On the automorphisms of cubic vertex-transitive graphs

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(joint work with P. Potočnik and G. Verret)

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Algebraic Graph Theory International Webinar

Automorphisms

Let Γ be a graph. Then

- We say Γ is cubic if it is simple, connected and 3-valent;
- We let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ;
- If a subgroup $G \leq \mathrm{Aut}(\Gamma)$ acts transitively on the vertices of Γ, then we say Γ is G-vertex-transitive.

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- $o(g) =$ the order of g;
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- $\mu(g) =$ the number of orbits of g.

Remark: An orbit of g is an orbit of $\langle g \rangle$, the cyclic group generated by g.

Automorphisms

Let Γ be a graph. Define

- $o(\Gamma) = \max\{o(g) \mid g \in \mathrm{Aut}(\Gamma)\}$;
- $\ell(\Gamma) = \max\{\ell(g) \mid g \in \mathrm{Aut}(\Gamma)\}$;
- $\mu(\Gamma) = \max\{\mu(g) \mid g \in \mathrm{Aut}(\Gamma), g \neq 1\}$.

Notice that

- $\ell(\Gamma) \mid o(\Gamma)$
- "small" $\mu(\Gamma)$ implies "large" $\ell(\Gamma)$ and $o(\Gamma)$.

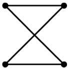
Automorphisms

- What can we say about a graph Γ with large $o(\Gamma)$?
- Can we give a "good" upper bound for $o(\Gamma)$ in terms of $|V(\Gamma)|$?
- Under which conditions does the equality $o(\Gamma) = \ell(\Gamma)$ hold?
- What is the relation between $\mu(\Gamma)$ and $o(\Gamma)$?

Split Praeger-Xu graphs

SPX(1,s)

Split Praeger-Xu graphs



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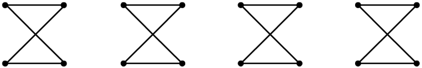

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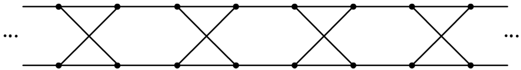

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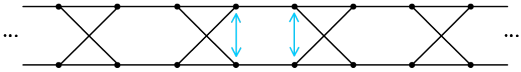

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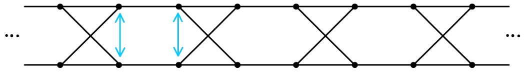

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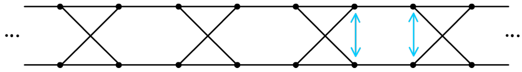

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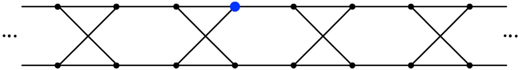

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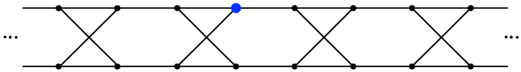

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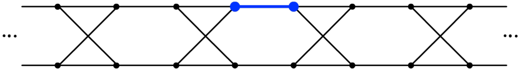

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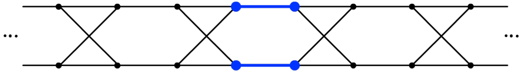

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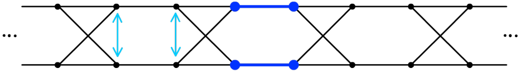

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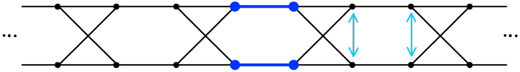


SPX(1,s)

