Orbits of fully regular maps on $SL(2, q)$ under the group of operators $\langle \mathbf{D}, \mathbf{P}, \mathbf{H}_j \rangle$

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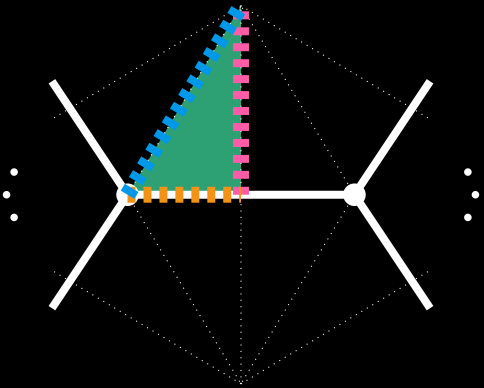
Motivation for looking at the orbit structure

Building “small” examples of super-symmetric (kaleidoscopic and self-dual and self-Petrie) maps of odd valency.

Given a particular orbit, construct a parallel product of maps, consisting of one of each (isomorphism class of) map within the orbit.

Fully regular (reflexible) maps



$$G \cong \langle x, y, z \mid x^2, y^2, z^2, (xy)^2, (yz)^k, (zx)^l, \dots \rangle$$

where k is the vertex degree, and l is the face length. Type (k, l).
In a fully regular (reflexible) map the automorphism group G acts regularly on flags.

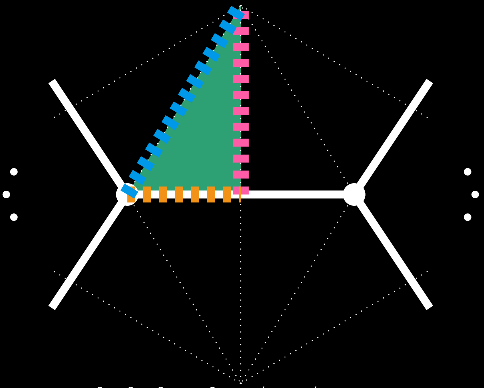
The set:

(*Isomorphism classes of*) Reflexible maps $\mathcal{M}$ with automorphism group $SL(2, q)$, for given $q = 2^\alpha$ and $\alpha \geq 2$.

Thanks to Conder, Potočnik and Širáň (2008) we know a lot about such reflexible maps:
There are known generating triples of matrices for x, y, z in terms of ξ_k and ξ_l, respectively primitive kth and lth roots of unity in the finite field $GF(q^2)$.
Up to conjugacy each such map $\mathcal{M}$ is determined by its *trace triple* $(\omega_k, \omega_l, \omega_m)$ where each $\omega_i := \xi_i + \xi_i^{-1} \in GF(q)$.

A nice coincidence: since we are working in a field of characteristic two we have $\omega_k + \omega_l = \omega_m$.

Fully regular (reflexible) maps, for the purposes of this talk:



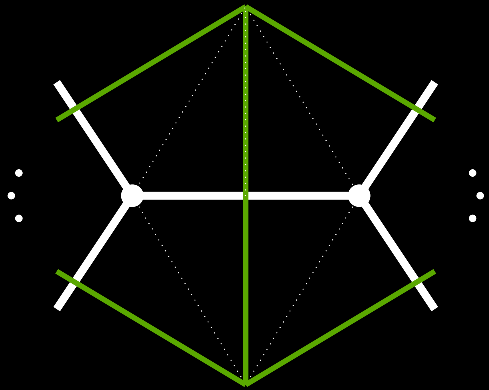
$G \cong \langle x, y, z \mid x^2, y^2, z^2, (xy)^2, (yz)^k, (zx)^l, (xyz)^m \ \dots \rangle \cong SL(2, 2^\alpha)$

where k is the vertex degree, and l is the face length, and m is the Petrie length.

k, l and m are odd.

