Regular Maps and Curves with Large Number of Automorphisms

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Joint work with E. Bujalance, M. Conder and A. Costa ....

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Complex Curves, Fuchsian Groups Riemann Surfaces

Given an orientable, **closed** surface X of genus $g \geq 2$

The equivalence:

$(X, \mathcal{M}(X)$, complex atlas) ($\mathcal{M}(X) = \langle x, y \rangle$, $p(x, y) = 0$, the field of meromorphic functions on X)

$X \cong \frac{\mathbb{H}}{\Delta}$, with Δ a (cocompact) **Fuchsian group**
Δ discrete subgroup of $PSL(2, \mathbb{R})$. **Riemann Uniformization Theorem (Koebe)**

Surface Fuchsian Group $\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g \mid \Pi[a_i, b_i] = 1 \rangle$

$(X, \mathcal{M}(X)$, **complex curve**) ($\mathcal{M}(X) = \mathbb{C}[x, y]/p(x, y)$, the field of rational functions on X)

The curve X given by the polynomial $p(x, y)$ and the meromorphic function $x : X \to \widehat{\mathbb{C}}$.

Meromorphic Functions and Abelian Integrals

We have seen in elementary Calculus: $arcsin(x) = \int_0^x \frac{du}{\sqrt{1-u^2}}$

Later in Complex Analysis

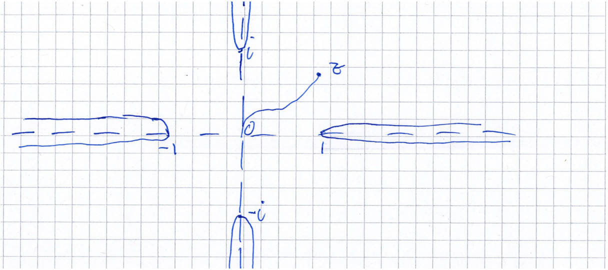
$\arcsin(z) = \int_0^z \frac{du}{\sqrt{1-u^2}}$ and $\quad L_a(z) = \displaystyle\int_0^z \frac{\sqrt{1+(a^2-1)u^2}}{\sqrt{1-u^2}}du$

But there are issues: the integrand is multi-valued, there are singularities in the integrand, and the value of the integrand depends on the homotopy class of the path of integration.

This happens in general for integrals of the form $\int_{z_0}^z R(x,y)du$ with $R(x,y)$ a rational function, y a locally defined function such that $F(x,y) = r_0(x)y^n + r_i(x)y^{n-1} + \cdots + r_n(x) = 0$, $r_i(x)$ rational functions; and, the path of integration does not pass through any singularity of the integrand.

$$L_a(z) = \int_0^z \frac{\sqrt{1+(a^2-1)x^2}}{\sqrt{1-x^2}}dx,\ R(x,y) = y, \text{with } F(x,y) = y^2 - \frac{1+(a^2-1)x^2}{1-x^2}$$

We make some 'branch cuts' on the plane such that the cuts are non-intersecting paths joining the branch points to ∞, the remainder of the complex plane is simply connected. The function y and, hence, the integrand can be analytically continued across a branch cut, away from singularities



Riemann Surfaces

We construct a **Riemann Surface** for the function $R(x, y)$, with y is a locally defined function such that $F(x, y) = r_0(x)y^n + r_i(x)y^{n-1} + \cdots + r_n(x) = 0$, $r_i(x)$ rational:

- Take n (n the degree of $F(x, y)$) copies of the complex plane with 'branch cuts'
- Each copy of the cut plane defines a branch of y
- Glue, using analytical continuation, the cut planes along the branch cuts (keeping y continuous)
- Compactify by adding points at infinity and points corresponding to branch points
- Resolve singularities

  We have a Riemann surface S with:
- Two important meromorphic functions $x : S \to \widehat{\mathbb{C}}$ and $y : S \to \widehat{\mathbb{C}}$
- The branch cuts decompose the surface into n polygons. The function x lifts the base cut plane bijectively to the polygons.
- x, y generate the field of meromorphic functions on X, $\mathbb{C}(S)$, is a degree n extension of the rational functions field $\mathbb{C}(S) = \mathbb{C}(x, y)/\langle F(x, y) \rangle$

