# Geometric approach to Berge's conjecture

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based on a joint work with Edita Máčajová

# Perfect matchings in cubic graphs

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**Definition**

The perfect-matching index of a bridgeless cubic graph $G$

$\pi(G) \quad = \quad$ the smallest # of perfect matchings that cover $E(G)$.

# Perfect matching index

## Observation

- $\pi(G) \geq 3$ for every bridgeless cubic graph $G$
- $\pi(G) = 3 \iff G$ is 3-edge-colourable.

Perfect matching index is only interesting for cubic graphs that have no 3-edge-colouring – that is, snarks.

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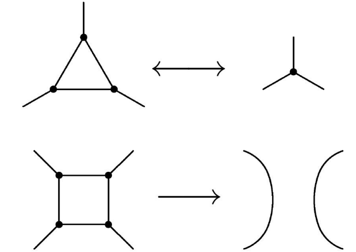
**Definition**

- A snark is a 2-connected cubic graph that has no 3-edge-colouring.
- A snark is nontrivial if it cyclically 4-edge-connected, with girth at least 5.

# Nontrivial snarks



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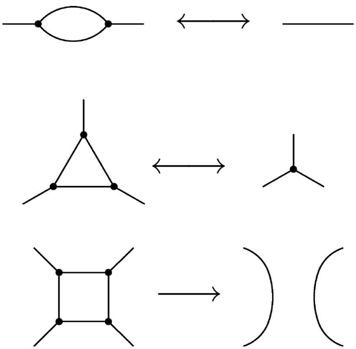

Similar simplifications for cycle-separating edge-cuts of size $\leq 3$

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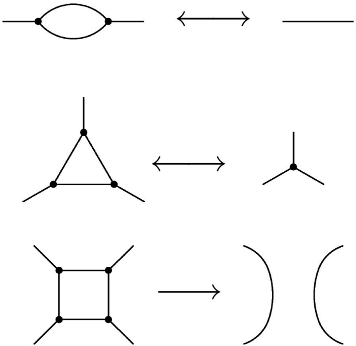

Similar simplifications for cycle-separating edge-cuts of size $\leq 3$

$\Longrightarrow$ 'nontrivial' usually means

- girth $> 4$, and
- cyclically 4-edge-connected.

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**Conjecture (Berge, communicated by Seymour, 1979)**

Every bridgeless cubic graphs contains a family of five perfect matchings that together cover all the edges.

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Every bridgeless cubic graphs contains a family of six perfect matchings which together cover each edge exactly twice.

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If these conjectures are true $\implies$ $\pi(G) \leq 5$ for every cubic graph $G$.

Snarks with $\pi \geq 5$ deserve a special interest:

- Berge's conjecture (of course)
- Several other conjectures reduce to snarks with $\pi(G) \geq 5$, e.g.:
  - Cycle double cover conjecture
  - Short cycle cover conjecture (the $7/5$-conjecture)

# Small snarks with $\pi \geq 5$

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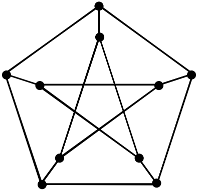
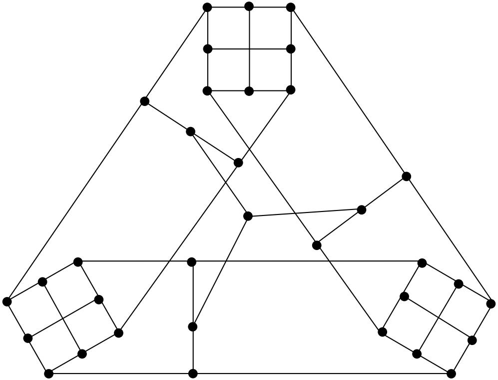
- There are exactly 64 326 024 such snarks.
- Only two of them have $\pi(G) > 4$:
  the Petersen graph and a snark of order 34.



# Windmill snark $W_{34}$ of order 34 with $\pi = 5$

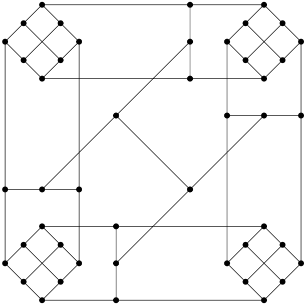


Brinkmann, Goedgebeur, Hägglund, Markström (2013)

# Families of snarks with $\pi = 5$

The windmill snark $W_{34}$ gave rise to several infinite families of snarks with $\pi = 5$:

- Windmill snarks of Esperet and Mazzuoccolo (2014)
- A similar family of Chen (2016)



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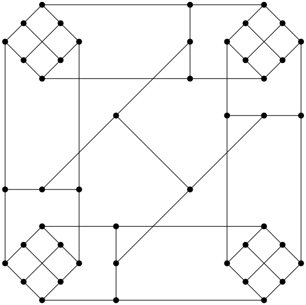
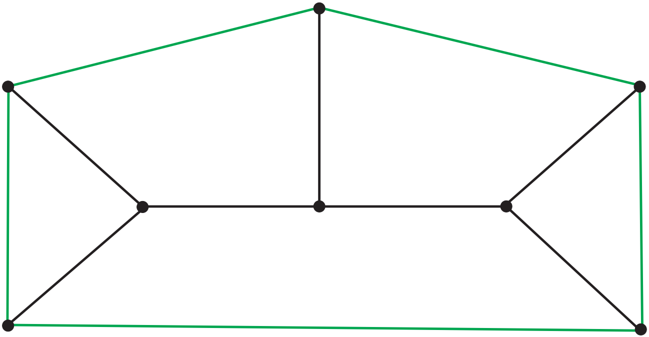