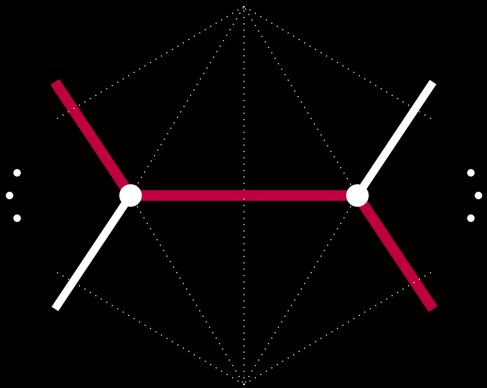
$$\mathcal{M} \sim (\omega_k, \omega_l, \omega_m)$$

Dual operator acting on $\mathcal{M} \sim (\omega_k, \omega_l, \omega_m)$



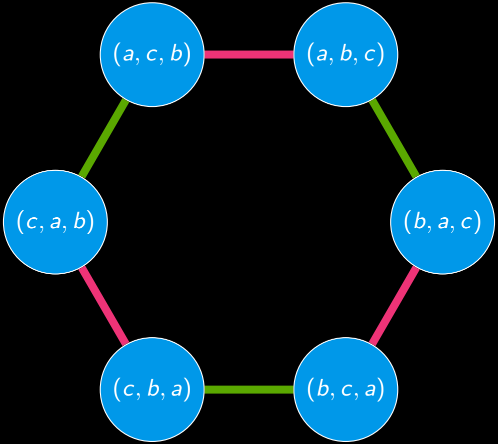
$\mathbf{D}$ is the dual operator, and $\mathcal{M}\mathbf{D}$ has trace triple $(\omega_l, \omega_k, \omega_m)$.

Petrie operator acting on $\mathcal{M} \sim (\omega_k, \omega_l, \omega_m)$



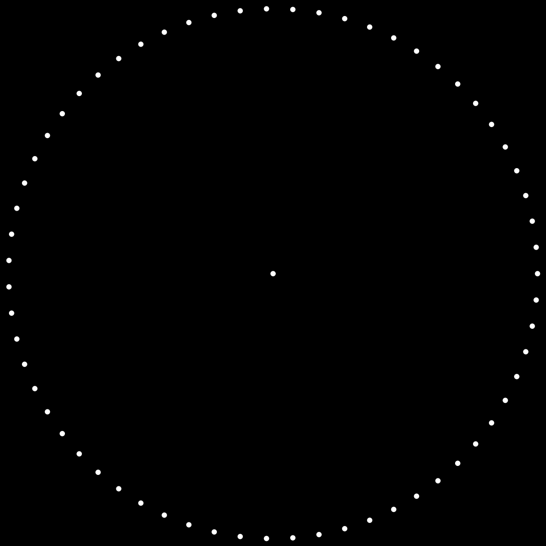
$\mathbf{P}$ is the Petrie operator, and $\mathcal{M}\mathbf{P}$ has trace triple $(\omega_k, \omega_m, \omega_l)$.

The orbit of trace triples under the action of the group $\langle \mathbf{D}, \mathbf{P} \rangle$ on $\mathcal{M} \sim (a, b, c)$

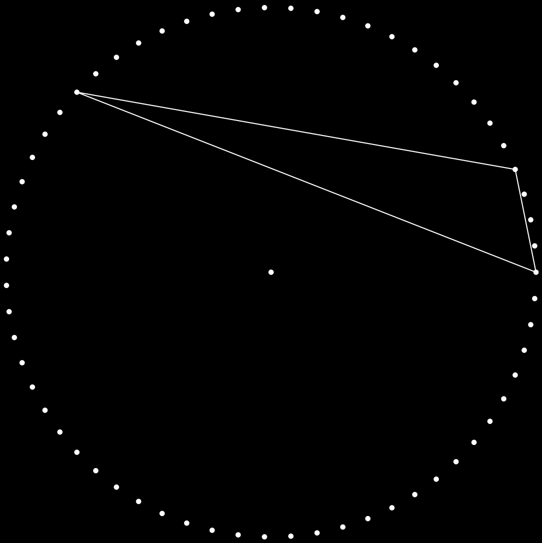


Each map is described by a(n ordered) triple of distinct non-zero field elements (which sum to zero).