Riemann Ph.D. Thesis: *Foundations for a general theory of functions of a complex variable*, 1851
Hurwitz: *Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, 1891

Since RUT

$(X$, complex atlas$) \cong \mathbb{H}/\Delta$, with Δ a (cocompact) Fuchsian group

Surface Fuchsian Group $\Gamma_g = \langle a_1, b_1, \dots, a_g, b_g \mid \Pi[a_i, b_i] = 1 \rangle$

- **Teichmüller** space $\mathcal{T}_g$, space of geometries on a surface of genus g
  $\mathcal{T}_g = \{\sigma : \Gamma_0 \to PSL(2, \mathbb{R}) \mid \sigma\, injective,\ \sigma(\Gamma_0) \text{ discrete } \}/PSL(2, \mathbb{R})$

  A Riemann surface with prescribed geometry is given by a marked polygon (and all its conjugate by a hyperbolic transformation) in the hyperbolic plane, or the **space of conjugacy classes of Fuchsian groups** isomorphic to the abstract group $\Gamma_0 = \langle a_1, b_1, \dots, a_g, b_g;\ a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle$.

- **Moduli** space $\mathcal{M}_g$, space (orbifold) of conformal structures on a surface of genus g

- **Mapping Class Group** (Teichmüller Modular Group)
  $M_g^+ = Diff^+(X)/Diff_0(X) = Out(\Gamma_g)$

- **Orbifold Universal Covering** $\mathcal{M}_g = \mathcal{T}_g/M_g^+$
  $\mathcal{B}_g$ **Branch Locus = Singular Locus of** $\mathcal{M}_g$ **as orbifold** (**Not** the singular set of $\mathcal{M}_{2,3}$ as algebraic variety, A. Costa- A. Porto for a proof with Fuchsian groups)

Nielsen Realization Theorem (Abikoff 1980, Macbeath for NEC groups)

$\mathcal{B}_g = \{X \in \mathcal{M}_g \,|\, Aut(X) \neq 1\}$

($\mathcal{B}_2$: surfaces with more automorphisms than the hyperelliptic involution)

$g = 1$ Euclidean case: $\mathcal{T}_1 = \mathbb{H}, M_1 = PSL(2,\mathbb{Z}), \mathcal{B}_1 = \{i, e^{i\pi/3}\}$, $\mathcal{M}_1$ hyperbolic triangle with a vertex at ∞, *the modular space, the nodal curve* $y^2 = x^3$.

Considering (X, dianalytic atlas, top. type $\mathbf{t}$) $\cong \mathbb{H}/\widehat{\Delta}$, with $\widehat{\Delta}$ an NEC group $\mathcal{T}_t^K$ and $\mathcal{M}_t^K$ the Teichmüller and moduli space of Klein surfaces of topological type $\mathbf{t}$.

$\mathcal{M}_t^K = \mathcal{T}_t^K / M(\widehat{\Delta})$, $\quad M(\widehat{\Delta}) = Out(\widehat{\Delta})$. Branch locus $\mathcal{B}_t^K$

Studies of branch locus and moduli spaces:

For $g = 1$ Schwarz

For $g = 2$ Bolza (1887, moduli of automorphic functions)

For hyperbolic surfaces Harvey, Natanzon, Macbeath (Macbeath-Singerman).

Number Fields, Triangular Groups and Dessins d'Enfants

Belyi Th: A plane complex curve X is defined over a number field iff there is a finite N-sheeted orbifold-covering = meromorphic function $\beta : X \to \widehat{\mathbb{C}}$ of the Riemann sphere **ramified on at most three points** $\{0, 1, \infty\}$ (the meromorphic function β is Belyi function).
In the case of Klein's Quartic, a 7-sheeted orbifold-covering of the Riemann sphere ramified at three points ($y^7 = x^2(x-1)$).

