Constructing small volume hyperbolic manifolds from Coxeter groups

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(in joint work with PhD student Gina Liversidge)

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Some background

A (Riemannian) manifold M of rank n is a topological space with the property that each point has a neighbourhood that is homeomorphic to some open subset of n-dimensional Euclidean space $\mathbb{R}^n$, and accordingly, M can be viewed as the analogue of a Riemann surface in higher dimension.

The volume of M is a topological invariant that can be calculated using the Riemannian metric ... analogous to the area of a fundamental region for a Riemann surface.

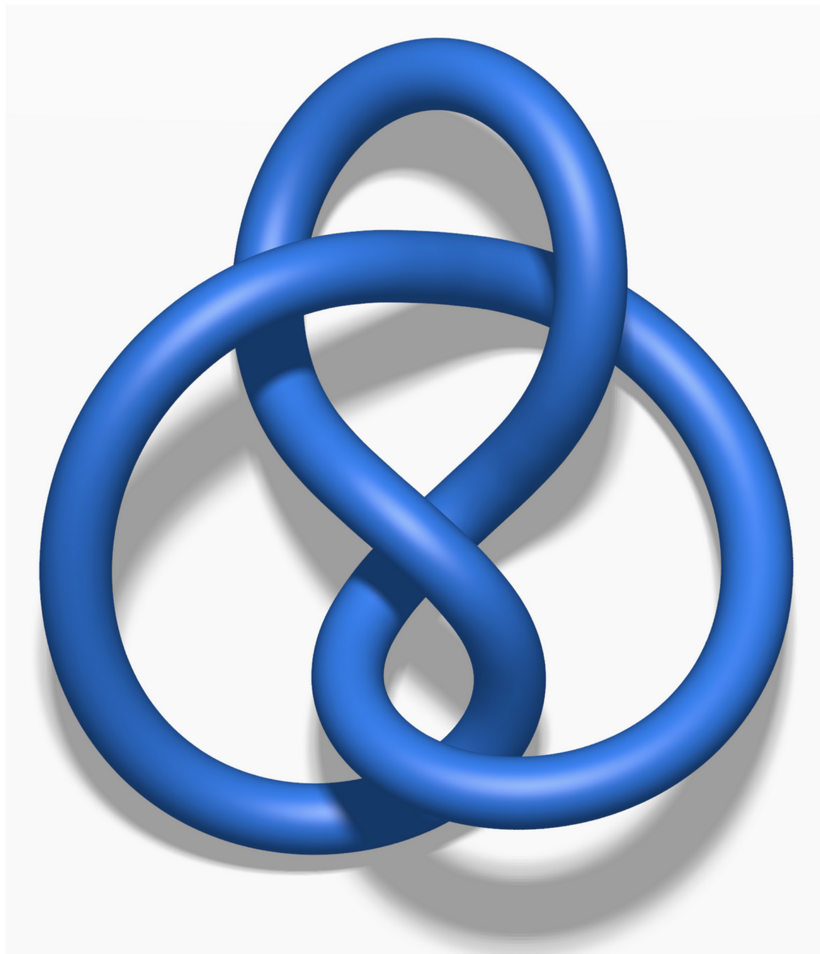
Much interest is placed on the smallest volume n-manifolds for particular values of n in various cases: compact, non-compact, orientable and non-orientable, and so on.

Some background (cont.)

The smallest volume of a compact orientable hyperbolic 2-manifold is 4π, achievable by any closed hyperbolic surface of genus 2 (and Euler characteristic -2).

For non-compact orientable 2-manifolds, the smallest volume is 2π, achieved by a once-punctured torus (with Euler characteristic -1).

The smallest volume of a compact orientable hyperbolic 3-manifold is realised uniquely by the Weeks-Matveev-Fomenko manifold (aka the 'Weeks manifold'), while the smallest volume of a non-compact orientable 3-manifold is achieved by the figure 8 knot complement and a 'sibling' of this obtained by Dehn surgery on the Whitehead link.



The figure 8 knot

What about 4-manifolds?

The volume of every hyperbolic 4-manifold is a constant multiple of its Euler characteristic: $\mathrm{vol}(M) = 4\pi 2\chi(M)/3$, and in the compact case, this is always even.

