ON THE DEFECT OF VERTEX-TRANSITIVE GRAPHS OF GIVEN DEGREE AND DIAMETER

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Overview

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Degree-Diameter Problem

Regular Δ-valent graph with diameter D is a (Δ, D)-graph.

For given degree $\Delta \geq 3$ and diameter $D \geq 2$ determine (the size of) the largest (Δ, D)-graph.

There are versions of the problem for vertex-transitive, Cayley, circulant, . . . , graphs.

Moore bound

The size of any (Δ, D)-graph G is at most $1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-2}$. This is the *Moore bound*

$$M(\Delta, D) = 1 + \Delta \frac{(\Delta - 1)^D - 1}{\Delta - 2}.$$

Graphs which attain the Moore bound are *Moore graphs*.

Difference $\delta(G) = M(\Delta, D) - |V(G)|$ is the *defect* of G.

Improvements to Moore bound

Following results are well known

- Moore graphs exist only for $(3, 2)$, $(7, 2)$ and (maybe) $(57, 2)$;
- Otherwise, $\delta(G) \geq 2$;
- That's all, folks
- (even for VT graphs).

(Of course there are result for particular values (Δ, D).)

Cages

The cage problem is to determine (the size of) the smallest k-regular graph G with girth g (for $k \geq 3$ and $g \geq 4$). The extremal graphs are called *cages*.

If g is odd, then the size of any k-regular graph G with girth g is at least $M(k, \frac{g-1}{2})$ (similar bound exists also for even g).

The difference $\varepsilon(G) = |V(G)| - M(k, \frac{g-1}{2})$ is the *excess* of G.

Degree-diameter/cage duality

Theorem. *Let G be a Δ regular graph. Then, any of the following conditions follows from the remaining ones.*

$$\begin{aligned} diam(G) &= D, \\ g(G) &= 2D+1, \\ |V(G)| &= M(\Delta, D). \end{aligned}$$

The cage problem and the degree-diameter problem are considered to be "dual".

Lower bound for VT cages

In 1982 N. Biggs proved:

Theorem. *For any odd k there exist infinitely many (odd) values g such that each vertex-transitive k-regular graph with girth g has excess at least g/k.*

In 2011 R. Jajcay and J. Širáň conjectured:

Conjecture. *Let G be a vertex-transitive k-regular graph with odd girth g. If G is not a Moore graph, then the excess of G is at least $k-1$.*

These results were the main motivation for our research.

Counting cycles

Let $C_G(v, n)$ denote the number of v-rooted cycles of length n in G.

- n divides $\sum_v C_G(v, n)$;
- If $C_G(v, n)$ does not depend on v, then n divides $|V(G)|C_G(v, n)$;
- If G is vertex-transitive, then $C_G(v, n)$ does not depend on v.

Cycles in Moore graphs (Friedman)

We say that an edge is *horizontal at distance* i from a vertex v if both its endpoints have distance i from v.

Let v be a vertex in a Moore (Δ, D)-graph.

- Each horizontal edge at distance D from v lies in *exactly* one v-rooted *cycle* of length $2D + 1$.
- Each v-rooted cycle of length $2D+1$ contains *exactly* one horizontal edge at distance D from v.
- The number of v-rooted cycles of length $2D + 1$ can be computed *exactly*.

Cycles in Moore graphs (Friedman)

Lemma. *Let G be a Moore (Δ, D)-graph. Then*

$$C_G(v, 2D+1) = \frac{1}{2}\Delta(\Delta-1)^D.$$

Let us denote

$$M_C(\Delta, D) = \frac{1}{2}\Delta(\Delta-1)^D.$$

Cycles in the cage problem (Biggs)

Let v be a vertex in a regular Δ-valent graph with girth $2D+1$ and excess ε.

• Each horizontal edge at distance D from v lies in *exactly* one v-rooted *cycle* of length $2D+1$.

• Each v-rooted cycle of length $2D+1$ contains *exactly* one horizontal edge at distance D from v.

• The number of v-rooted cycles of length $2D+1$ can be *estimated* with *precision* which depends on Δ, ε but not on D.

Cycles in the cage problem (Biggs)

Lemma. *Let G be a regular Δ-valent graph with girth $2D+1$ and excess at most ε. Then*

$$0 \leq M_c(\Delta, D) - C_G(v, 2D+1) \leq \frac{1}{2}\Delta\varepsilon.$$

We are not assuming that G is vertex-transitive.

The proof is about 6 lines long.

Cycles in the degree-diameter problem

Let v be a vertex in a (Δ, D)-graph with defect δ.

- Each horizontal edge at distance D from v lies in *at least* one v-rooted *closed walk* of length $2D+1$.
- Each v-rooted cycle of length $2D+1$ form v contains *at most* one horizontal edge at distance D from v.
- Every exception to the Moore case contributes to the defect.
- Everything bad/interesting happens near distance D.
- The number of v-rooted cycles of length $2D+1$ can be *estimated* with *precision* which depends on Δ, δ but not on D.