The meromorphic function β induces a cell-decomposition $\mathcal{H}$ of the Riemann surface X: the dessin d'enfant (map or hypermap). The preimages of 0 providing the *hypervertices,* the preimages of 1 the *hyeredges* and the preimages of ∞ the *hyperfaces*.
In the case of Klein's Quartic, the tessellation is the well-known tessellation with 168 triangles, each one representing an element in $PSL(2,7)$.

Translating into Fuchsian groups: $\beta : \mathbb{H}/\Gamma_g \to \mathbb{H}/\Delta(l,m,n)$, where $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.
The dessin has type (l,m,n), and (orbifold-covering) monodromy

$$\theta_\beta : \Delta(l,m,n) \to G = Mon(\mathcal{H})$$

In the case of Klein's Quartic, $\theta_y : \Delta(2,3,7) \to PSL(2,7) = Mon(\mathcal{H})$.

Only interested in uniform dessin: $\Gamma_g = H = \theta_\beta^{-1}(Stb(1))$ (a surface group).

Some classic curves and dessins d'enfants:

- Wiman's curve of type I in genus g, $y^2 = (x^{2g+1} - 1)$, $G = C_{4g+2}$, and quotient orbifold uniformised by $\Delta(2, 2g+1, 4g+2)$
- Wiman's curve of type II in genus g, $y^2 = x(x^{2g} - 1)$, $G = C_{4g} \rtimes_{2g-1} C_2$, and quotient orbifold uniformised by $\Delta(2, 4, 4g)$. In genus two this curve is Bolza's curve with $G_2 = GL(2, 3)$ and Fuchsian gr. $\Delta(2, 3, 8)$,
- Accola-Maclachlan's curve in genus g, $y^2 = x(x^{2g+2} - 1)$, $G = (C_{2g+2} \times C_2) \rtimes C_2$, the quotient uniformised by $\Delta(2, 4, 2g+2)$

  For genera $g \equiv 3 \bmod 4$ there is one more curve: Kulkarni curve $y^{2g+2} = x(x-1)^{g-1}(x+1)^{g+2}$, and gr. $G = \langle x, y : x^{2g+2} = y^4 = (xy)^2 = 1;\ y^2xy^2 = x^{g+2} \rangle$, the quotient uniformised by $\Delta(2, 4, 2g+2)$.
- Picards's curve $y^3 = (x^4 - 1)$, $G = C_{12}$, the quotient uniformised by $\Delta(4, 3, 12)$
- Klein's Quartic $y^7 = x^2(x-1)$, $G = PSL(2, 7)$, the quotient uniformised by $\Delta(2, 3, 7)$
- Bring's curve (genus 4) $y^5 = (x^3 - 1)$, $G = \Sigma_5$, the quotient uniformised by $\Delta(2, 4, 5)$
- Wiman's sextic (genus 6) $G = \Sigma_5$, the quotient uniformised by $\Delta(2, 4, 6)$ and $x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) = 12x^2y^2z^2$

Wiman 1895, 1896, Accola 1968, Maclachlan 1969, Kulkarni 1991, 1997

- Two curves in genus 8 with $G = PGL(2,7)$, the quotient uniformised by $\Delta(2,3,8)$
- Macbeath's three curves in genus 14 with $G = PSL(2,13)$, the quotient uniformised by $\Delta(2,3,7)$
- Given a prime $p \equiv 1 \bmod 5$, four curves of genus $p+1$ with $G = C_p \rtimes C_{10}$, the quotient uniformised by $\Delta(2,5,10)$
- Given a prime $p \equiv 1 \bmod 8$, four curves of genus $p+1$ with $G = C_p \rtimes C_8$, the quotient uniformised by $\Delta(2,8,8)$
- Given a prime $p \equiv 1 \bmod 3$, two curves of genus $p+1$ with $G = (C_p \rtimes C_6) \times C_2$, the quotient uniformised by $\Delta(2,6,6)$
- Given a prime $p \geq 3$, one curve of genus $g = (p-1)^2$, $x^p y^p - x^p - y^p + 1 = 0$ with $8(g+1+2\sqrt{g})$ automorphisms and $G = (C_p \times C_p) \rtimes D_4$, the quotient uniformised by $\Delta(2,4,2p)$.
- One curve in genus $g = 3^{n-1}$ with $G = C_{3^{n-1}} \rtimes GL(2,3)$, the quotient uniformised by $\Delta(2,3,8)$

Macbeath 1969, Conder 2009, Conder-Kulkarni 1992, Gromadzki-Maclachlan 1993, González-Diez 1995, Ying 2006, Wootton 2007, Belolipetsky-Jones 2005, Conder-Siran-Tucker 2010, Costa-I-Ying 2010.