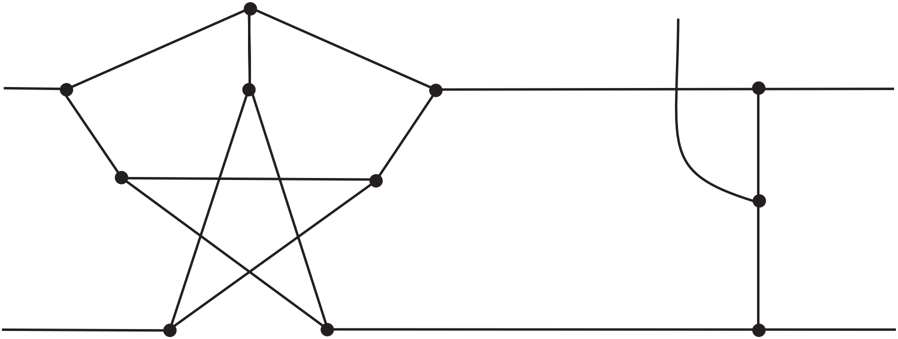
- Treelike snarks of Abreu, Kaiser, Labbate & Mazzuoccolo (2016)

# Treelike snarks



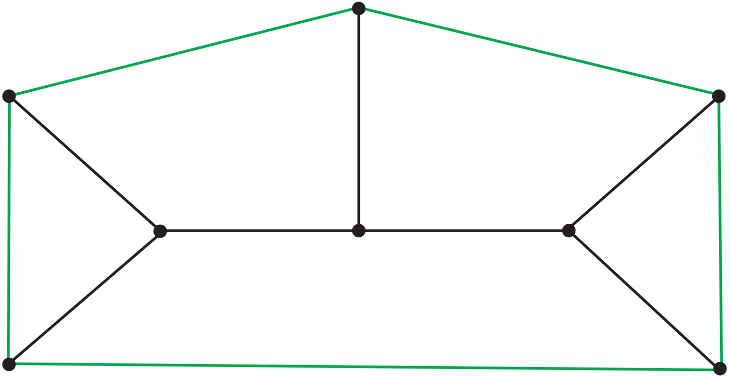
Halin graph

# Treelike snarks



Petersen fragment

# Treelike snarks



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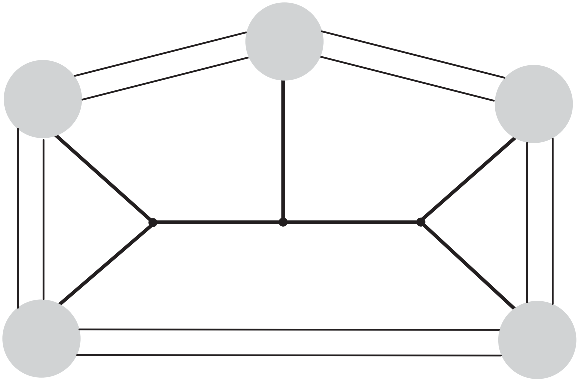

# Treelike snarks

The proof that treelike snarks have $\pi \geq 5$ heavily depends on computer-aided arguments.

### 8.1 The pattern set of the Petersen fragment

The pattern set of $F_0$ (42 patterns):

| | | |
|---|---|---|
| A A AB AC AD | A B CD AB AB | A BC D BC BC |
| A A AB C D | A B CD AC AC | A BC D BD BD |
| A AB A AC AD | A B CD C C | A BC D D D |
| A AB A BC BD | A B CD CD CD | AB AB AB AC AD |
| A AB AC A AD | A BC A AB BD | AB AC AB AB AD |
| A AB AC B BD | A BC B AB AD | AB AC AB BC CD |
| A AB AC C CD | A BC B BC CD | AB AC AD A A |
| A B AB AB CD | A BC BD A AB | AB AC AD AB AB |
| A B AB AC BD | A BC BD BC C | AB AC AD AD AD |
| A B AC A D | A BC BD BD D | AB AC AD B B |
| A B AC AB BD | A BC D A A | AB AC AD BC BC |
| A B C A AD | A BC D AB AB | AB AC AD BD BD |
| A B C C CD | A BC D AD AD | AB AC AD D D |
| A B CD A A | A BC D B B | AB CD AC AB BC |

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We present a covering $\mathcal{C} = \{P_1, P_2, P_3, P_4\}$ of a cubic graph $G$ with four perfect matching by the mapping

$$\xi_{\mathcal{C}} \colon E(G) \to \mathbb{Z}_2^4$$

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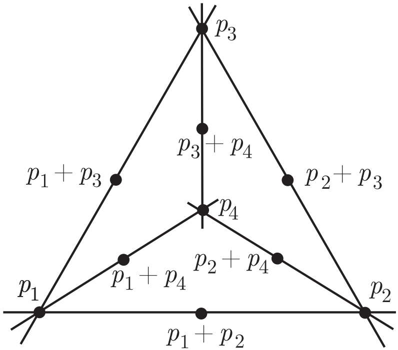
where $\quad \xi_{\mathcal{C}}(e)_i = 1 \iff e \notin P_i.$

- $\xi_{\mathcal{C}}$ is a flow.
- The three values around every vertex of $G$ form a line in $\mathbb{P}_3(\mathbb{F}_2) = PG(3, 2)$.