Stabilizers

Theorem

Let Γ *be a cubic vertex-transitive graph with automorphism group* G *and let* G_v *be the stabilizer of a vertex* v*. If* $g \in G_v$ *then*

$$o(g) \leq 6.$$

Theorem

Let Γ *be a cubic vertex-transitive graph and let* $g \in \mathrm{Aut}(\Gamma)$*. If* X *and* Y *are two orbits of* g*, then*

$$\frac{1}{6} \leq \frac{|X|}{|Y|} \leq 6.$$

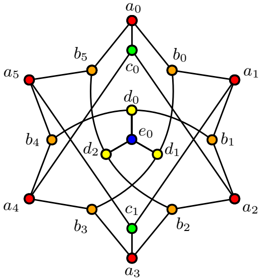
In particular,

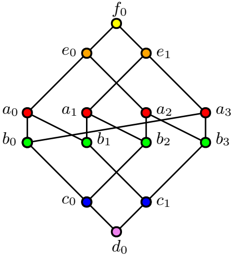
$$\frac{\ell(g)}{6} \leq |X|.$$

Orbit sizes

Theorem

Let Γ be a cubic G-vertex-transitive graph not isomorphic to $K_{3,3}$, and let $g \in G$. If X is a vertex-orbit of g, then $|X| = \frac{\ell(g)}{k}$ for some $k \in \{1, 2, 3, 4, 6\}$.





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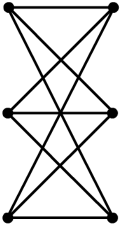
Theorem

If Γ is a cubic G-vertex-transitive graph other than $K_{3,3}$, then $o(g) = \ell(g)$ for all $g \in G$.



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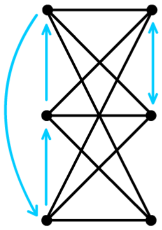

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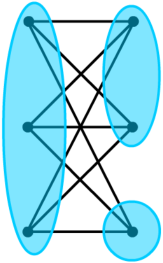

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Corollary

If Γ *is a cubic* G*-vertex-transitive graph, then* $o(\Gamma) = \ell(\Gamma)$*.*

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Corollary

If Γ is a cubic G-vertex-transitive graph, then $o(\Gamma) = \ell(\Gamma)$.

Corollary

If Γ is a cubic G-vertex-transitive graph of order n, then

$$o(g) \leq n$$

for all $g \in G$.

Orbit sizes

Theorem

Let G be a transitive permutation group on a set Ω, let $\omega \in \Omega$, let p be a prime and let k be an integer coprime to p such that $\exp(G_\omega) = kp^\alpha$ for some $\alpha \geq 1$. Then

$$o(g) \leq k\ell(g)$$

for every $g \in G$. In particular, if G_ω is a p-group, then for every $g \in G$ we have $o(g) = \ell(g)$.

Theorem

If Γ be a 4-valent G-vertex-transitive graph then $o(g) \leq 9n$ for all $g \in G$.

Orbit sizes

Question

For $d > 4$, does there exist a constant c_d such that $o(\Gamma) < c_d \cdot n$ for every d-valent G-vertex-transitive graph of order n?

Regular orbits

- Let G be a permutation group acting on a set Ω and let $g \in G$. We say an orbit X of g is a regular orbit if the $|X| = o(g)$ (i.e. X is of maximal size).

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- Every automorphism of a cubic vertex-transitive graph other than $K_{3,3}$ admits a regular orbit.
- This is not true in general, as a complete graph of order n can have an automorphism whose order greatly surpasses n.
- Graphs admitting automorphisms with regular orbits have been studied in the context of multicirculant graphs.

Definition

If Γ is a graph and and $g \in \mathrm{Aut}(\Gamma)$ we say two orbits of g, X and Y, are adjacent if there is a vertex in X that is adjacent to some vertex in Y.

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Theorem

Let Γ be a cubic vertex-transitive graph not isomorphic to $K_{3,3}$, K_4, the cube graph Q_3, the Petersen graph, the Möbius-Kantor graph, the Pappus graph, the Heawood graph.

Then, for every $g \in \mathrm{Aut}(\Gamma)$, either

1. *$\langle g \rangle$ is transitive on $\mathrm{V}(\Gamma)$, or*
2. *every regular orbit of g is adjacent to another regular orbit.*

Regular orbits

Theorem

Let Γ be a cubic G-vertex-transitive graph of order n and let $g \in G$. Then $\mu(g) \leq \frac{17n}{6\,o(g)}$.