Fuchsian Groups

- Δ (cocompact) discrete subgroup of $PSL(2, \mathbb{R})$
- A (compact) Riemann Surface (Orbifold) of genus $g \geq 2$ $\quad X = \frac{\mathbb{H}}{\Delta}$
- Δ has presentation:

  generators: $x_1, ..., x_r, a_1, b_1, ..., a_h, b_h$

  relations: $x_i^{m_i}, i = 1 : r,\ x_1 ... x_r a_1 b_1 a_1^{-1} b_1^{-1} ... a_h b_h a_h^{-1} b_h^{-1}$

  x_i: generator of the maximal cyclic subgroups of Δ
- $X = \frac{\mathbb{H}}{\Delta}$: orbifold with r cone points and underlying surface of genus g
- Algebraic structure of Δ and geometric structure of X are determined by the signature $\quad s(\Delta) = (h; m_1, \ldots, m_r)$
- Δ is the orbifold-fundamental group of X.

- Area of Δ: area of a fundamental region P
  $\mu(\Delta) = 2\pi(2h - 2 + \sum_1^r (1 - \frac{1}{m_i}))$
  But for us $\mu(\Delta) = (2h - 2 + \sum_1^r (1 - \frac{1}{m_i}))$, (-Euler characteristic of the hyperbolic orbifold)
- **X hyperbolic equivalent to $P/\langle$pairing$\rangle$**
- **Poincaré's Th: $\Delta = \langle$pairing$\rangle$** (Maskit, 1971)
- **Riemann-Hurwitz Formula:** If Λ is a subgroup of finite index, N, of a Fuchsian group Δ, then $N = \frac{\mu(\Lambda)}{\mu(\Delta)}$ (Euler characteristic is multiplicative under coverings)
- RUT: Any Riemann surface of genus $g \geq 2$ is uniformized by a surface Fuchsian group $\Gamma_g = \langle a_1, b_1, ..., a_g, b_g \,;\, a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1} \rangle$

Morphisms and Orbifold-Coverings

G finite group of automorphisms of $X_g = \mathbb{H}/\Gamma_g$,

Γ_g a surface Fuchsian group if there exist

Δ Fuchsian group and epimorphism $\theta : \Delta \to G$ with $Ker(\theta) = \Gamma_g$

θ is the monodromy of the regular covering $f : \mathbb{H}/\Gamma_g \to \mathcal{H}/\Delta$

$$\begin{tikzcd} & \mathbb{H} \arrow[dl] \arrow[dd] \\ X/ = \mathbb{H}/\Gamma_g \arrow[dr] & \\ & X/G = \mathbb{H}/\Delta \end{tikzcd}$$

Δ: lifting to $\mathbb{H}$ of G

Generally, an inclusion $i : \Lambda \to \Delta$ of Fuchsian groups determines a covering of Riemann orbifolds:

A morphism (orbifold-covering) $f : X = \mathbb{H}/\Lambda \to Y = \mathbb{H}/\Delta$,

Covering f determined by (the transitive) monodromy $\theta : \Delta \to \Sigma_{|\Delta:\Lambda|}$,
$\Lambda = \theta^{-1}(Stb(1))$
(symbol $\leftrightarrow$ Λ-coset $\leftrightarrow$ sheet for f $\leftrightarrow$ copy of fund. polygon for Δ)

Theorem (Singerman 1970) Λ (and so i) determined θ (and Δ): If $s(\Delta) = (h; m_1, \dots, m_r)$, then $s(\Lambda) = (h'; m'_{11}, \dots, m'_{1s_1}, \dots, m'_{r1}, \dots, m'_{rs_r})$ iff $\theta : \Delta \to \Sigma_{|\Delta:\Lambda|}$ s.t.