A picture of the finite field $GF(64)$, generated by β

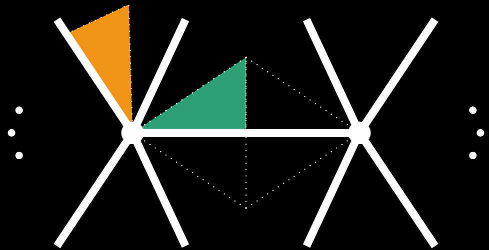


A visual representation of an orbit under $\langle \mathbf{D}, \mathbf{P} \rangle$



Each orbit is shown as a triangle with its corners at distinct non-zero field elements.

Hole operators $\mathbf{H}_j$



$\mathbf{H}_j : x \to x$, $\mathbf{H}_j : y \to y$, and $\mathbf{H}_j : z \to (zy)^j y$

In particular, $\mathbf{H}_j : R \to R^j$ where R is the rotation around the vertex.

For our purposes, we need the operators to form a group. The hole operator $\mathbf{H}_j$ acting on any particular map $\mathcal{M}$ has an inverse operator only when the valency of $\mathcal{M}$ and j are coprime, so here we require $(q^2 - 1, j) = 1$.

What do all these operators do to the trace triple of $\mathcal{M} \sim (\omega_k, \omega_l, \omega_m)$?

Dual operator: $\mathcal{M}\mathbf{D} \sim (\omega_l, \omega_k, \omega_m)$.

Petrie operator: $\mathcal{M}\mathbf{P} \sim (\omega_k, \omega_l, \omega_m)$

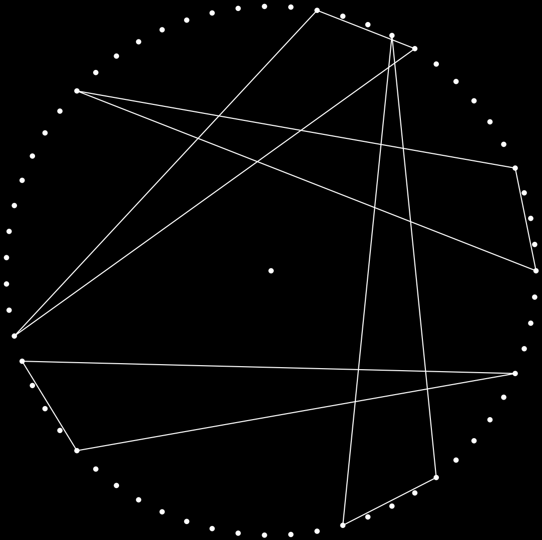
Hole operators: $\mathbf{H}_j : R \to R^j$, where $R = \begin{pmatrix} \xi_\kappa & 0 \\ 0 & \xi_\kappa^{-1} \end{pmatrix}$ and so the first element in the trace triple for the image map will be $\xi_k^j + \xi_k^{-j}$.

Lemma

Let $\mathcal{M} \sim (\omega_k, \omega_l, \omega_m)$ be a given fully regular map where $G \cong SL(2, q)$ and $q = 2^\alpha$ for $\alpha \geq 2$. Let j be coprime to k. Then $\mathcal{M}\mathbf{H}_j \sim (\xi_k^j + \xi_k^{-j}, \frac{\omega_l}{\omega_k}(\xi_k^j + \xi_k^{-j}), \frac{\omega_m}{\omega_k}(\xi_k^j + \xi_k^{-j}))$.

In plain language: the ratios within the trace triple are preserved!!!

An easy corollary - back to the picture:



The maximum size of any orbit is $6(q-1)$.

Do these biggest orbits always occur?