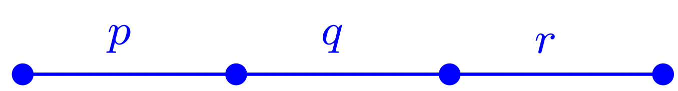
It is known that there exist non-compact orientable hyperbolic 4-manifolds of minimal Euler characteristic 1, but the compact case is more challenging. A well-studied example of a compact orientable hyperbolic 4-manifold was the Davis manifold (1985), with Euler characteristic 26.

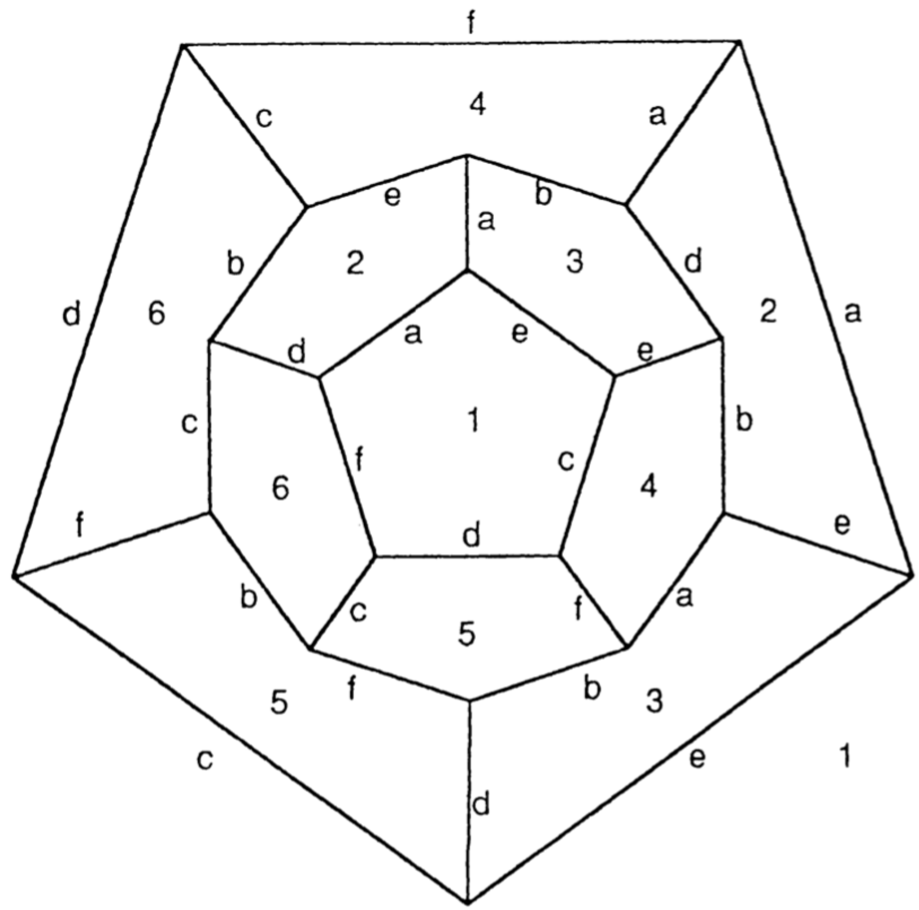
Currently for $n = 4$ the smallest known volume in the compact case is achieved by a non-orientable example found by MC & Colin Maclachlan (2005), with Euler characteristic 8. Its orientable double cover has Euler characteristic 16, and so both give improvements on the Davis manifold.

Connection with hyperbolic reflection groups

One of the first examples of a compact orientable hyperbolic 3-manifold (the Weber-Seifert manifold) was constructed in the 1930s from the identification of opposite faces of a dodecahedron. (There are three ways to do this consistently, and the other two give the Poincaré homology sphere and 3-dimensional real projective space.)