Cycles in the degree-diameter problem

Lemma. *Let* $\Delta \geq 3$ *and* $\delta \geq 1$ *be positive integers. Let* G *be a* (Δ, D)*-graph defect at most* δ*. Then, for a sufficiently large* D*, we have*

$$|M_c(\Delta, D) - C_G(v, 2D+1)| \leq 2\Delta(\Delta-1)^2\delta^2.$$

We are not assuming that G is vertex-transitive.

The proof is about 6 pages long.

The bound can be improved, but the proof is even longer.

The condition

If G is a VT (Δ, D) graph then $2D+1$ divides $|V(G)|C_G(v, 2D+1)$.

Theorem. *Let* $\Delta \geq 3$, $\delta \geq 1$ *and* $D \geq 1 + 3\log(\delta)/\log(\Delta - 1)$ *be positive integers. If no number from the set*

$$\{(M(\Delta, D)-i)(M_C(\Delta, D)-j);\ 0 \leq i \leq \delta,\ 0 \leq |j| \leq 2\Delta(\Delta-1)^2\delta^2\}$$

is a multiple of $2D+1$*, then the defect of any vertex-transitive* (Δ, D)*-graph is greater than* δ*.*

A corollary

Corollary. *If p is a prime divisor of $2D+1$ which divides no number from the set*

$$\{\Delta(\Delta-1)^D + i;\ |i| \leq 4\Delta(\Delta-1)^2\delta^2\},$$

then the defect of any vertex-transitive (Δ, D)-graph is greater than δ.

The first result

Lemma. *Let* $\Delta \geq 3$ *and* $\delta \geq 1$ *be fixed. Then there exists a constant* K *(depending only on* Δ*) and a diameter* $D < K\delta^2 \log(\delta)$ *such that any vertex-transitive* (Δ, D)*-graph has defect greater than* δ*.*

- The smallest r such that $\Delta(\Delta-1)^{(r-1)/2} > 4\Delta(\Delta-1)^2\delta^2$.
- A prime p such that $2\Delta(\Delta-1)^{(r-1)/2} < p < 4\Delta(\Delta-1)^{(r-1)/2}$.
- $2D+1 = rp$.

Theorem. *For any degree* $\Delta \geq 3$*, there exist infinitely many* D*s such that for any VT* (Δ, D)*-graph the defect* δ *satisfies* $\delta > D^{\frac{1}{2+o(1)}}$*.*

What about fixed defect

- From the previous Theorem it follows that for any fixed $\Delta \geq 3$ and $\delta \geq 1$ the set of diameters such that each vertex-transitive (Δ, D)-graph has defect greater than δ is infinite.

- How big is this set?

- For technical reasons we will consider the complementary set

$$S(\Delta, \delta) = \{D;\ \text{some VT } (\Delta, D)\text{-graph has defect at most } \delta\}.$$

We will show that $S(\Delta, \delta)$ has zero asymptotic density.

Asymptotic density

For $A \subseteq \mathbb{N}$ let $A(n) = |\{a \in A;\ a \leq n\}|$.

The *upper asymptotic density* of A is $\overline{d}(A) = \overline{\text{lim}}\frac{A(n)}{n}$.

The *lower asymptotic density* of A is $\underline{d}(A) = \underline{\text{lim}}\frac{A(n)}{n}$.

If $\overline{d}(A) = \underline{d}(a)$ then the *asymptotic density* of A is

$$d(A) = \lim \frac{A(n)}{n}.$$

Some examples

An infinite arithmetic sequence with difference b has asymptotic density $\frac{1}{b}$.

Sets with asymptotic density 0 are considered to be small.

The set of all primes has asymptotic density 0.

There are sets which do not have asymptotic density.

For example the set of positive integers with leading digit equal to 2.

Sets with asymptotic density 0

Let $\mathcal{I}_d$ be the set of all sets with asymptotic density 0. Then

- It contains $\emptyset$.
- It is hereditary.
- It is closed under finite unions.

(• It contains all singletons.)

In other words, $\mathcal{I}_d$ is an (admissible) ideal in Boolean algebra $P(\mathbb{N})$.

Some auxiliary sets

Given integers a, b, c, d and q satisfying the property $a \neq 0 \neq c$ and $q > 2$, let

$$A(q,a,b,c,d) = \{n \in \mathbb{N};\ n \text{ is odd and } n|(aq^{(n-1)/2}+b)(cq^{(n-1)/2}+d)\}.$$

Note that $2S(\Delta,\delta)+1$ is a subset of the union of finite number of $A(\Delta-1,\Delta,b,\Delta,d)$s.

