Skew morphisms of finite groups

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Slovak University of Technology

Algebraic Graph Theory International Webinar
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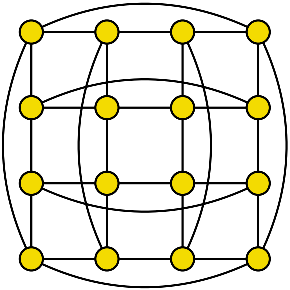
Part 1: Introducing regular Cayley maps



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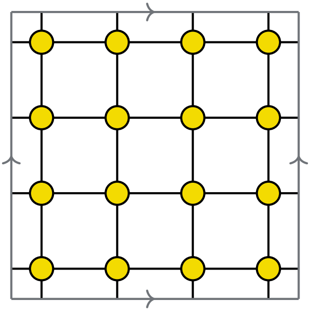

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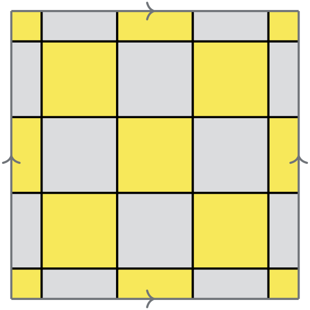

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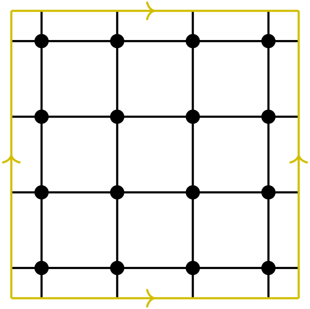

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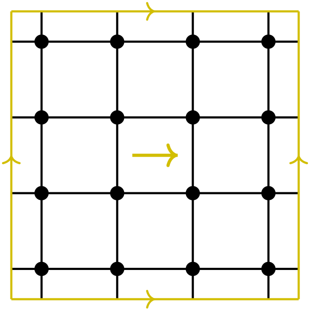

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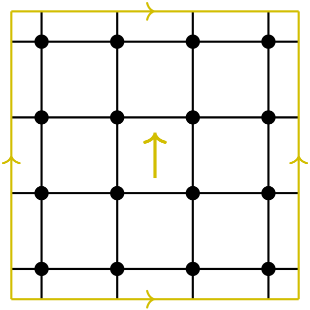

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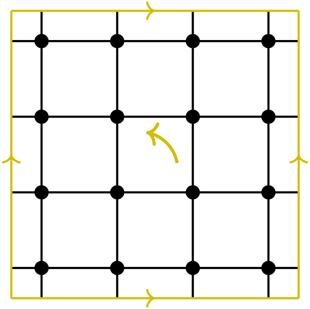

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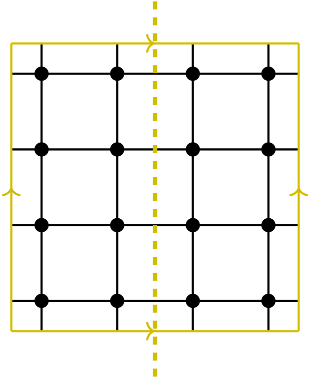
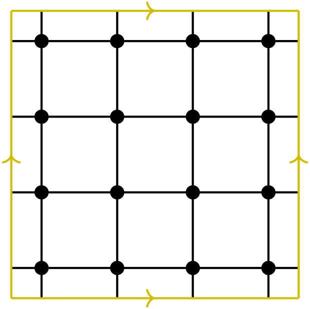


An **automorphism** of a map $\mathcal{M}$ is a bijection that preserves the map structure. Orientation-preserving automorphisms form a subgroup $\mathrm{Aut}^+(\mathcal{M})$ of $\mathrm{Aut}(\mathcal{M})$.



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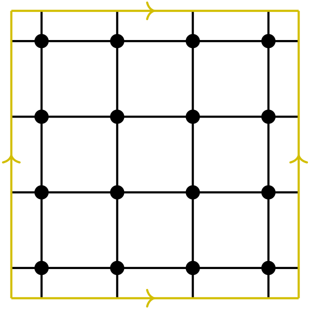

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If $|\mathrm{Aut}^+(\mathcal{M})| = |\mathcal{D}(\mathcal{M})|$ we say that $\mathcal{M}$ is **orientably regular**.
(Equivalently, $\mathcal{M}$ is orientably regular if $\mathrm{Aut}^+(\mathcal{M})$ is regular on darts.)
Note that the stabiliser in $\mathrm{Aut}^+(\mathcal{M})$ of every vertex is cyclic.