In the late 1970s, John Milnor took this and the work of William Thurston further, in a study of hyperbolic volume, considering hyperbolic 3-manifolds constructible from hyperbolic reflection groups $\Gamma_{p,q,r}$ with Coxeter-Dynkin diagram





Dodecahedral face identifications for the homology sphere

Table of hyperbolic reflection groups $\Gamma_{p,q,r}$ (by Milnor)

p, q, r	$V_3(\Sigma_{p,q,r})$	L.C.M.	least $\#(\Gamma/\Pi)$	$V_3(H^3/\Pi)$
3, 5, 3	.0390503	120	?	
4, 3, 5	.035885	240	?	
5, 3, 5	.0933255	120	120	11.199064 (a)
$3, 3, 6^\infty$	$.0422892 = V_3^{\max}/24$	24	24	$1.0149416 = V_3^{\max}$ (b)
$4, 3, 6^\infty$	$.1057231 = (5/48)V_3^{\max}$	48	48	$5.074708 = 5V_3^{\max}$
$3, 4, 4^\infty$	$.0763305 = (1/6)Л(\pi/4)$	48	48	$3.66386 = 8Л(\pi/4)$ (c)
$5, 3, 6^\infty$	.1715017	120	?	
$3^\infty, 6, 3^\infty$	$.1691569 = V_3^{\max}/6$	12	12	$2.029883 = 2V_3^{\max}$ (d)
$6^\infty, 3, 6^\infty$	$.2537354 = V_3^{\max}/4$	24	?	
$4^\infty, 4, 4^\infty$	$.2289914 = (1/2)Л(\pi/4)$	16	16	$3.66386 = 8Л(\pi/4)$ (e)

(a) Weber-Seifert manifold (b) Gieseking manifold (non-orientable) (c) Non-orientable

(d) $S^3 \setminus$  (e) Non-orientable

Unresolved cases

Table of hyperbolic reflection groups $\Gamma_{p,q,r}$ (by Milnor)

p, q, r	$V_3(\Sigma_{p,q,r})$	L.C.M.	least $\#(\Gamma/\Pi)$	$V_3(H^3/\Pi)$
3, 5, 3	.0390503	120		?
4, 3, 5	.035885	240		?
5, 3, 5	.0933255	120	120	11.199064 (*a*)
$3, 3, 6^\infty$	$.0422892 = V_3^{\max}/24$	24	24	$1.0149416 = V_3^{\max}$ (*b*)
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(*a*) Weber-Seifert manifold (*b*) Gieseking manifold (non-orientable) (*c*) Non-orientable

(*d*) $S^3 \smallsetminus$  (*e*) Non-orientable

In 1992 Peter Lorimer found two further small-volume 3-manifolds constructible from the [3, 3, 5] and [5, 3, 5] Coxeter groups (and tessellated by regular dodecahedra), and later considered the [4,3,5] case, but he was suffering from ill health and made an unfortunate big mistake in that work (which was nevertheless published as a 74-page paper in the International Journal of Theoretical Physics in 2002).

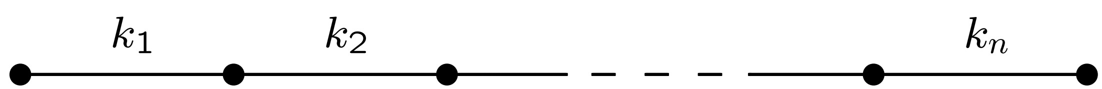
In 2004 Brent Everitt found all of the orientable spherical and hyperbolic 3-manifolds that arise by identifying the faces of a Platonic solid, again using Coxeter groups. This work partially filled in two of the four incomplete entries in Milnor's table, namely the cases [3,5,3] and [5,3,6].

Coxeter groups

The $[k_1, k_2, ..., k_n]$ Coxeter group is the finitely-presented group generated by $x_1, x_2, \ldots, x_{n+1}$ subject to the relations

- $x_i^2 = 1$ for $1 \le i \le n+1$,
- $(x_i x_{i+1})^{k_i} = 1$ for $1 \le i \le n$,
- $(x_i x_j)^2 = 1$ for $1 \le i < j \le n+1$ with $|j - i| \ge 2$,

often represented by the Coxeter-Dynkin diagram



Examples:

- Dihedral groups $D_k = \langle\, x, y \mid x^2, y^2, (xy)^k \,\rangle$
- Full triangle groups

$$\Delta(2, k, m) = \langle\, a, b, c \mid a^2, b^2, c^2, (ab)^k, (bc)^m, (ac)^2 \,\rangle.$$

Construction of manifolds from Coxeter groups

Very briefly, if Λ is a torsion-free subgroup of finite index in the $[k_1, k_2, ..., k_n]$ Coxeter group Γ, where $n \geq 3$, then an n-manifold $\mathcal{M}$ can be constructed from features of the natural permutation representation of Γ on the right cosets of Λ, and then various properties of $\mathcal{M}$ (such as its volume) can be computed from these and the parameters $k_1, k_2, ..., k_n$.