i) Riemann-Hurwitz $\frac{\mu(\Lambda)}{\mu(\Delta)} = |\Delta : \Lambda|$

ii) $\theta(x_i)$ product of s_i cycles each of length $\frac{m_i}{m'_{i1}}, \dots, \frac{m_i}{m'_{is_i}}$

We say the morphism is uniform if $\Lambda = \Gamma_g$. In this case $\theta(x_i)$ product of $\frac{|\Delta:\Lambda|}{m_i}$ cycles each of length m_i

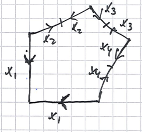
The polygonal graph of the tessellation of X given by the $|\Delta : \Lambda|$ copies of Y is the dual of the Schreier coset-graph of $i : \Lambda \to \Delta$.

Example: Surfaces of genus 2 with 8 automorphisms. They admit an action of D_4 with monodromy $\theta : \Delta(0; 2, 2, 2, 4) \longrightarrow D_4$

$\theta(x_1) = a = (1,3,5,7)(2,4,6,8)$

$\theta(x_2) = s = (1,2)(4,7)(3,8)(6,5)$

$\theta(x_3) = sa = (1,4)(2,3)(5,8)(6,7)$

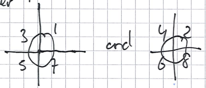


$Y_2 = \mathbb{H}/\Delta$

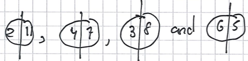
(of course $\theta(x_4) = a^2 = (1,5)(2,6)(3,7)(4,8)$)

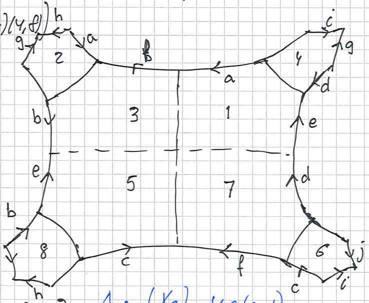
No singular pts

For order 4



For one of order 2





The area is $2\pi 8(\frac{1}{4}) = 4\pi$, so genus is 2 $\quad$ $\text{Area}(X_g) = 4\pi(g-1)$

(Uniform) Maps and Hypermaps and Fuchsian Groups

A Belyi function $\beta : X \to \widehat{\mathbb{C}}$ induces a cell-decomposition $\mathcal{H}$ of the Riemann surface X: the dessin d'enfant (map or hypermap). The preimages of 0 providing the *hypervertices*, the preimages of 1 the *hyeredges* and the preimages of ∞ the *hyperfaces*.

Geometrically, $\beta : X = \mathbb{H}/\Gamma_g \to \widehat{\mathbb{C}} = \mathbb{H}/\Delta(l, m, n)$ as a covering of Riemann orbifolds X and $\widehat{\mathbb{C}}(l, m, n)$ uniformized by Γ_g and $\Delta(l, m, n)$ respectively.

Fuchsian groups: $i : \Gamma_g \to \Delta(l, m, n)$, where $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. The dessin has type (l, m, n), and (orbifold-covering) monodromy

$$\theta_\beta : \Delta(l, m, n) \to G = Mon(\mathcal{H}) \leq \Sigma_{|\Delta:\Gamma_g|}$$

When $m = 2$ we have a map of type $\{l, n\}$.

Monodromy group $\mathcal{H}$: $Mon(\mathcal{H}) = \langle a, s\, ; a^l = s^m = (as)^n = \cdots = 1\rangle \leq \Sigma_N = \Sigma_{|\Delta:H|}$

The group $H(\cong \Gamma_g) = \theta_\beta^{-1}(Stb(1)) \leq \Delta(l, m, n)$, the hypermap group.

$Mon(\mathcal{H})$ acts transitively on the H-cosets. The permutation a has as cycles the cycles around hypervertices, s the cycles around (hyper)edges, and as the cycles around hyperfaces (always consistent with the orientation of X).