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We say that Γ is a Cayley graph for $\mathbf{B}$.

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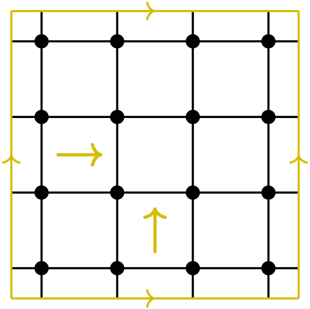
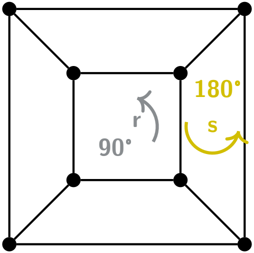
CAYLEY GRAPH $\not\Rightarrow$ AN EMBEDDING IS A CAYLEY MAP

CAYLEY MAP $\Rightarrow$ THE UNDERLYING GRAPH IS CAYLEY

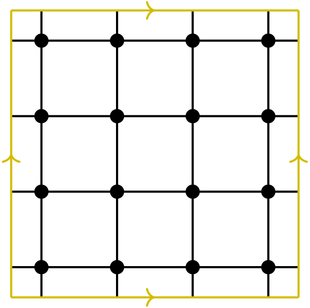


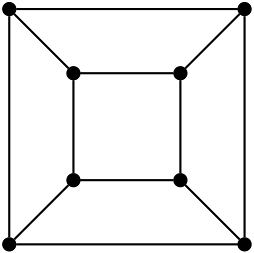
Cayley map for $\langle \uparrow, \rightarrow \rangle = \mathbf{B} \cong \mathbb{Z}_4 \times \mathbb{Z}_4$



Cayley map for $\langle \mathbf{r}, \mathbf{s} \rangle = \mathbf{B} \cong \mathrm{D}_4$

A **regular Cayley map** is a Cayley map that is also orientably regular.





Part 2: The structure of $\mathrm{Aut}^+(\mathcal{M})$ for regular Cayley maps

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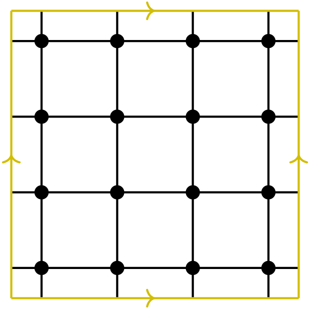
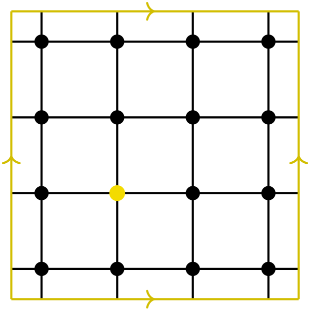
There is a subgroup **B** of $\mathrm{Aut}^+(\mathcal{M})$ regular on the vertices of $\mathcal{M}$.

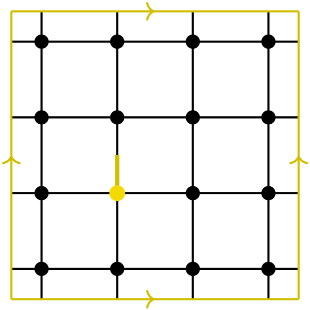
The structure of $\mathrm{Aut}^+(\mathcal{M})$ for regular Cayley maps





Let **C** be a vertex stabiliser in $\mathrm{Aut}^+(\mathcal{M})$, then:

C is cyclic and core-free in $\mathrm{Aut}^+(\mathcal{M})$,



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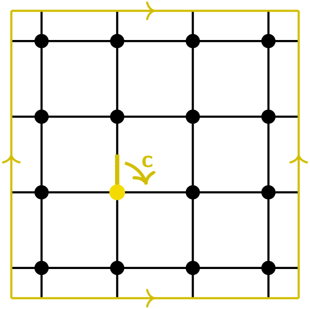

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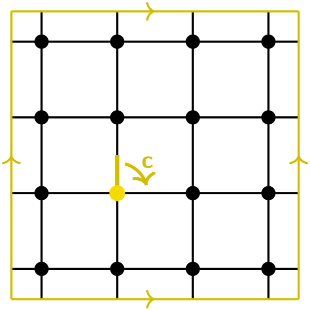
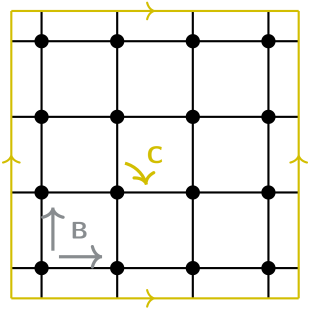