An observation

For $A \subseteq \mathbb{N}$ and prime p let

$$A_p = \{a \in A;\ p|a \text{ and } p^2 \nmid a\}.$$

Let $A = 2\mathbb{N} - 1$. Then

$$\begin{aligned} d(A) &= \frac{1}{2}, \\ d(A_2) &= 0, \\ d(A_p) &= \frac{p-1}{2p^2} \text{ (for } p \text{ odd)}, \\ \sum_p d(A_p) &= \infty. \end{aligned}$$

Niven's criterion

Theorem. *Let Q be a set of primes such that*

$$\sum_{p \in Q} \frac{1}{p} = \infty.$$

Let $A \subseteq \mathbb{N}$ be such that

$$\sum_{p \in Q} \overline{d}(A_p) < \infty.$$

Then

$$d(A) = 0.$$

A theorem of Erdős

Theorem. *Let* $\{q_i\}_{i=1}^{\infty}$ *be a set of integers such that* $q_i \nmid q_j$, *unless* $i = j$. *Then* $\sum_{i=1}^{\infty} \frac{1}{q_i \log q_i}$ *converges.*

As a special case we obtain the following:

Lemma. *If* $\{p_i\}_{i=1}^{\infty}$ *is the sequence of all primes, then the sum* $\sum_{i=1}^{\infty} \frac{1}{p_i \log p_i}$ *converges.*

The second result

Theorem. *For any $\Delta > 2$ and $\delta \geq 1$, the asymptotic density of the set of all $D \geq 2$ for which there exists a vertex-transitive (Δ, D)-graph with defect not exceeding δ is 0.*

Let $A = A(q,a,b,c,d)$. If $p \nmid acq$, then $\overline{d}(A_p) \leq \frac{2\log q}{p \log p}$.

Each A has asymptotic density 0 by Niven + Erdős. Therefore $S(\Delta, \delta)$ has asymptotic density 0, as well.

Reference: G. Exoo, R. Jajcay, M. Macaj and J. irn: On the defect of vertex-transitive graphs of given degree and diameter, Journal of Combinatorial Theory, Series B Volume 134, 2019, Pages 322-340, https://doi.org/10.1016/j.jctb.2018.07.002

Some computations

Using bounds on $M_c(\Delta, D) - C_G(v, 2D+1)$ we computed for how many diameters from $\{1 \ldots 1000000\}$ there may exist a vertex-transitive cubic graph with defect exactly δ (we ignore the fact that odd defects are not possible). In the column denoted i there are numbers of such diameters in the set $\{200000 * (i - 1) + 1 \ldots 200000 * i\}$.

Some computations

$\delta \backslash i$	1	2	3	4	5
1	23485	20693	19797	19268	18906
2	18705	16211	15492	15079	14758
3	18259	15784	15073	14682	14355
4	24093	19986	19015	18476	18009
5	20180	17214	16481	16003	15669
6	20196	17229	16495	16004	15667
7	24432	20460	19499	18897	18553
8	23514	19785	18860	18354	17929
9	24486	20577	19637	19133	18663
10	33829	27180	25646	24894	24285

Girth

We know that any (Δ, D) graph G with positive defect contains a cycle of length smaller than $2D+1$. Our methods enable us to bound the number of v-based cycles of length $2D+1-k$ in G independently on D, too. It turns out that such bounds are much better (and much easier to prove) when we assume that $2D+1-k$ is the girth of G.

Girth

Lemma. *Let* $\Delta \geq 3$, $\delta \geq 1$ *and* $k \geq 1$ *be positive integers. Let* G *be a* (Δ, D)*-graph with defect* δ *and girth* $2D + 1 - 2k$*. Then*

$$0 \leq C_G(v, 2D + 1 - 2k) \leq \lfloor \frac{\delta}{2(\Delta - 1)^{k-1}} \rfloor.$$

Lemma. *Let* $\Delta \geq 3$, $\delta \geq 1$ *and* $k \geq 1$ *be positive integers. Let* G *be a* (Δ, D)*-graph with defect* δ *and girth* $2D + 1 - (2k - 1)$*. Then*

$$0 \leq C_G(v, 2D + 1 - (2k - 1)) \leq \lfloor \frac{\Delta\delta}{2(\Delta - 1)^{k-1}} \rfloor.$$

Note that for fixed defect the girth cannot be much smaller than $2D + 1$.

Girth

Lemma. *Let* $\Delta \geq 3$, δ, D *and* k *be positive integers and let* b_k *be a bound on the number of* v*-rooted cycles of length* $2D+1-k$ *in a (vertex-transitive)* (Δ, D)*-graph with defect* δ *and girth* $2D+1-k$. *If*

$$b_k \cdot GCD(M(\Delta, D) - \delta, 2D+1-k) < 2D+1-k,$$

then there is no vertex transitive (Δ, D) *graph with defect* δ *and girth* $2D+1-k$.