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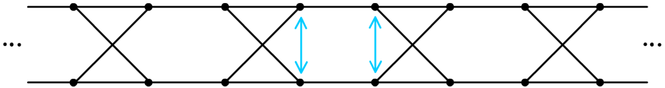
Theorem (P. Potočnik, P. Spiga)

Let Γ be a cubic G-vertex-transitive graph of order n and let $g \in G$ be a non-trivial automorphism fixing more than $\frac{n}{3}$ vertices. Then one of the following occurs:

1. *$n \leq 20$ and Γ is one of six exceptional graphs;*
2. *Γ is a split Praeger-Xu $SPX(r, s)$;*

Regular orbits



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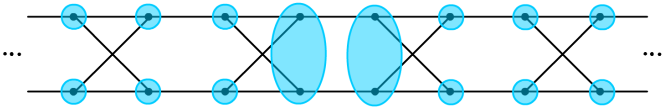

Regular orbits

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- No non-trivial automorphism of $SPX(1, s)$ has more than $2\frac{n}{o(g)} - 2$.

- If Γ is a cubic graph and $g \in \mathrm{Aut}(\Gamma)$ is transitive, then $\mu(g) = 1 = 2\frac{n}{o(g)} - 1$.

Regular orbits

Theorem

Let Γ be a cubic G-vertex-transitive graph of order n and let $g \in G$. Then $\mu(g) \leq \frac{17n}{6\, o(g)}$.

Conjecture

Let Γ be a cubic G-vertex-transitive graph of order n and let $g \in G$. Then $\mu(g) \leq 2\frac{n}{o(g)} - 1$.

Regular orbits

Theorem

Let Γ be a cubic G-vertex-transitive graph of order n and let $g \in G$. If Γ is not isomorphic to $K_{3,3}$ or a split Praeger-Xu graph $SPX(r, s)$, then at least $\frac{5}{12}n$ vertices are contained in a regular orbit of g.

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Conjecture

Let Γ be a cubic G-vertex-transitive graph of order n and let $g \in G$. If Γ is not isomorphic to $K_{3,3}$ or a split Praeger-Xu graph $SPX(r,s)$, then at least $\frac{2}{3}n$ vertices are contained in a regular orbit of g.

Multicirculants

Definition

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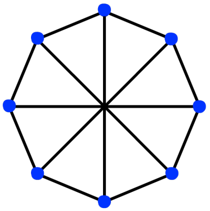
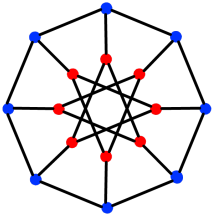
Definition

For a positive integer k, we say a graph Γ is a k-multicirculant if $\mathrm{Aut}(\Gamma)$ admits a semiregular automorphism g with exactly k orbits.

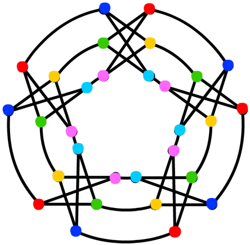
Multicirculants



1-multicirculant
(circulant)



2-multicirculant
(bicirculant)



6-multicirculant

Multicirculants

Some results involving multicirculant graphs...

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- P. Spiga, Semiregular elements in cubic vertex-transitive graphs and the restricted Burnside problem, Math. Proc. Cambridge Phil. Soc. 157 (2014), 45–61.
- M. Giudici, I. Kovács, C.-H. Li, G. Verret, Cubic arc-transitive k-multicirculants, J. Combin. Theory Ser. B 125 (2017), 80–94
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Conjecture (D. Marušič)

Every vertex-transitive graph admits a non-trivial semiregular automorphism.

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Every vertex-transitive graph admits a non-trivial semiregular automorphism.