Let **C** be a vertex stabiliser in $\mathrm{Aut}^+(\mathcal{M})$, then:

C is cyclic and core-free in $\mathrm{Aut}^+(\mathcal{M})$, **C** $= \langle$**c**$\rangle$, and $|$**C**$| = \deg v$



Let $\mathbf{G} = \mathrm{Aut}^+(\mathcal{M})$, and let **B** and **C** be defined as in previous slides, then:

- **C** is cyclic and core-free in **G**
- $\mathbf{B} \cap \mathbf{C} = \{1_G\}$
- $|B||C| = |V(\mathcal{M})||\deg v| = |G|$
- $\mathbf{G} = \mathbf{BC}$

Part 3: Skew product groups and skew morphisms

A group G is a **skew product group** for B if there exists C such that:

- $G = BC$
- $B \cap C = \{1_G\}$
- C is cyclic and core-free in G

We say that C is a skew complement (of B in G).

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Any pair (B, c) such that c generates a skew complement of B in G is called a **skew generating pair** for G.

Let (B, c) be a skew generating pair of B in G, and let $C = \langle c \rangle$. Note that every $g \in G$ is uniquely expressible in a form $g = bc'$ with $b \in B$ and $c' \in C$.

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Then for every $b \in B$ there exists a unique $b' \in B$ and a unique $j \in \{1, \ldots, |C| - 1\}$ such that $cb = b'c^j$.
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$$= \cdots = \varphi(a)\varphi^{\pi(a)}(b)c^{\sum_{i=0}^{\pi(a)-1} \pi(\varphi^i(b))}$$

A **skew morphism** of a group B is a permutation φ of B such that

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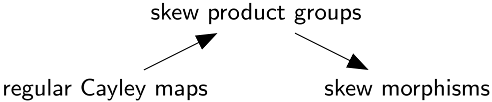
Each value i_a is unique modulo $|\langle\varphi\rangle|$.
This gives a function $\pi\colon B \to \{1, \ldots, |\langle\varphi\rangle| - 1\}$, the **power function** of φ.

Regular Cayley maps for **B**, skew product groups for **B**, skew morphisms of **B**:



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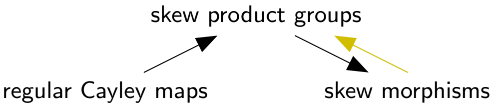

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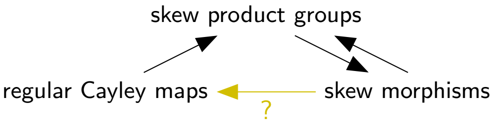

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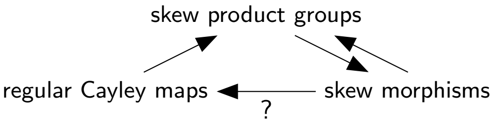

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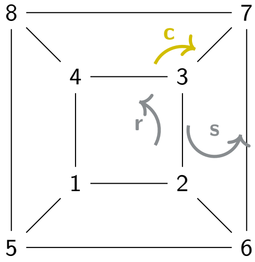
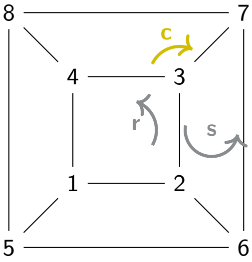
Part 4: Example



$\mathbf{r} = (1,2,3,4)(5,6,7,8)$, $\mathbf{s} = (1,8)(2,7)(3,6)(4,5)$ and $\mathbf{c} = (1,8,6)(2,4,7)$



r $= (1, 2, 3, 4)(5, 6, 7, 8)$, **s** $= (1, 8)(2, 7)(3, 6)(4, 5)$ and **c** $= (1, 8, 6)(2, 4, 7)$

Group $G = \langle r, s, c\rangle$ is a skew product group for $\mathrm{D}_4 \cong \langle r, s\rangle$, with a skew complement $C = \langle c\rangle$.
Also $(\langle r, s\rangle, c)$ is a skew generating pair for G.

$$r = (1,2,3,4)(5,6,7,8),\ s = (1,8)(2,7)(3,6)(4,5),\ c = (1,8,6)(2,4,7)$$

We will compute $\varphi(r)$:

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We have $\varphi = (r, r^3s, r^3)(r^2, s, r^2s)$, and $\pi(r) = \pi(r^3s) = \pi(r^3) = 2$.