Theorem

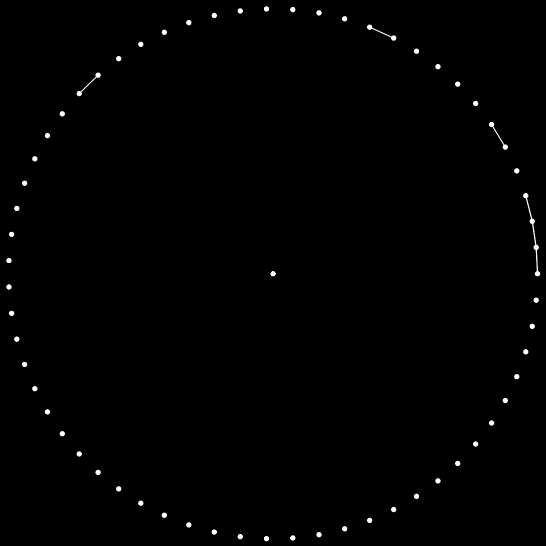
Let $G \cong SL(2,q)$ where $q = 2^\alpha$ and $\alpha \geq 3$. Under the action of $\langle \mathbf{D}, \mathbf{P}, \mathbf{H}_j \,|\, (j, q^2-1) = 1) \rangle$, there is an orbit consisting of $6(q-1)$ distinct isomorphism classes of fully regular map if and only if $\alpha \geq 5$.

Proof - sketchy outline.

Thanks to the Fresher's dream: if $\mathcal{M} \sim (\omega_k, \omega_l, \omega_m)$, then $\mathcal{M}\mathbf{H}_2$ has trace triple $(\omega_k^2, \omega_l\omega_k, \omega_m\omega_k)$.

Let $\mathcal{M} \sim (1, \beta, 1+\beta)$ where β is a primitive element of the field $GF(q)$. Then there is an algorithm which allows us to get from $\mathcal{M}$ to a map with trace triple $(f, f\beta, f(1+\beta))$, for any $f = \beta^a$, using operators from $\langle \mathbf{D}, \mathbf{H}_2 \rangle$. Including the action of $\mathbf{P}$, this yields $6(q-1)$ distinct trace triples.

The picture again:



$(\beta^{24}, \beta^{23}, ...)$ and $(1, \beta, ...)$ are in the same orbit.

Is this really $6(q-1)$ mutually non-isomorphic maps?

We must check the absence of map isomorphisms within the set of $6(q-1)$ trace triples.
First look at each set of $q-1$ maps which all have the same ratio.

Lemma

Let $\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$ and $\mathcal{M}'(\omega'_k, \omega'_\ell, \omega'_m)$ be two maps each with automorphism group $G = SL(2, 2^\alpha)$, distinct trace triples, and yet the same triple trace ratio $1 : \delta : 1 + \delta$.
The maps $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic to each other if and only if δ is in a proper subfield $GF(2^c) < GF(2^\alpha)$ and $\omega_k^{2^c} = \omega'_k$.

So we have six sets of $((q-1)$ mutually non-isomorphic maps).

Is this really $6(q-1)$ mutually non-isomorphic maps?

What about possible isomorphisms between these sets?

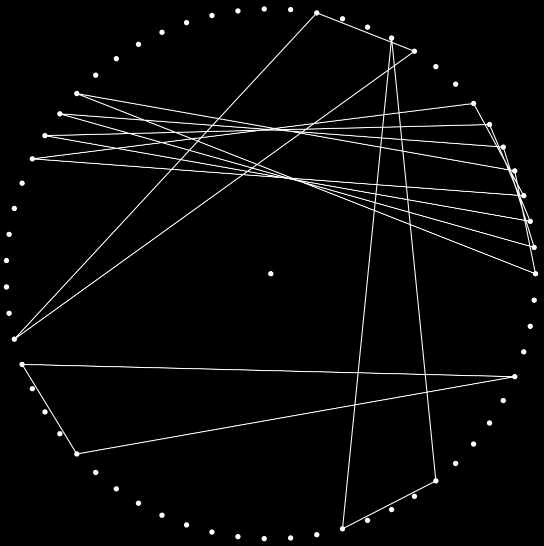
Lemma

Let $\mathcal{M}(\omega_k, \omega_\ell, \omega_m)$ and $\mathcal{M}'(\omega'_k, \omega'_\ell, \omega'_m)$ be two maps each with automorphism group $G = SL(2, 2^\alpha)$ and yet distinct triple trace ratios $1 : \gamma : 1+\gamma$ and $1 : \gamma' : 1+\gamma'$ respectively.
Then $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic maps if and only if there exists c such that $1 \leq c < \alpha$ and $\gamma^{2^c} = \gamma'$ and $\omega_k^{2^c} = \omega'_k$.