Example: Torsion-free subgroups of the $[3, 7]$ Coxeter group (which is isomorphic to the full $(2, 3, 7)$ triangle group) produce regular maps of type $\{3, 7\}$ on surfaces and algebraic curves (or compact Riemann surfaces) of genus $g > 1$ with automorphism group of largest possible order $168(g - 1)$.

Torson-free subgroups of Coxeter groups

Geometric theory (involving fixed points of group actions) shows that every finite subgroup of an infinite $[k_1, k_2, \ldots, k_n]$ string Coxeter group $\Gamma_{k_1,k_2,\ldots,k_n}$ is conjugate to a finite subgroup of one of the following $n+1$ subgroups (each of which is obtained by deleting one of the generators x_j):

- $\langle x_1, x_2, \ldots, x_{n-1}, x_n \rangle \cong \Gamma_{k_1,k_2,\ldots,k_{n-1}}$
- $\langle x_1, x_2, \ldots, x_{n-1}, x_{n+1} \rangle \cong \Gamma_{k_1,\ldots,k_{n-2}} \times C_2$
- $\langle x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1} \rangle \cong \Gamma_{k_1,\ldots,k_{j-2}} \times \Gamma_{k_{j+1},\ldots,k_n}$ for $3 \le j \le n-1$
- $\langle x_1, x_3, \ldots, x_n, x_{n+1} \rangle \cong C_2 \times \Gamma_{k_3,\ldots,k_{n-1},k_n}$
- $\langle x_2, x_3, \ldots, x_n, x_{n+1} \rangle \cong \Gamma_{k_2,\ldots,k_{n-1},k_n}$.

This gives an iterative process for finding representatives of conjugacy classes of maximally finite subgroups of $\Gamma_{k_1,k_2,\ldots,k_n}$.

Important observation

If Λ is a torsion-free subgroup of a group Γ, then in the right multiplicative permutation representation of Γ on right cosets of Λ, the orbits of every finite subgroup of Γ must all be regular (for otherwise some non-trivial element of a finite subgroup will lie in a conjugate of Λ). Hence if the index $|\Gamma : \Lambda|$ is finite, it must be divisible by the LCM of the orders of representatives of all finite subgroups of Γ.

[Note: In his study of the $[4, 3, 5]$ case, Peter Lorimer sadly made the mistake of claiming that the index should be just a multiple of 120, but the LCM is 240, and so he found over 200 classes of index 120 subgroups that are not torsion-free.]

Finding torsion-free subgroups of small index

One way is to use the `LowIndexSubgroups` routine in MAGMA – e.g. ask for conjugacy classes on subgroups of index 240 in the $[4, 3, 5]$ Coxeter group, and check among those for the ones in which the orbits of the finite subgroups are regular.

This can be time-consuming when the index is large.

Another way is to use Peter Dobcsányi's `lowx` program, which skips unwanted branches of the search tree.

Another way is to take a union of regular orbits of the largest finite subgroup and try to link those together to form a transitive permutation representation of the whole group.

All three ways can be fruitful, but usually one is better than the others.

Small volume compact 4-manifolds

In hyperbolic 4-space $\mathbb{H}^4$, there are five compact Coxeter simplices (whose faces are geodesic, with dihedral angles between faces of co-dimension 1 being submultiples of π).

One of them is associated with the $[5, 3, 3, 5]$ Coxeter group, and gives rise to the Davis manifold, of characteristic 26, via a torsion-free subgroup of index 14400.

A study of the other four simplices (by Colin Maclachlan & MC) showed that a better choice was the $[5, 3, 3, 3]$ group – for which the associated Coxeter group has a torsion-free subgroup of index 115200 that gives rise to a non-orientable compact 4-manifold with Euler characteristic 8.

How did we find it?

If x_1, x_2, x_3, x_4, x_5 are canonical generators for the $[5, 3, 3, 3]$ Coxeter group, then consider these finite subgroups:

- $\langle x_1, x_2, x_3, x_4 \rangle \cong \Gamma_{5,3,3}$, of order 14400,
- $\langle x_1, x_2, x_3, x_5 \rangle \cong \Gamma_{5,3} \times C_2 \cong A_5 \times C_2 \times C_2$, of order 240,
- $\langle x_1, x_2, x_4, x_5 \rangle \cong \Gamma_5 \times \Gamma_3 \cong D_5 \times D_3$, of order 60,
- $\langle x_1, x_3, x_4, x_5 \rangle \cong C_2 \times \Gamma_{3,3} \cong C_2 \times S_4$, of order 48,
- $\langle x_2, x_3, x_4, x_5 \rangle \cong \Gamma_{3,3,3} \cong S_5$, of order 120.