Recall that $\varphi = (r, r^3s, r^3)(r^2, s, r^2s)$. The underlying graph of $\mathcal{M}$ will be $\mathrm{Cay}(\mathrm{D}_4, \{r, r^3s, r^3\})$, the local clockwise orientation of darts around each vertex is consistent with φ.

1

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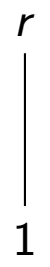

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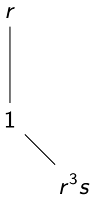

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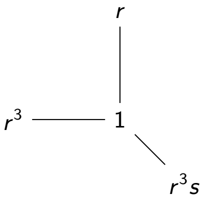

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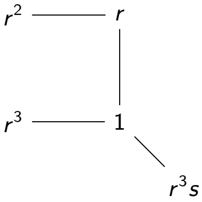

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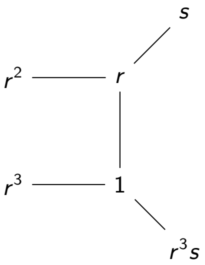

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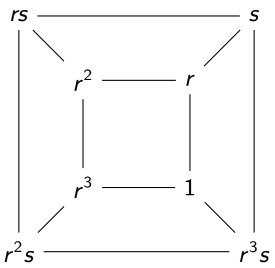

Part 4: What is currently known

B	Regular Cayley maps	Skew product groups	Skew morphisms
$\mathbb{Z}_n$			
$\mathbb{Z}_p$			
$\mathbb{Z}_{pq}$			
$\mathbb{Z}_{p^e}$			
$\mathbb{Z}_{2^e}$			
D_n			
simple groups			
$\mathrm{Sym}(n)$, $\mathrm{PGL}(2, p)$, ...			
non-abelian characteristically simple groups			

Kovács, Nedela, 2011   Conder, Tucker, 2014
Kovács, Nedela, 2017   Du, Hu, Lucchini, 2019
Kovács, Kwon, 2021   B, Conder, Verret, 2021+
Chen, Du, Li, 2021+   Kan, Kovács, Kwon, 2021+

B	Regular Cayley maps	Skew product groups	Skew morphisms
$\mathbb{Z}_n$	✓		
$\mathbb{Z}_p$			
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$\mathbb{Z}_p$	✓	✓	✓
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$\mathbb{Z}_{p^e}$			
$\mathbb{Z}_{2^e}$			
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$\mathbb{Z}_n$	✓		
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$\mathbb{Z}_{pq}$	✓	(✓)	✓
$\mathbb{Z}_{p^e}$			
$\mathbb{Z}_{2^e}$			
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$\mathbb{Z}_{2^e}$			
D_n			
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Part 5: Skew morphisms of simple groups

The **core** of B in G is the maximal normal subgroup of G contained in B.

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Let $G = BC$ be a skew product group for a (non-abelian) **simple** group B:

- B is **normal** in G
- B is **core-free** in G

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Since B is core-free, G is a permutation group on the coset space $(G : B)$ with a regular cyclic subgroup C.

A group is **monolithic** if it has a unique minimal subgroup, and this subgroup is not abelian.

All non-abelian simple and almost simple groups are monolithic. The smallest example that is not almost simple is $(\mathrm{Alt}(5) \times \mathrm{Alt}(5)) \rtimes \mathbb{Z}_2$.

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B, Conder, Verret '21+

Let G be a group with core-free subgroups B and C such that $G = BC$, where B is monolithic with monolith A, and C is cyclic. Then G has a unique minimal normal subgroup N, and this normal subgroup N contains A.

B, Conder, Verret '21+

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By Li, Praeger, 2012 we have the following:

B, Conder, Verret '21+

Let $G = BC$ be a skew product group of a monolithic group B. If B is core-free in G, then one of the following occurs:

(1) $G \cong \mathrm{Alt}(n)$, $B \cong \mathrm{Alt}(n-1)$ and $C \cong \mathrm{C}_n$ for some odd $n \geq 7$,
(2) $G \cong \mathrm{PSL}(2, 11)$, $B \cong \mathrm{Alt}(5)$ and $C \cong \mathrm{C}_{11}$,
(3) $G \cong \mathrm{M}_{23}$, $B \cong \mathrm{M}_{22}$ and $C \cong \mathrm{C}_{23}$,
(4) $G = \mathrm{Sym}(n)$, $B = \mathrm{Sym}(n-1)$ and $C = \mathbb{Z}_n$, with $n \geq 6$,
(5) $G = \mathrm{M}_{11}$, $B = \mathrm{M}_{10}$ and $C = \mathbb{Z}_{11}$.