(It is possible to show that the set of diameters for which there exist a VT D-regular graph with defect at most δ has asymptotic density 0 from the results about girth, too.)

Further computations

Using results about girth in vertex-transitive (Δ, D) graphs we computed for how many diameters from $\{1 \dots 1000000\}$ there may exist a vertex-transitive cubic graph with defect exactly δ (again we ignore the fact that odd defects are not possible). In the column denoted i there are numbers of such diameters in the set $\{200000 * (i-1) + 1 \dots 200000 * i\}$.

We did not use the condition on the number of cycles of length $2D + 1$.

Further computations

δ	1	2	3	4	5
1	0	0	0	0	0
2	14	0	0	0	0
3	8	1	0	0	0
4	34047	30273	29067	28277	27796
5	10	1	1	0	0
6	70	1	1	3	1
7	26	3	2	2	1
8	56	3	0	0	1
9	19	3	1	0	0
10	45247	40388	38884	37937	37247

Further computations

δ	1	2	3	4	5
11	43	1	1	2	1
12	70	2	4	1	0
13	59	5	2	0	1
14	189	7	6	4	4
15	61	8	2	1	0
16	144	12	9	6	1
17	71	1	3	1	4
18	181	10	6	5	2
19	105	9	4	0	2
20	136	10	4	5	4

Further computations

δ	1	2	3	4	5
21	56	3	3	4	1
22	51806	46192	44585	43470	42780
23	56	3	1	1	0
24	187	7	5	8	5
25	217	17	14	8	10
26	246	9	11	4	3
27	91	5	1	2	2
28	247	19	6	4	7
29	139	9	6	7	5
30	449	22	16	6	11

Comments

Further computations indicate that it is much easier to satisfy the girth conditions for defects of the form $M(\Delta, d)$. It seems to be related to Fermat's little theorem.

Combined computations

δ	1	2	3	4	5
1	0	0	0	0	0
2	7	0	0	0	0
3	3	0	0	0	0
4	5126	3615	3276	3068	2982
5	5	0	0	0	0
6	31	0	0	1	0
7	16	0	0	0	0
8	26	1	0	0	0
9	13	0	0	0	0
10	9036	6002	5428	5110	4885

Combined computations

δ	1	2	3	4	5
11	25	0	1	0	0
12	42	0	1	0	0
13	37	0	0	0	0
14	103	0	1	1	0
15	35	0	0	0	0
16	88	0	1	0	0
17	50	0	0	0	1
18	110	1	1	2	0
19	68	0	0	0	0
20	70	1	0	0	0
21	40	1	1	0	0

Combined computations

δ	1	2	3	4	5
21	40	1	1	0	0
22	12048	7550	6764	6259	5845
23	39	1	0	0	0
24	111	0	0	0	1
25	133	5	0	1	2
26	150	1	1	1	0
27	63	0	0	0	0
28	163	1	1	0	1
29	97	1	0	0	0
30	286	4	2	0	1

Open problems

It seems, that we are close to the limits of the cycle counting techniques. Therefore the most important open problem is to come with new ideas.

However, there are still some places when it is possible to find improvements of existing results.

The main problem

For each $\Delta \geq 3$ and $D \geq 2$ find the largest (general, vertex-transitive, Cayley,...) (Δ, D) graph.

The vertex-transitive problem

We used the vertex-transitivity of G only to show that $C_G(v, n)$ does not depend on v.

Find a better way to use vertex-transitivity.

Number theory problems

Let f be a function of two variables and let

$S(\Delta, f) = \{D;$ some VT (Δ, D)-graph has defect at most $f(\Delta, D)\}$.

Show that $S(\Delta, D)$ is small. To be small can be defined as

- to be empty, or
- to be finite, or
- to have convergent series of reciprocal values, or
- to have zero density, or
- to have infinite complement.

The uniform problem

There are sets of asymptotic density 0 which contain arbitrarily long sequences of consecutive integers, e.g., $\cup\{n!+1 \ldots n!+n\}$.

There exists a notion of uniform (or Banach) density which forbids this behavior for sets of density 0.

Prove that the set of diameters of vertex transitive Δ-regular graphs with defect (at most) δ has uniform density 0.

The continuous problem

Plesník in 1973 showed that in Moore graphs the numbers $C_G(v,n)$ do not depend on v for $n \leq 4D+1$ and computed the corresponding values $P_C(\Delta, D, n)$. Is it possible to bound the distance between $C_G(v,n)$ and $P_C(\Delta, D, n)$ in a (Δ, D)-graph G with defect δ independently of D?

The positive problem

Are there (vertex-transitive) (Δ, D) graphs with

$$C_G(v, 2D+1) > M_C(\Delta, D)?$$

Thank You