# Generalisation: theory of tetrahedral flows

E. Máčajová & MŠ: Cubic graphs that cannot be covered with four perfect matchings, JCTB 150 (2021), 144–176.




Tetrahedron spanned by four points of $\mathbb{P}_3(\mathbb{F}_2)$ in general position

# Tetrahedra in $\mathbb{P}_3(\mathbb{F}_2)$

The 3-dimensional projective space $\mathbb{P}_3(\mathbb{F}_2) = PG(3,2)$ over $\mathbb{F}_2$ is a pair $(P, L)$ where

- $P = \mathbb{Z}_2^4 - \{0\}$ (points)
- $L = \{$ 3-element subsets $\{x, y, z\}$ of $P$ with $x + y + z = 0$ $\}$ (lines)

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## Definition

A tetrahedron $T(p_1, p_2, p_3, p_4)$ in $PG(3,2)$ is a configuration of six lines and ten points spanned by four points $p_1$, $p_2$, $p_3$, and $p_4$ in general position.

# Flows on graphs

## Definition

A flow on a graph $G$ is a mapping $\phi\colon D(G) \to A$, where $A$ is an abelian group, such that

- $\phi(x^{-1}) = -\phi(x)$ for every dart $x \in D(G)$
- $\sum_{x \in D(v)} \phi(x) = 0$ (Kirchhoff's law).

A flow $\phi$ is nowhere-zero if $\phi(x) \neq 0$ for each dart $x \in D(G)$.

If $A \cong \mathbb{Z}_2^n$, we can regard a flow as a mapping $\phi\colon E(G) \to A$.

# Tetrahedral flows

## Definition

A tetrahedral flow, more specifically a $T$-flow, on a cubic graph $G$ is a mapping

$$\phi\colon E(G) \to P(T)$$

to the point set $P(T)$ of a tetrahedron $T$ s. t., at each vertex $v$ of $G$, *the three values that meet at $v$ form a line of $T$* (Kirchhoff's law).

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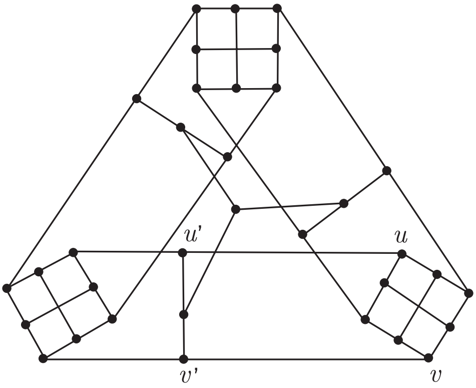
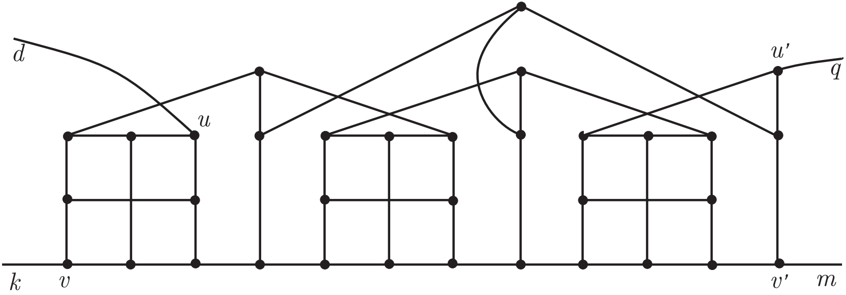
## Theorem (Máčajová & S.)

*There is a 1-to-1 correspondence between coverings of $G$ with four perfect matchings and $T$-flows, where $T$ is any fixed tetrahedron in $\mathbb{P}_3(\mathbb{F}_2)$.*

# The windmill snark $W_{34}$ has no tetrahedral flow

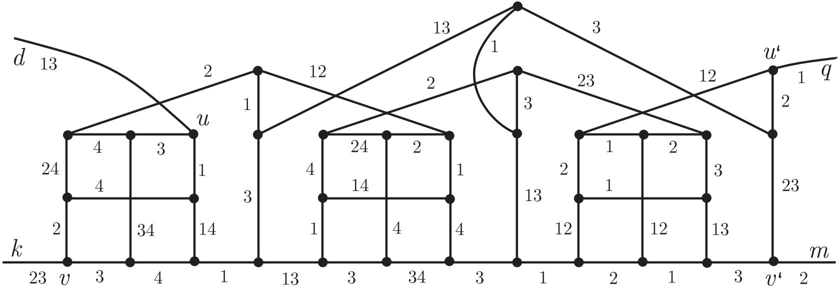


# The dipole $D_{34}$



# Tetrahedral flows through $D_{34}$

Transition $\{p_1 + p_3, p_2 + p_3\} \to \{p_1, p_2\}$:



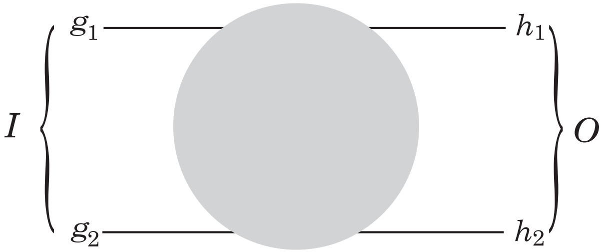

(encoding of colours: $i \mapsto p_i$, $ij \mapsto p_i + p_j$)

# Transition relation for dipoles

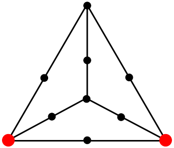
## Definition

A dipole $D = D(I, O)$ admits a transition $\{x, y\} \to \{x', y'\}$, if there exists a tetrahedral flow $\xi$ on $D$ such that

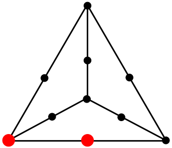
- $\{\xi(g_1), \xi(g_2)\} = \{x, y\}$, and
- $\{\xi(h_1), \xi(h_2)\} = \{x', y'\}$.