Theorem (D. Marušič, R. Scapellato, 1998)

Every cubic vertex-transitive graph admits a non-trivial semiregular automorphism.

The functions η and κ

Let Γ be a cubic vertex-transitive graph and let $S \subset \mathrm{Aut}(\Gamma)$ be the set of non-trivial semiregular automorphisms of Γ. We define

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and

$$\kappa(\Gamma) = \frac{|\mathrm{V}(\Gamma)|}{max\{o(g) \mid g \in S\}}$$

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and

$$\kappa(\Gamma) = min\{k \in \mathbb{Z} \mid \Gamma \text{ is a } k\text{-multicirculant}\}$$

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- $\eta(\Gamma) = 1$ if and only if $\kappa(\Gamma) = 1$
- if $1 < \eta(\Gamma) \leq 2$ then either
  - Γ is the cube graph and $\eta(\Gamma) = \frac{4}{3}$;
  - Γ is the Petersen graph and $\eta(\Gamma) = \frac{5}{3}$;
  - Γ is the Heawood graph and $\eta(\Gamma) = \frac{7}{4}$;
  - Γ is the Möebius-Kantor graph and $\eta(\Gamma) = \frac{4}{3}$;
  - Γ is the Pappus graph and $\eta(\Gamma) = \frac{3}{2}$.

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  - Γ is the Möebius-Kantor graph and $\eta(\Gamma) = \frac{4}{3}$;
  - Γ is the Pappus graph and $\eta(\Gamma) = \frac{3}{2}$.
  - Γ admits an automorphism with two orbits of size $\frac{n}{2}$ each, and thus $\eta(\Gamma) = \kappa(\Gamma) = 2$
- in particular, if $\eta(\Gamma) \leq 2$ then $\kappa(\Gamma) \leq 2$, with the exception of the Pappus graph with κ-value 3.
- What is the value of $\kappa(\Gamma)$ if $2 < \eta(\Gamma) \leq 3$?

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- The group $\mathrm{Aut}(\Gamma)$ acts on the arcs of Γ.

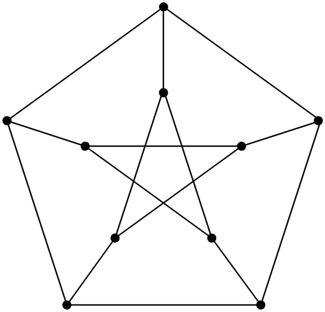
Definition

Let Γ be a graph and $G \leq \mathrm{Aut}(\Gamma)$. Let V/G and A/G denote the set of G-orbits on the vertices and the arcs of Γ, respectively.

The quotient Γ/G is the multigraph with vertex-set V/G and arc-set A/G where:

1. the initial vertex of an arc $(x, y)^G$ is u^G if and only if $x \in u^G$;
2. the inverse of an arc $(x, y)^G$ is $(u, v)^G$ if and only if there exists $g \in G$ such that $v^g = x$ and $u^g = y$.

Quotient graphs



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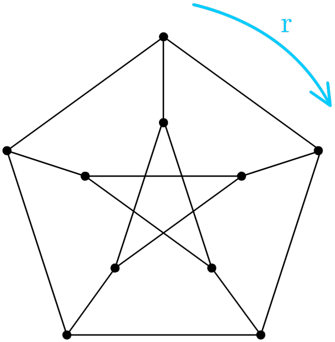
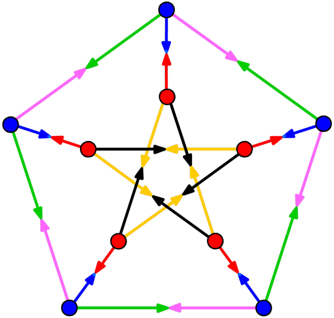