Two dessins d'enfants of type (l, m, n) are isomorphic if their hypermap groups are conjugate in $\Delta(l, m, n)$, i.e. they defined the same complex structure of $X = \mathbb{H}/H = \mathbb{H}/\Gamma_g$.

Some Confusion

A morphism of dessins is given by an inclusion of subgroups $H_1 \leq H_2$ of triangle Fuchsian groups, inducing a morphism of Riemann orbifolds and a covering of the embedded graphs.

$Aut(\mathcal{H}) \leq Mon(\mathcal{H}) = G$. (I only consider orientation preserving automorphisms)
Notice that for any dessin d'enfant $\mathcal{H}$ on a Riemann surface $X = \mathbb{H}/\Gamma_g$, one has $Aut(\mathcal{H}) \leq Aut(X)$.

A dessin d'enfant $\mathcal{H}$ with hypermap group H is (orientably) regular iff $Aut(\mathcal{H})$ acts transitively on the H-cosets, i.e. $Aut(\mathcal{H}) = Mon(\mathcal{H}) = G$. That is, the monodromy $\theta_\beta : \Delta(l, m, n) \to G = Mon(\mathcal{H})$ is a quotient, $H = \Gamma_g \trianglelefteq \Delta(l, m, n)$ and the covering $\beta : X = \mathbb{H}/\Gamma_g \to \widehat{\mathbb{C}} = \mathbb{H}/\Delta(l, m, n)$ is regular.

Of course $Mon(\mathcal{H}) \leq Aut(X)$.

A (orientably) regular dessin d'enfant $\mathcal{H}$ with monodromy group $Aut(\mathcal{H}) = \Delta(l, m, n)/H = \langle a, s \rangle$ is reflexible if the function defined by $a \to a^{-1}$, $s \to s^{-1}$ is an automorphism of G, otherwise is said to be chiral.

Reflexible dessin $\equiv$ dessin on a *symmetric* surface (admitting an anticonformal involution)

I learnt this from the **Southampton Extended Group** (David & Gareth 1978, people in Southampton, Porto, Auckland, Slovakia, Madrid, USA, Frankfurt, ...)

Regular maps of genus g with $4g$ conformal automorphisms

With a few exceptions an oriented regular dessin d'enfant $\mathcal{H}$ with $Aut(\mathcal{H})$ of order $4g$ is a (medial) truncation of the regular map $\mathcal{W}$ determining Wiman's curve of type II $y^2 = x(x^{2g} - 1)$.

$\mathcal{W}$ is a map of type $\{4, 4g\}$ with $Aut(\mathcal{W}) = C_{4g} \rtimes_{2g-1} C_2$ for $g \geq 3$.
For $g = 2$, $\mathcal{W}$ is the map of type $\{3, 8\}$ with $Aut(\mathcal{W}) = GL(2, 3)$ on Bolza's curve, the curve of genus 2 with largest number of automorphisms (Wiman 1895, Kulkarni 1993).

Algorithm ($Aut(\mathcal{H}) \leq Aut(X)$): we will determine first actions of groups of order $4g$ on a curve of genus g; i.e. determine groups Δ and monodromies θ such that $\theta : \Delta \to G$ with $Ker(\theta) = \Gamma_g$ and $|G| = 4g$.

With the following exceptions:

- A family in genus 3 with quotient uniformised by $\Delta(2,2,3,3)$ and $Aut(X) = A_4$
- A family in genus 6 with quotient uniformised by $\Delta(2,2,3,4)$ and $Aut(X) = \Sigma_4$
- A family in genus 15 with quotient uniformised by $\Delta(2,2,3,5)$ and $Aut(X) = A_5$
- Picard's curve in genus 3 with quotient uniformised by $\Delta(3,4,12)$ and $Aut(X) = C_{12}$
- A curve in genus 6 with quotient uniformized by $\Delta(3,8,8)$ and $Aut(X) = C_3 \rtimes C_8$, a truncation of the map $\{6,8\}$ with $Aut = \langle r, s \,;\, a^6 = (as)^2 = (as^{-1}a)^2 = 1, a^3 = s^4 \rangle$
- A curve in genus 6 with quotient uniformized by $\Delta(4,6,6)$ and $Aut(X) = \mathrm{SL}(2,3)$, a truncation of the map $\{6,8\}$ with $Aut = C_8 \rtimes C_6$
- A curve in genus 6 with quotient uniformized by $\Delta(4,6,6)$ and $Aut(X) = D_4 \times C_3$
- A curve in genus 12 with quotient uniformised by $\Delta(4,6,8)$ and $Aut(X) = \langle 2,3,4 \rangle$
- A curve in genus 12 with quotient uniformised by $\Delta(4,6,8)$ and $Aut(X) = (C_3 \rtimes C_8) \rtimes C_2$
- A curve in genus 30 with quotient uniformised by $\Delta(4,6,10)$ and $Aut(X) = C_{15} \rtimes D_4$
- A curve in genus 30 with quotient uniformised by $\Delta(4,6,10)$ and $Aut(X) = \mathrm{SL}(2,5)$

All the exceptional maps and hypermaps are reflexible, embedded in symmetric Riemann surfaces.

Actual actions, all genera

The possible signatures of Fuchsian groups for all genera are $\Delta(3,6,2g)$, $\Delta(2,4g,4g)$, $\Delta(4,4,2g)$ and $\Delta(2,2,2,2g)$.

- There is NO action $\theta : \Delta(3,6,2g) \to G_{4g}$
- The unique action; $\theta : \Delta(2,4g,4g) \to C_{4g} = \langle a\,;\, a^{4g} = 1\rangle$ defined by $\theta(x_1) = a^{2g}$, $\theta(x_2) = a^{2g-1}$.
- The unique action;
  $\theta : \Delta(4,4,2g) \to G_{4g} = \langle a, t\,;\, a^{2g} = t^4 = 1, t^2 = a^g, t^3at = a^{-1}\rangle$ defined by $\theta(x_1) = t$, $\theta(x_2) = t^3a$.
  G_{4g} is the central product of C_{2g} by C_4.
- The unique action $\theta : \Delta(2,2,2,2g) \to D_{2g} = \langle a, s\,;\, a^{2g} = s^2 = sa^2 = 1\rangle$ defined by $\theta(x_1) = s$, $\theta(x_2) = sa^{g-1}$, $\theta(x_3) = a^g$.

The only curves X with $|Aut(X)| = 4g$ form an equisymmetric, uniparametric family $\mathcal{S}$ given by the monodromy $\theta : \Delta(2,2,2,2g) \to D_{2g} = \langle a, s\,; a^{2g} = s^2 = sa^2 = 1\rangle$ defined by $\theta(x_1) = s$, $\theta(x_2) = sa^{g-1}$, $\theta(x_3) = a^g$. This family does exist for all genera $g \geq 2$.

As a Riemann surface, $\mathcal{S}$ is the Riemann Sphere with three punctures. One of the punctures is Wiman's curve of type II, the other two are nodal curves. Within $\mathcal{S}$ there are three real-uniparametric families of symmetric Riemann surfaces building three arcs, connected by Wiman's curve and the two nodal curves.

To see that Wiman's curve of type II (a symmetric Riemann surface) is in the closure of the family one checks that $\Delta(2,4,4g)$ contains a subgroup with signature $(0;2,2,2,2g)$ and that the action with monodromy above extends to an action of $C_{4g} \rtimes_{2g-1} C_2$ with with quotient orbifold uniformized by $\Delta(2,4,4g)$.

All the surfaces in the family $\mathcal{S}$ (and Wiman's curve) are hyperelliptic.