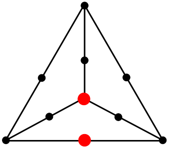
# Geometric shapes of point pairs



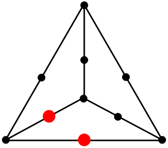
line segment



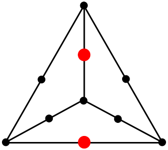
half–line



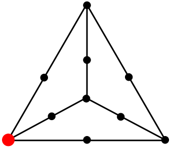
altitude



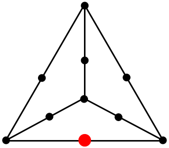
angle



axis



double point

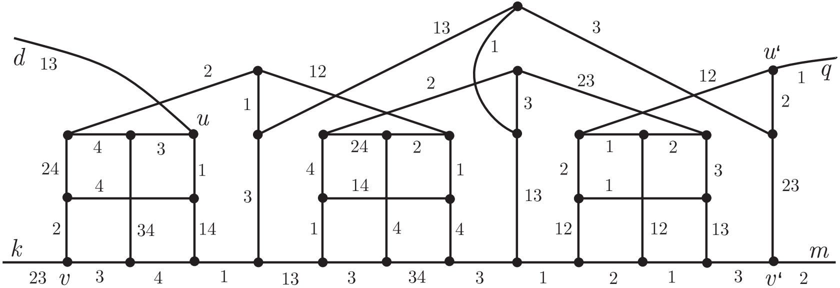


double point

The (merged) set of shapes $\Sigma = \{\mathtt{ls}, \mathtt{hl}, \mathtt{alt}, \mathtt{ang}, \mathtt{ax}, \mathtt{dpt}\}$.

# Tetrahedral flows through $D_{34}$

Transition `ang` $\to$ `ls`:




(encoding of colours: $i \mapsto p_i$, $ij \mapsto p_i + p_j$)

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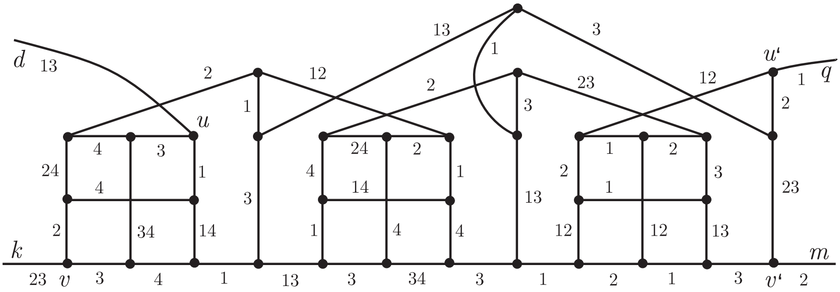


(encoding of colours: $i \mapsto p_i$, $ij \mapsto p_i + p_j$)

## Observation

Every transition through $D_{34}$ is of the form `ang` $\to$ `ls`.
Consequently, $\pi(W_{34}) \geq 5$.

# Collinearity destroying dipoles

**Theorem**

*Let $X$ be a dipole obtained from a cubic graph with $\pi \geq 5$ by removing two adjacent vertices, and let $\{x, y\} \to \{x', y'\}$ be a non-degenerate transition through $X$. Then at most one of the pairs $\{x, y\}$ and $\{x', y'\}$ is collinear in $T$.*

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We call $X$ a such a dipole a collinearity destroying dipole, or briefly a decollineator.

**Theorem**

*The following statements are equivalent for a dipole $X$:*

- *$X$ is a decollineator.*
- *$X$ has no transitions of type*
  $\mathtt{ls} \to \mathtt{ls}$ *or* $\mathtt{hl} \to \mathtt{hl}$.

# Admissible transitions

**Theorem**

*All transitions through an arbitrary dipole $X$ have the form*

$$\mathtt{s} \to \mathtt{s}$$

*for some* $\mathtt{s} \in \Sigma$ *except possibly those of the form*

$$\mathtt{ls} \to \mathtt{ang} \quad \textit{or} \quad \mathtt{ang} \to \mathtt{ls}.$$

*In particular, all transitions involving a half-line, an altitude, an axis, or a double point must have one of the following forms:*

$$\mathtt{hl} \to \mathtt{hl}, \quad \mathtt{alt} \to \mathtt{alt}, \quad \mathtt{ax} \to \mathtt{ax}, \quad \textit{and} \quad \mathtt{dpt} \to \mathtt{dpt}.$$

# Results obtained by using the geometric theory

- New constructions of snarks with $\pi \geq 5$
- NP-completeness of determining the perfect matching index of a nontrivial snark
- Disproof of a conjecture about real-valued flows on snarks $\pi \geq 5$ [Abreu, Kaiser, Labbate, Mazzuoccolo (2016)], [Fiol, Mazzuoccolo, Steffen (2018)]
- Disproof of a conjecture about the length of a shortest cycle cover of cubic graphs [Brinkmann, Goedgebeur, Hägglund, Markström (2013)]

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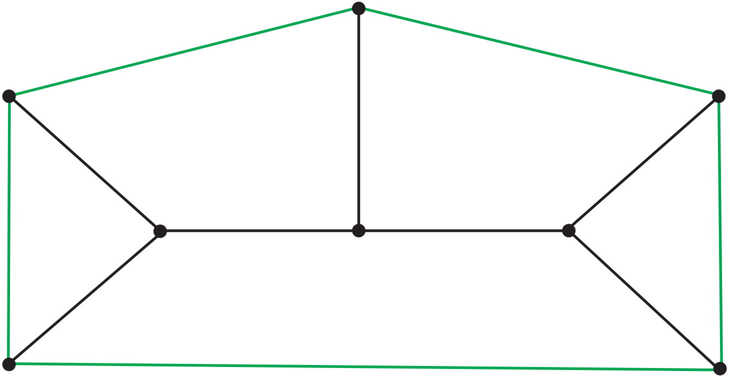

