies:
    - $\langle i, j \rangle \sim \langle i, j \pm 1 \rangle$ for all $i \in \mathbb{Z}_m, j \in \mathbb{Z}_n$;
    - $\langle i, j \rangle \sim \langle i+1, j \rangle$ for all $i \in \mathbb{Z}_m \setminus \{m-1\}, j \in \mathbb{Z}_n$ with i and j of different parity;
    - $\langle m-1, j \rangle \sim \langle 0, j+\ell \rangle$ for all $j \in \mathbb{Z}_m$ of the same parity as m.

The Honeycomb toroidal graphs





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- He asked several questions, among which two are about symmetries of $\mathrm{HTG}(m, n, \ell)$:
  - When is $\mathrm{HTG}(m, n, \ell)$ a GRR?
  - What is the automorphism group of $\mathrm{HTG}(m, n, \ell)$?

At last... symmetries

Theorem (Š, 202?)

Let m, n, ℓ be integers with $n \geq 4$ even, $0 \leq \ell \leq n/2$ of the same parity as m, and $\Gamma = \mathrm{HTG}(m, n, \ell)$. If G is "the" regular generalized dihedral subgroup of $\mathrm{Aut}(\Gamma)$, then Γ is not a normal Cayley graph of G if and only if one of the following holds:

- $\Gamma = \mathrm{HTG}(1, 6, 3)$ *($K_{3,3}$) is* 3*-AR;*
- $\Gamma \in \{\mathrm{HTG}(2, 4, 0), \mathrm{HTG}(2, 4, 2), \mathrm{HTG}(1, 8, 3)\}$ *($\mathrm{GP}(4, 1)$) is* 2*-AR;*
- $\Gamma = \mathrm{HTG}(1, 14, 5)$ *(Heawood) is* 4*-AR;*
- $\Gamma \in \{\mathrm{HTG}(1, 16, 5), \mathrm{HTG}(2, 8, 4)\}$ *($\mathrm{GP}(8, 3)$) is* 2*-AR;*
- $\Gamma = \mathrm{HTG}(3, 6, 3)$ *(Pappus) is* 3*-AR;*
- $mn = 4n'$ *for some integer* $n' > 2$*, either* $n = 4$ *or* $\Gamma \in \{\mathrm{HTG}(1, 4n', 2n' - 1), \mathrm{HTG}(2, 2n', 2)\}$*, and* $\Gamma \cong \mathrm{GPr}(n')$*, is not AT and has vertex-stabilizers of order* $2^{n'-1}$*.*

At last... symmetries

Theorem (Š, 202?)

For all other $\mathrm{HTG}(m, n, \ell)$ $\mathrm{Aut}(\Gamma)$ *can be determined via:*

(c1) $\gcd(n, \ell + m) = 2m$ *and* $2mn \mid (\ell^2 + 2m\ell - 3m^2)$,

(c2) $\gcd(n, \ell - m) = 2m$ *and* $2mn \mid (\ell^2 - 2m\ell - 3m^2)$,

(c3) $\ell \in \{0, n/2\}$,

(c4) $\gcd(n, \ell + m) = 2m = \gcd(n, \ell - m)$ *and* $2mn \mid (\ell^2 + 3m^2)$.

- *Γ is 2-AR iff any two (and thus all) of (c1), (c2), (c3) and (c4) hold, which occurs iff Γ is one of $\mathrm{HTG}(m, 2m, m)$ with $m \geq 4$, and $\mathrm{HTG}(m, 6m, 3m)$ with $m \geq 2$;*
- *Γ is 1-AR iff (c4), but none of (c1), (c2) and (c3) holds;*
- *Γ is not AT with vertex-stabilizers of order 2 iff precisely one of (c1), (c2) and (c3) holds;*
- *Γ is a GRR iff none of (c1), (c2), (c3) and (c4) holds.*