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B, Conder, Verret '21+

Let $G = BC$ be a skew product group of a ~~monolithic~~ non-abelian simple group B. If B is core-free in G, then one of the following occurs:

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(5) $G = \mathrm{M}_{11}$, $B = \mathrm{M}_{10}$ and $C = \mathbb{Z}_{11}$.

B, Conder, Verret '21+

Groups $\mathrm{Alt}(5)$, $\mathrm{Alt}(6)$ and M_{22} admit 240, 1440 and 1774080 proper skew morphisms while, for even $n \geq 8$, $\mathrm{Alt}(n)$ admits $n!$ proper skew morphisms. No other non-abelian simple group admit a proper skew morphism.

B, Conder, Verret '21+

Every proper skew morphism of a non-abelian finite simple group B gives rise to a non-balanced regular Cayley map for B. Moreover, every non-balanced regular Cayley map for a non-abelian finite simple group is one for either $\mathrm{Alt}(5)$ with valency 11, or M_{22} with valency 23, or $\mathrm{Alt}(n)$ with valency $n+1$ for some even $n \geq 6$.

Part 6: Skew morphisms of (small) cyclic groups

Lucchini '98

Let C be a cyclic proper subgroup of a group G. If C is core-free in G, then $|C| < |G : C|$.

Let G be a skew product group for a cyclic group B:

- $G = BC$
- $B \cap C = \{1_G\}$
- C is cyclic and core-free in G

Lucchini '98

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Let G be a skew product group for a cyclic group B:

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- $B \cap C = \{1_G\}$
- C is cyclic and core-free in G

Then $|C| < |G : C| = |B|$, and hence $|\varphi| < |B|$.

Let $B = \langle b \rangle$, and let φ be a skew morphism of B.

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Let $\bar{\ }$ be the canonical projection $G \to G/K$.

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Then $\overline{G} = \overline{B}\,\overline{C}$ is a skew product group for C $(\cong \overline{C})$.

A **quotient** of φ is the skew morphisms $\overline{\varphi}$ of C induced by $(C, \overline{b})$.

Previous largest complete list of skew morphisms of cyclic group goes up to order 60.

$$\varphi$$
$$\big\downarrow$$
$$\overline{\varphi}$$

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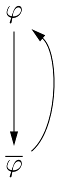

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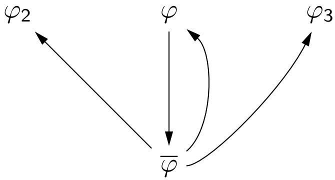

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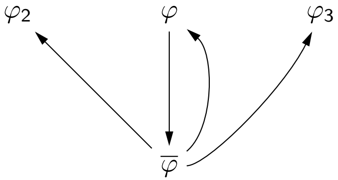

Using quotients we found skew morphisms of all cyclic groups up to order 161.
The list is available at `https://drive.google.com/file/d/1vTNXwaCqdaoZjh1MP-5TBbbTQ9Q44fIy`

Some open problems

Let $\alpha \in \mathrm{Aut}(B)$ and $\varphi, \psi \in \mathrm{Skew}(B)$, then:

- ✗ $\varphi\psi \in \mathrm{Skew}(B)$
- ✗ $\varphi\alpha \in \mathrm{Skew}(B)$
- ✓ $\alpha^{-1}\varphi\alpha \in \mathrm{Skew}(B)$
- ✗ $\psi^{-1}\varphi\psi \in \mathrm{Skew}(B)$

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Q: Does there exist a group B which admits a proper skew morphism that is central in $\mathrm{Aut}(B)$?

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Q: Does there exist a group B which admits a proper skew morphism that is central in $\mathrm{Aut}(B)$?

Q: Does there exist a skew morphism φ of a group B such that B is core-free in the skew product group induced by φ, and φ has non-trivial centraliser in $\mathrm{Aut}(B)$?

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Q: Does there exist a group B which admits a proper skew morphism that is central in $\mathrm{Aut}(B)$?

Q: Does there exist a skew morphism φ of a group B such that B is core-free in the skew product group induced by φ, and φ has non-trivial centraliser in $\mathrm{Aut}(B)$?

Q: Does every non-simple almost simple group admit a proper skew morphism?