$G = \langle r \rangle$

Quotient graphs



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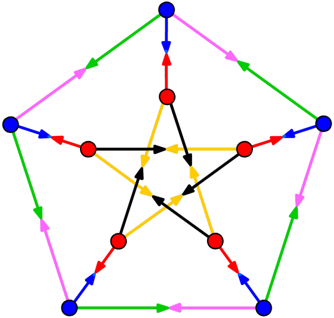

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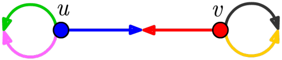


Quotient graphs

Definition

Let Γ be a cubic G-vertex-transitive graph and let $g \in G$. Let $\lambda : \mathrm{V}(\Gamma/g) \to \mathbb{Q}$ be given by

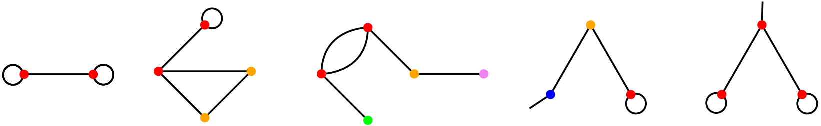
$$\lambda(v) = \frac{o(g)}{|v^G|}.$$

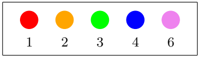
The pair $(\Gamma/g, \lambda)$ is a called labelled quotient.

Observe that:

- $(\Gamma/g, \lambda)$ is a diagramatic representation of the partition of the vertices an arcs of Γ induced by the action of g;
- if Γ is not isomorphic to $K_{3,3}$, then $Im(\lambda) = \{1, 2, 3, 4, 6\}$.

Quotient graphs





Quotient graphs

Lemma

Let Γ be a cubic G-vertex-transitive graph of order n and let $g \in G$. If $o(g) \geq \frac{n}{3}$ then $\mu(g) \leq 5$.

Lemma

Let Γ be a cubic G-vertex-transitive graph and let $g \in G$. Then

- *An orbit of size $o(g)$ can only be adjacent to orbits of size $\frac{o(g)}{i}$ for $i \in \{1, 2, 3\}$;*
- *An orbit of size $\frac{o(g)}{2}$ can only be adjacent to orbits of size $o(g)$ or $\frac{o(g)}{j}$ with $j \in \{1, 2\}$;*
- *An orbit of size $\frac{o(g)}{4}$ can only be adjacent to orbits of size $\frac{o(g)}{j}$ with $j \in \{1, 2\}$;*
- *An orbit of size $\frac{o(g)}{j}$ with $j \in \{3, 6\}$ is adjacent to exactly one orbit, which has size $3\frac{o(g)}{j}$.*

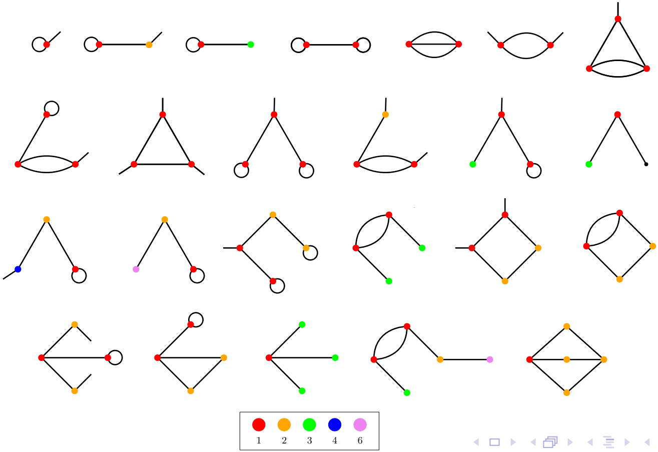
If in addition Γ has more than 20 vertices, then:

- *Two orbits of size $\frac{o(g)}{3}$ cannot have a common neighbour;*
- *ETC...*

Orbit partitions

Theorem

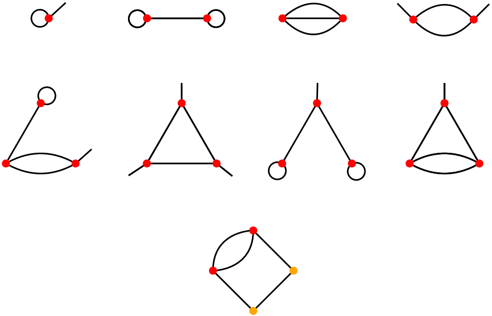
Let Γ be a cubic G-vertex-transitive graph of order n and let $g \in G$ be such that $o(g) \geq n/3$. Then Γ/g is one of the following.