(Bujalance-Costa-I 2017, Conder's homepage, and MAGMA for genera 273, 276, 420, 429 and 841)

Dessins d'Enfants on Wiman's Curve of Type II

The regular dessins with $4g$ automorphisms existing for all genera ($g \geq 2$) given by the **non-maximal** actions:

- $\theta_1 : \Delta(2, 4g, 4g) \rightarrow C_{4g} = \langle a\,;\, a^{4g} = 1 \rangle$, $\theta_1(x_1) = a^{2g}$, $\theta_1(x_2) = a^{2g-1}$ and
- $\theta_2 : \Delta(4, 4, 2g) \rightarrow G_{4g} = \langle a, t\,;\, a^{2g} = t^4 = 1, t^2 = a^g, t^3 at = a^{-1} \rangle$ defined by $\theta_2(x_1) = t$, $\theta_2(x_2) = t^3 a$

The actions extend to a maximal action of $C_{4g} \rtimes_{2g-1} C_2 = \langle a, s\,;\, a^{4g} = s^2 = (sa)^4 = 1, sas = a^{2g-1} \rangle = \langle a, t\,;\, a^{4g} = t^4 = (ta)^2 = 1, t^2 = a^{2g} t^3 at = a^{2g-1} \rangle$ giving Wiman's map $\mathcal{W}$ on Wiman's curve $X_W; y^2 = x(x^{2g} - 1)$ of type II.

(Bujalance-Conder, Kulkarni 1993, Bujalance-Costa-I 2017)

Wiman's map $\mathcal{W}$ on Wiman's curve $X_W; y^2 = x(x^{2g} - 1)$.

For $g = 2$ Wiman's map is the Bolza's map of type $\{3, 8\}$ and automorphism group $GL(2, 3)$ (Bolza 1887, Wiman 1895, Singerman 1970, Girondo 2003, ...)

In genus $g \geq 3$ is the map of type $\{4, 4g\}$ with $Mon(\mathcal{W}) = Aut(X_W) = \langle as, s \rangle$ where

$$as = (1, 2, 3, \ldots, 4g)(4g + 1, 4g + 2, 4g + 3, \ldots, 4g + 4g)$$

and

$$s = \prod_{k=0}^{g-1} (4g + 4k + 1, 4g - 4(g - k) + 1)(4g + 4k + 2, 4g - 4(g - k) + 2)$$

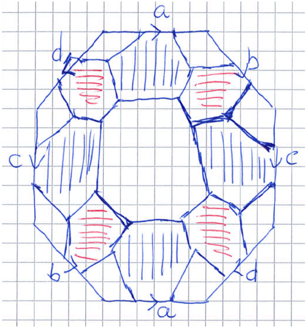
$$(4g + 4k + 3, 4g - 4(g - k) + 3)(4g + 4(k + 1), 4g - 4(g - k - 1)).$$

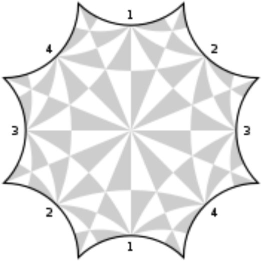
The regular dessins d'enfants given by these actions are medial truncations $\mathcal{T}_1, \mathcal{T}_2$ of the regular map $\mathcal{W}$ of type $\{4, 4g\}$ with $(Aut(\mathcal{W}) = C_{4g} \rtimes_{2g-1} C_2)$. Their monodromy groups are:

- $Mon(\mathcal{T}_1) = \langle (1, 2, 3, \ldots, 4g), \prod_{i=1}^{g} (i, 2g + i) \rangle$
- $Mon(\mathcal{T}_2) = \langle (1, 2, 3, \ldots, 2g)(2g + 1, 2g + 2, \ldots, 4g), (1, 2g + 1, g + 1, 3g + 1) \prod_{i=2}^{g} (i, 4g + 2 - i, g + i, 3g + 2 - i) \rangle$

The extended group of automorphisms of the map of type $\{4g, 4g\}$ is D_{4g}.

The extended group of automorphisms of the hypermap of type $(4, 4, 2g)$ is $G_{4g} \rtimes C_2 = \langle a, t, \sigma \,;\, \sigma^2 = a^{2g} = t^4 = (\sigma a)^2 = (\sigma t)^2 = 1, t^2 = a^g, t^3 a t = a^{-1} \rangle$: the semidirect product of the central product of C_{2g} and C_4 by C_2.





The hypermap $(4,4,4)$ and the map $\{3,8\}$ in genus 2

THANK YOU