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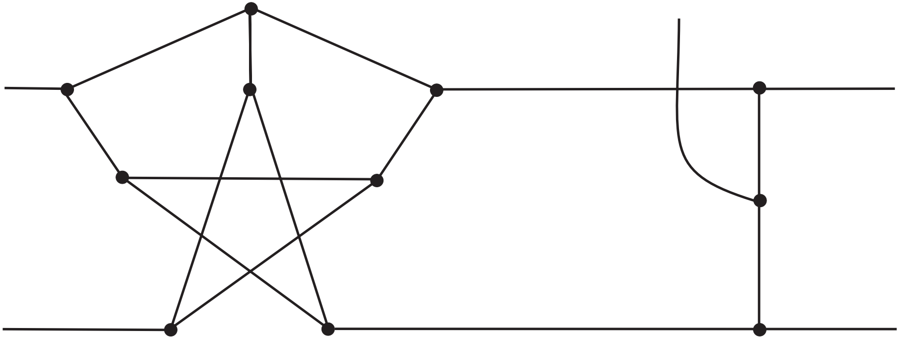

Petersen fragment $\rightarrow$ general Halin fragment

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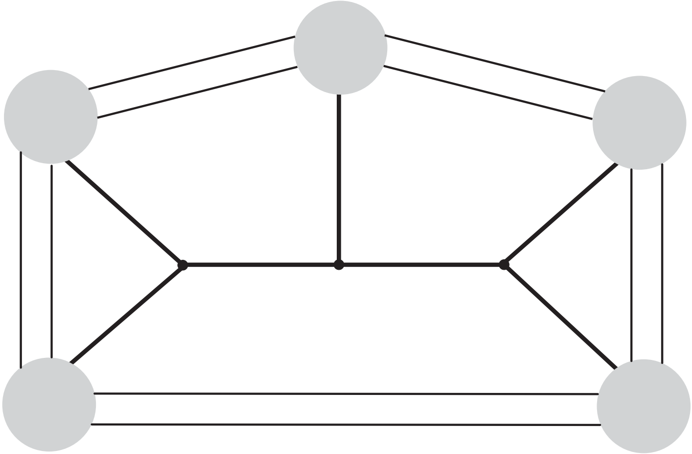

A Halin snark

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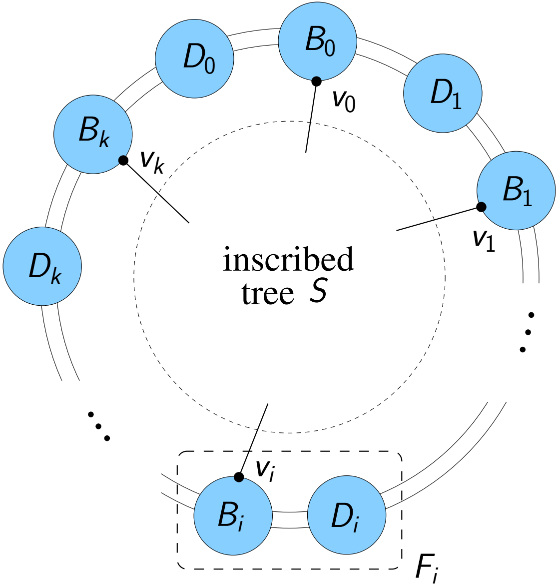

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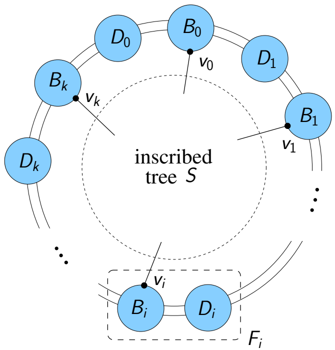

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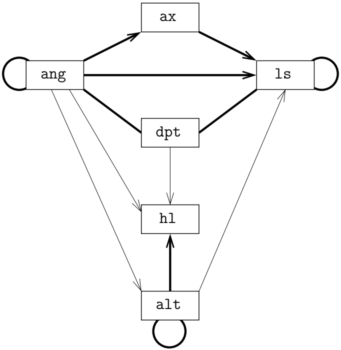

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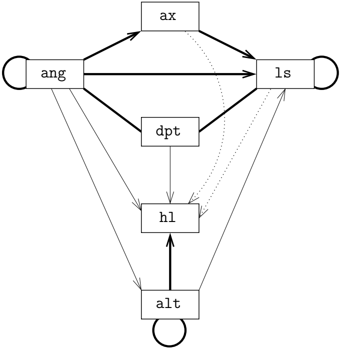
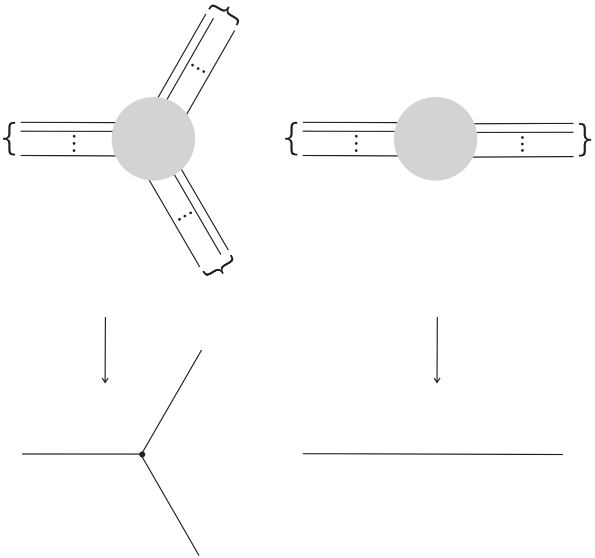