Orbit partitions

Theorem

Let Γ *be a cubic* G*-vertex-transitive graph of order* $n > 20$ *and let* $g \in G$ *be such that* $o(g) \geq n/3$. *Then* Γ/g *is one of the following.*

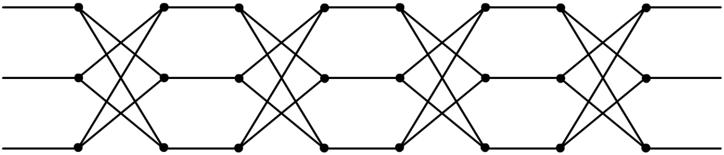


Orbit partitions

Theorem

Let Γ be a cubic G-vertex-transitive graph of order $n > 20$ and let $g \in G$ be such that $o(g) \geq n/3$. If g has orbit of size $o(g)/2$ then $\Gamma \cong SDW(r, 3)$, where $r > 3$ is odd.

Moreover, $\kappa(SDW(r,3)) = 6$ if $3 \mid r$ and $\kappa(SDW(r,3)) = 2$ otherwise.



The functions η and κ

Let Γ be a cubic G-vertex-transitive graph of order n. Then

- $\eta(\Gamma) = 1$ if and only if $\kappa(\Gamma) = 1$
- if $1 < \eta(\Gamma) \leq 2$ then either
  - Γ is the cube graph and $\eta(\Gamma) = \frac{4}{3}$;
  - Γ is the Petersen graph and $\eta(\Gamma) = \frac{5}{3}$;
  - Γ is the Heawood graph and $\eta(\Gamma) = \frac{7}{4}$;
  - Γ is the Möebius-Kantor graph and $\eta(\Gamma) = \frac{4}{3}$;
  - Γ is the Pappus graph and $\eta(\Gamma) = \frac{3}{2}$.
  - Γ admits an automorphism with two orbits of size $\frac{n}{2}$ each, and thus $\eta(\Gamma) = \kappa(\Gamma) = 2$
- in particular, if $\eta(\Gamma) \leq 2$ then $\kappa(\Gamma) \leq 2$, with the exception of the Pappus graph with κ-value 3.
- What is the value of $\kappa(\Gamma)$ if $2 < \eta(\Gamma) \leq 3$?

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- in particular, if $\eta(\Gamma) \leq 2$ then $\kappa(\Gamma) \leq 2$, with the exception of the Pappus graph with κ-value 3.
- if $\eta(\Gamma) \leq 3$ then $\kappa(\Gamma) \in \{1, 2, 3, 6\}$.

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- Let $f\colon \mathbb{N} \to \mathbb{N}$ be defined by

  $$f(r) = \max\{\kappa(\Gamma) : \Gamma \text{ a cubic vertex-transitive graph with } \eta(\Gamma) \leq r\}.$$

  We know that $f(1) = 1$, $f(2) = 3$ and $f(3) = 6$. Is the function f well-defined? What is its asymptotic behaviour as $n \to \infty$?

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  We know that $f(1) = 1$, $f(2) = 3$ and $f(3) = 6$. Is the function f well-defined? What is its asymptotic behaviour as $n \to \infty$?
- For a positive integer r, is the number of graphs Γ with $\eta(\Gamma) \leq r$ but $\eta(\Gamma) \notin \mathbb{Z}$ finite?

THANK YOU!

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