Yes, we do this by showing that there are more primitive elements in the field than there are solutions to the various ways in which field automorphisms might break things: A pair of maps from the six distinct sets are isomorphic only if $\beta^{2^c} \in \{\beta^{-1}, 1+\beta, \beta(1+\beta)^{-1}, 1+\beta^{-1}, (1+\beta)^{-1}\}$ for some c such that $1 \leq c < \alpha$.

Count them up, and it all works out (for big enough q). □

Back to the picture:



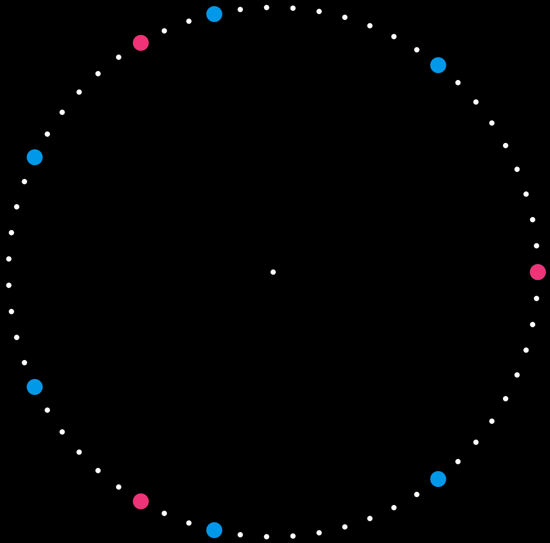
The full orbit has $6(q-1)$ mutually non-isomorphic maps.

Why focus on $6(q-1)$ - what about smaller possible orbits?

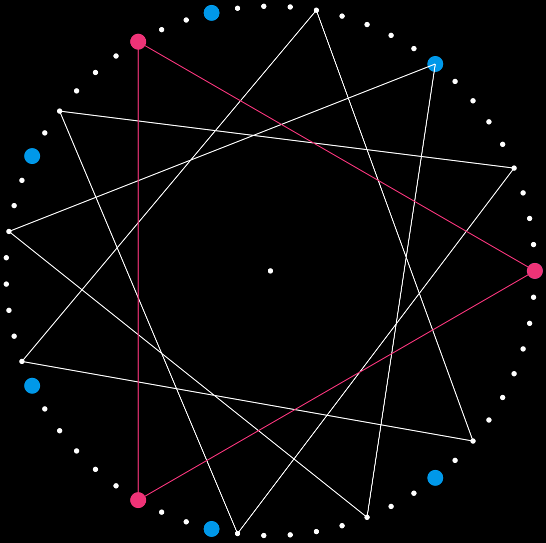
When an orbit of $\langle \mathbf{D}, \mathbf{P} \rangle$ does not contain six mutually non-isomorphic maps, there could also be orbits of size $3(q-1)$, $2(q-1)$, or even, in extremis, $q-1$.

When the ratio of a triple "collapses" into a smaller (sub)field, field automorphisms come into play, and some apparently distinct trace triples correspond to the same (isomorphism class of) map.

When field automorphisms reduce the count of non-isomorphic maps



A remarkable example:



The more observant of you...

...will have noticed that there are some gaps to be filled.

Some matters have been rather glossed over but are perfectly sound,

and others (like proving that the smaller sets are also in fact orbits, and do not decompose yet further)...

... I hope/intend to fill in soon!!!

This is the end of the talk...

Thanks for turning up! :-)