# Construction 2: Superposition



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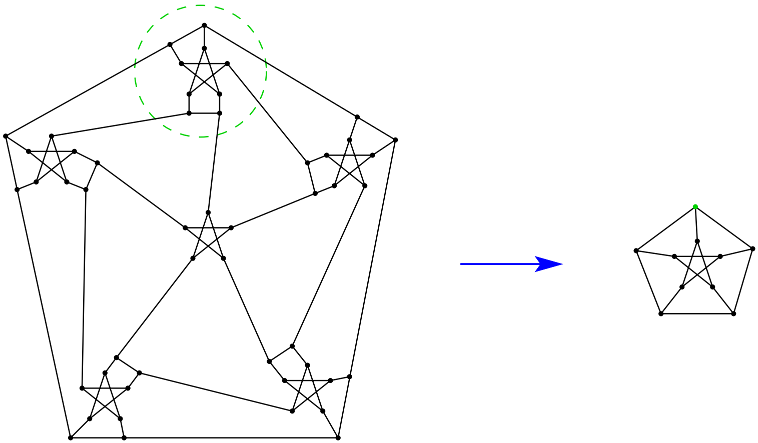

# Superposition mappings

Let $G$ and $H$ be graphs.

A graph mapping $f\colon G \twoheadrightarrow H$ is a mapping from a subdivision $G'$ of $G$ onto a subdivision $H'$ of $H$ s.t.

- vertex $\mapsto$ vertex
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We assume $G$ and $H$ to be cubic

# Superposition mappings and flows

- Every $f\colon G \to H$ is continuous if graphs are regarded as 1-dimensional cell complexes. $\Longrightarrow$
- For every abelian group, an $A$-flow $\phi$ on $G$ induces an $A$-flow $\phi_*$ on $H$.
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- However: $\phi$ nowhere-zero $\not\Longrightarrow$ $\phi_*$ nowhere-zero
  $\phi$ tetrahedral $\not\Longrightarrow$ $\phi_*$ tetrahedral
- We need control over the induced flows

# Superposition mappings and tetrahedral flows

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A superposition mapping $f\colon G \twoheadrightarrow H$ will be called $T$-continuous if for every tetrahedral flow $\phi$ on $G$ the induced $\mathbb{Z}_2^4$-flow $\phi_*$ is a again tetrahedral.

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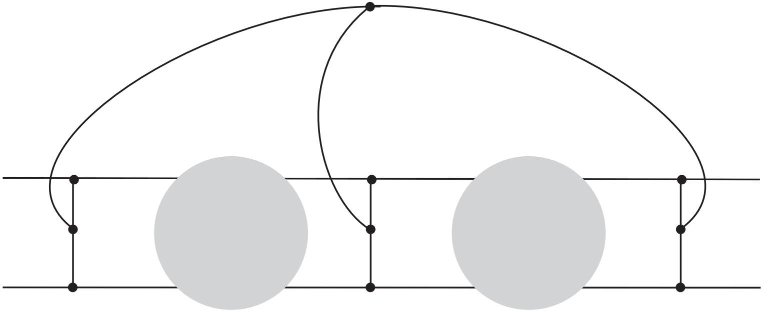
$T$-Continuous mappings can be easily constructed by choosing suitable supervertices and superedges.

# Superedges: Halin dipoles

**Theorem**

*Every transition through a Halin dipole has one of the following forms:*

$$\texttt{ls} \to \texttt{ls} \quad \textit{or} \quad \texttt{hl} \to \texttt{hl}.$$



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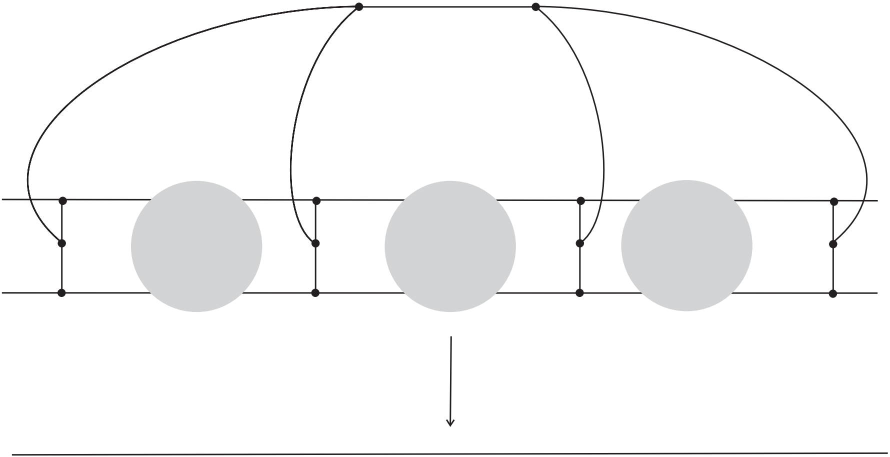
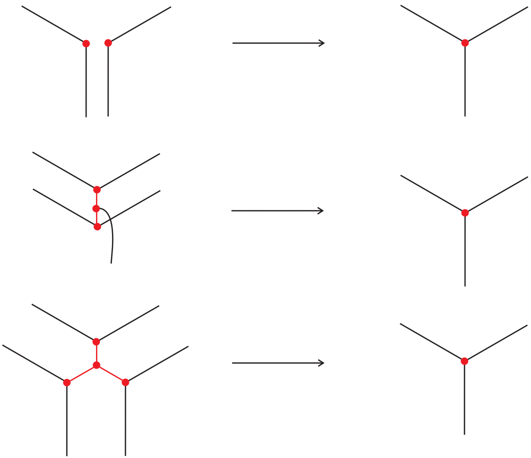