The index of any torsion-free subgroup is therefore divisible by $\mathrm{LCM}(14400, 240, 60, 48, 120) = 14400$.

We found that the intersection of two 'almost' torsion-free subgroups of index 120 and 960 was a torsion-free subgroup of index 115200. **Question**: Is there one of smaller index?

Aside: Recent work with Ruth Kellerhals

In this work, we constructed cusped hyperbolic n-manifolds for $3 \leq n \leq 5$ by considering ideal $(n-3)$-rectified regular simplices giving rise to a tesselation of n-dimensional hyperbolic space $\mathbb{H}^n$, via torsion-free subgroups of small index in the Coxeter groups $[3, 3, ..., 3, 6]$ of rank up to 6.

The case $n = 3$ gives Gieseking's single-cusped 3-manifold.

When $n = 4$, we found a torsion-free subgroup Λ of minimum index 720 in the arithmetic Coxeter pyramid group $\Gamma = [\infty, 3, 3, 3, 6]$, with quotient space $\mathbb{H}^4/\Lambda$ giving a non-orientable 4-cusped hyperbolic manifold of characteristic 1 commensurable with the orientable manifold of characteristic 1 constructed by Riolo and Slavich in a different way.

For $n = 5$, we found an orientable 2-cusped manifold that is closely related to an ideal birectified 6-cell with dihedral angles $\pi/3$ and $\pi/2$, which can be barycentrically decomposed into $6! = 720$ Coxeter pyramids with symbol $[6, 3, 3, 3, 3, 6]$.

The fundamental group of this 5-manifold is isomorphic to a torsion-free subgroup of minimum possible index 2880 in the $[6, 3, 3, 3, 3, 6]$ Coxeter group, and has a presentation in terms of three orientation-preserving isometries, where two are loxodromic elements of equal translation length, and the third is parabolic. Also it is not commensurable with the fundamental group of the small volume non-orientable 5-manifold found by Ratcliffe and Tschantz (22 years ago).

Finally, we return to small volume 3-manifolds obtainable from Coxeter groups – correcting the work by Peter Lorimer and completing the study begun by John Milnor.

Manifolds from the [4,3,5] Coxeter group

If x_1, x_2, x_3, x_4 are canonical generators for the $[4,3,5]$ Coxeter group, then consider these finite subgroups:

- $\langle x_1, x_2, x_3 \rangle \cong [4,3] \cong \Delta(2,4,3) \cong S_4 \times C_2$, of order 48,
- $\langle x_1, x_2, x_4 \rangle \cong \langle x_1, x_2 \rangle \times \langle x_4 \rangle \cong D_4 \times C_2$, of order 16,
- $\langle x_1, x_3, x_4 \rangle \cong \langle x_1 \rangle \times \langle x_3, x_4 \rangle \cong C_2 \times D_5$, of order 20,
- $\langle x_2, x_3, x_4 \rangle \cong [3,5] \cong \Delta(2,3,5) \cong A_5 \times C_2$, of order 120.

The index of any torsion-free subgroup is therefore divisible by $\mathrm{LCM}(48, 16, 20, 120) = 240$.

PhD student Gina Liversidge and MC carried out two computations (over 4 days) that revealed exactly 14 conjugacy classes of torsion-free subgroups of index 240 in the $[4,3,5]$ Coxeter group – and up to 14 new dodecahedral spaces.

What kind of computations?

• Use of the `LowIndexSubgroups` routine in MAGMA works, but takes over two months!

• Peter Dobcsányi's `lowx` program takes much less time

• Another quick and successful approach involved starting with two regular permutation representations of the $[3, 5]$ Coxeter group (isomorphic to $A_5 \times C_2$, of order 120), and then finding all ways of joining these together to form a suitable transitive permutation representation of the $[4, 3, 5]$ Coxeter group of degree 240.