Induced flow:

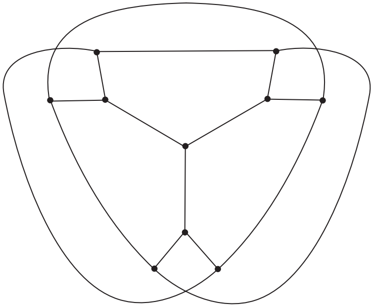
- half-line $\{p_1, p_1 + p_2\} \mapsto p_1 + p_1 + p_2 = p_2$ corner point
- line segment $\{p_1, p_2\} \mapsto p_1 + p_2$ midpoint

# Supervertices: from bipartite graphs



For a supervertex we take a cubic bipartite graph$-K_{1,3}$
with connectors corresponding to the leaves of $K_{1,3}$.

# Small examples of superposition with $\tau \geq 5$



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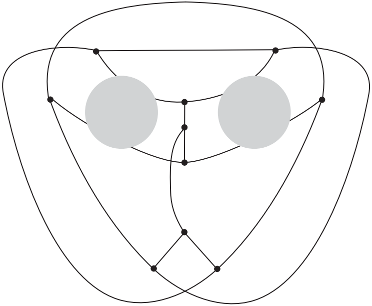

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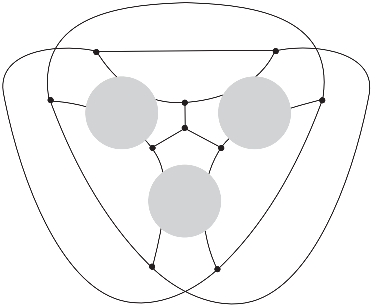

# Superposition using different tetrahedra

**Theorem**

*There exist infinitaly many dipoles, whose transition relation is*

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- angle $\{p_1 + p_2, p_1 + p_3\} \mapsto p_2 + p_3$ midpoint

Both $p_1 + p_2 + p_3$ and $p_2 + p_3$ are points of the tetrahedron dual to $T$!

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**Definition**

Let $T = T(p_1, p_2, p_3, p_4)$ and let $b = p_1 + p_2 + p_3 + p_4$ ('barycentre'). The tetrahedron dual to $T$ is defined by

$$T^* = T(p_1 + b, p_2 + b, p_3 + b, p_4 + b).$$

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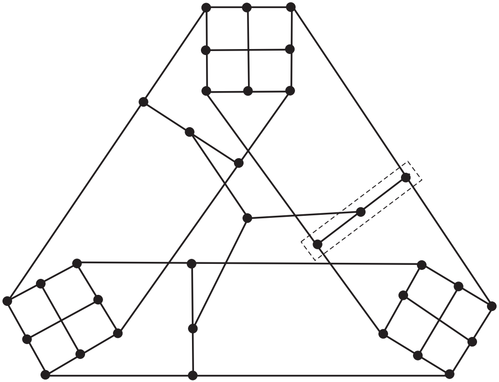

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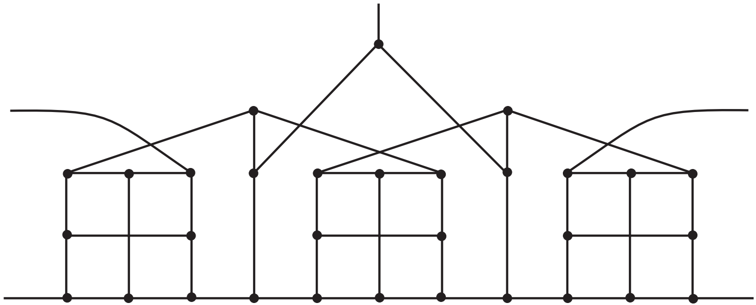

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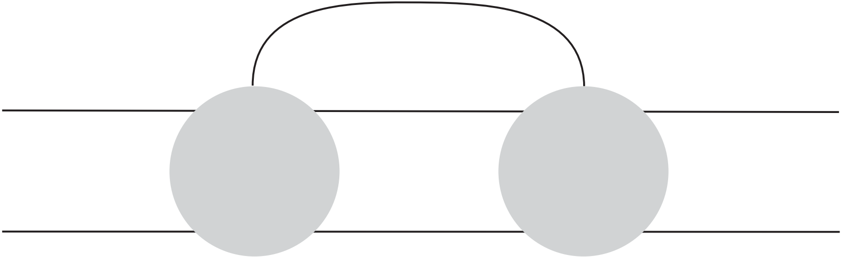

**Observation**

Each transition through this dipole has the form

$$\texttt{alt} \to \texttt{alt} \quad \text{or} \quad \texttt{ang} \to \texttt{ang}.$$

## Real-valued flows on snarks with $\pi \geq 5$

### Definition

- A nowhere-zero real-valued $r$-flow on a graph $G$, where $r \geq 2$ is a real number, is an $\mathbb{R}$-flow $\phi$ such that
$$|\phi(x)| \in [1, r-1]$$
for each dart $x \in D(G)$.
- The real flow number $\Phi_{\mathbb{R}}(G)$ of $G$ is the infimum of all $r \in \mathbb{R}$ such that $G$ has has a nowhere-zero $r$-flow.

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By Tutte's 5-flow conjecture, $\Phi_{\mathbb{R}}(G) \leq 5$.

# Real-valued flows on snarks with $\pi \geq 5$: a conjecture

Abreu, Kaiser, Labbate, Mazzuoccolo (2016) and
Fiol, Mazzuoccolo, Steffen (2018) conjectured that

"*. . . snarks critical with respect to Berge's conjecture are also critical with respect to Tutte's* 5*-flow conjecture* ."

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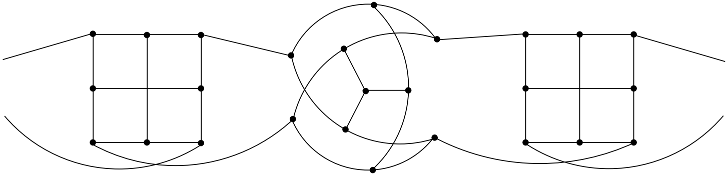
**Theorem (Máčajová & S.)**

*Every Halin snark has $\pi(G) \geq 5$.*

# Disproving the conjecture: construction

Let $G$ be a bipartite cubic graph. Construct a new graph $\tilde{G}$ as follows:

- replace each vertex of $G$ with a pair of vertices
- replace each edge of $G$ with the following dipole $X$



- Assemble $\tilde{G}$ in such a way that the input of $X$ goes to one partite set and the output goes to the other.

# Disproving the conjecture: theorem

**Theorem (Máčajová & S.)**

*If $G$ is a bipartite cubic graph, then* $\pi(\tilde{G}) \geq 5$ *and*

$$4 + \frac{1}{2} < \Phi_{\mathbb{R}}(G) \leq 4 + \frac{2}{3}.$$

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**Proof.**

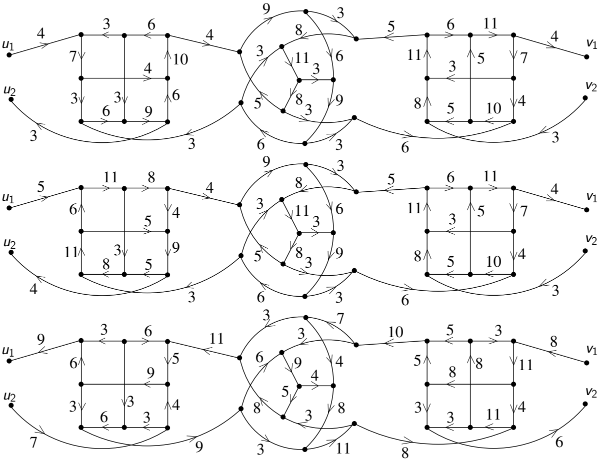
- The proof of the lower bound uses the structure of $X$ and its transition relation

$$\{\mathtt{ls} \longleftrightarrow \mathtt{ang}, \mathtt{ang} \to \mathtt{ang}, \mathtt{alt} \to \mathtt{alt}\}.$$

- The upper bound is established by finding a 12-flow with absolute value not smaller than 3.

□

# Disproving the conjecture: the flow




- $\Phi_{\mathbb{R}}(\tilde{G}) \leq 4 + \frac{2}{3}$

# Snarks with $\pi \geq 5$ and the short cycle cover conjecture

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- Although an optimisation problem in nature, it has strong structural implications.
- Implies the cycle double cover conjecture (Jamshy & Tarsi, 1992).
- Related to Berge's conjecture, etc.

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- Among the nontrivial snarks up to 36 vertices only two snarks fail to have a cycle cover of length $\frac{4}{3} \cdot |E(G)|$: the Petersen graph and $W_{34}$.

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**Conjecture (Brinkmann, Goedgebeur, Hägglund, Markström, 2013)**

For every cyclically 4-edge-connected cubic graph $G$ with $m$ edges

$$scc(G) \leq \frac{4}{3} \cdot m + o(m).$$

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Further evidence for this conjecture can be found in results by Hägglund & Markström (2013) and Steffen (2015).

# Disproving the conjecture

**Theorem (Máčajová & MS)**

*There exists a family $(G_n)_{n\geq 2}$ of cyclically 4-edge-connected cubic graphs of order $46n$ such that*

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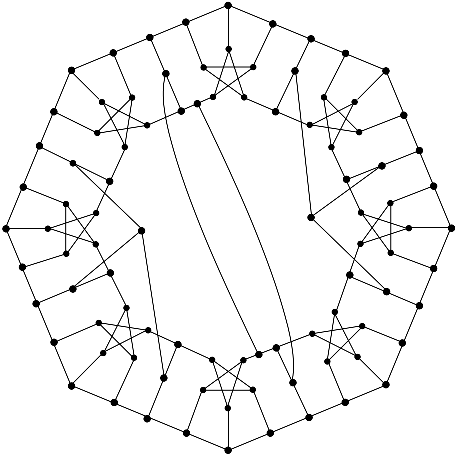
**Proof.**

- The family $(G_n)_{n\geq 2}$ was found with the help of geometric machinery.
- Bounding the length of the shortest cycle cover requires a careful structural analysis.

□

# The smallest member of the family $(G_n)$ on 92 vertices



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- Deciding whether there exist nontrivial snarks of order 38 and 40 with $\pi \geq 5$ seems to be quite challenging.

# Final remarks

**Theorem**

*There exist rich families of nontrivial snarks with* $\pi \geq 5$ *on* $n$ *vertices for every* $n \geq 42$.

- Deciding whether there exist nontrivial snarks of order 38 and 40 with $\pi \geq 5$ seems to be quite challenging.
- The only known cyclically 5-edge-connected snark with $\pi \geq 5$ remains the Petersen graph. Any other examples?

Thank you for listening