Manifolds from the [6,3,6] Coxeter group

If x_1, x_2, x_3, x_4 are canonical generators for the $[6,3,6]$ Coxeter group, then the subgroups $\langle x_1, x_2, x_3 \rangle$ and $\langle x_2, x_3, x_4 \rangle$ are isomorphic to the full $(2,3,6)$ triangle group, and hence are infinite, so we consider (only) these finite subgroups:

- $\langle x_1, x_2, x_4 \rangle \cong D_6 \times C_2$, of order 24,
- $\langle x_1, x_3, x_4 \rangle \cong C_2 \times D_6$, of order 24,
- $\langle x_2, x_3 \rangle \cong D_3$, of order 6.

Note: Each of the finite 2-generator subgroups $\langle x_i, x_j \rangle$ is contained in at least one of these. The index of any torsion-free subgroup is therefore divisible by $\mathrm{LCM}(24,6) = 24$.

An easy computation shows there are 12 conjugacy classes of torsion-free subgroups of index 24 in the $[6,3,6]$ Coxeter group – and up to 12 associated hyperbolic 3-manifolds.

Outcome

The last two pieces of work (done in 2022) fill the remaining two gaps in Milnor's table.

In fact, we can now go further and find all the 3-manifolds of Coxeter type $[p, q, r]$ with minimum volume (with regard to the type), when $p, q, r \geq 3$ and both $[p, q]$ and $[q, r]$ are spherical or Euclidean − that is, with $1/2 + 1/p + 1/q \geq 1$ and $1/2 + 1/q + 1/r \geq 1$.

This extends and completes the work by Weber & Seifert, Milnor, Lorimer and Everitt.

A summary table (up to duality) follows. In the first four cases, the Coxeter group is finite, and the manifold is unique, while in the remaining eleven cases, the first homology group of each of the manifolds is known as well.

Type	Minimum $\lvert\Gamma:\Lambda\rvert$	# of TF subgroup classes
$[3,3,3]$	120	1 (the 4-simplex)
$[3,3,4]$	384	1 (the 16-cell)
$[3,3,5]$	14400	1 (the 600-cell)
$[3,4,3]$	1152	1 (the 24-cell)
$[4,3,4]$	48	18
$[3,3,6]$	24	1 (the Gieseking manifold)
$[3,4,4]$	48	13
$[3,5,3]$	120	7
$[3,6,3]$	12	2
$[4,3,5]$	240	14
$[4,3,6]$	48	11
$[4,4,4]$	16	12
$[5,3,5]$	120	12 (one is Weber-Seifert)
$[5,3,6]$	120	77
$[6,3,6]$	24	12

Final notes

- In each case, compactness of a manifold is determined entirely by the triple (p, q, r) ... or more specifically in most cases, by compactness of the associated Coxeter simplex.

- All of the spherical and Euclidean manifolds we found are compact, while the hyperbolic manifolds are compact only for the triples $(4, 3, 5)$, $(3, 5, 3)$ and $(5, 3, 5)$.

- Isomorphisms among the resulting 3-manifolds are yet to be determined.

Thanks for listening!

Abstract

It is well known that various hyperbolic manifolds of small volume can be constructed using torsion-free subgroups of minimum possible index in certain string Coxeter groups. Examples include the 600-cell and the Weber-Seifert and Gieseking manifolds, obtainable from the $[3,3,5]$, $[5,3,5]$ and $[3,3,6]$ Coxeter groups. Constructions for certain 3-manifolds were developed in some notes by Milnor in the late 1970s on computing volumes, and in papers by Lorimer (1992) and Everitt (2004) using the identification of faces of a Platonic solid, and Colin Maclachlan and the speaker took a different approach (in 2005) to find the compact 4-manifold of currently smallest known volume, via subgroups

of the $[3, 3, 3, 5]$ Coxeter group. Milnor's notes did not completely resolve some cases for hyperbolic 3-manifolds, and a subsequent paper by Lorimer (2002) unsuccessfully attempted to deal with the case of the $[4, 3, 5]$ Coxeter group. We complete and extend these pieces of work by determining all of the small volume 3-manifolds constructible from torsion-free subgroups of minimum possible index in the $[p, q, r]$ Coxeter groups for which $p, q, r \geq 3$ and each of the pairs (p, q) and (q, r) is spherical or Euclidean (that is, with $1/p + 1/q \geq 1$ and $1/q + 1/r \geq 1$). This is quite recent joint work with PhD student Georgina Liversidge.