Regular and chiral maps with given valency or given type

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[Let me know if you want a copy of my slides]

Orientably-regular maps

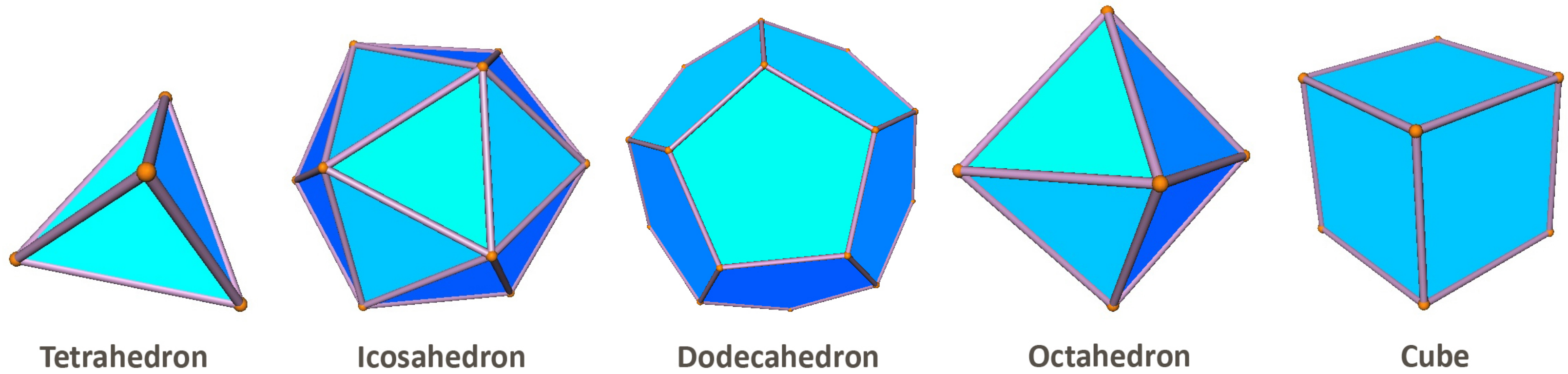
A map is an embedding of a connected graph or multigraph on a closed surface, breaking it up into simply-connected regions called the faces of the map.

A map M on an orientable surface is orientably-regular if the group of all of its orientation-preserving automorphisms is transitive on the arcs (incident vertex-edge pairs) of M.

In that case, every vertex has the same degree/valency k and every face of the map has the same size m, and we call $\{m, k\}$ the type of the map.

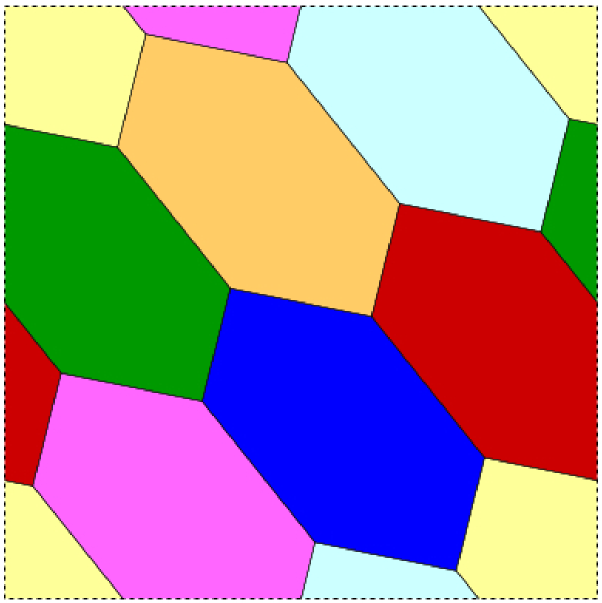
Orientably-regular maps are sometimes just called 'regular'. Those that admit also orientation-reversing automorphisms are 'reflexible', while the remaining ones are 'chiral'.

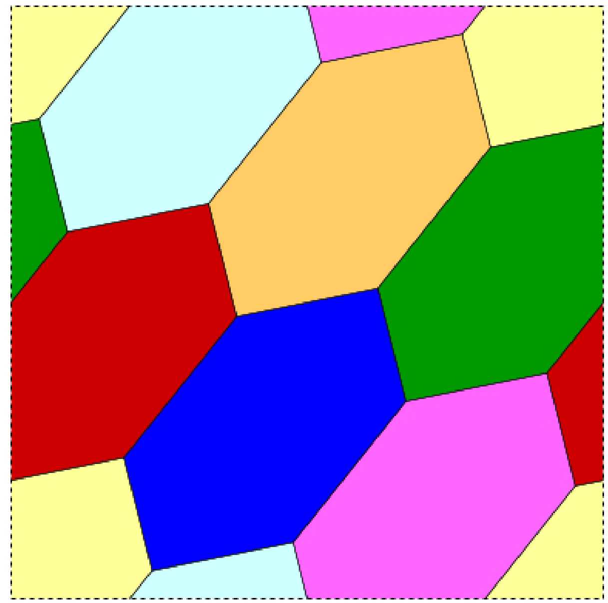
The Platonic solids give rise to reflexible maps on the sphere — with types $\{3,3\}$, $\{3,5\}$, $\{5,3\}$, $\{3,4\}$ and $\{4,3\}$:



Regular maps on the torus (genus 1) have types $\{3,6\}$, $\{4,4\}$ and $\{6,3\}$, and infinitely many of these maps are reflexible, while infinitely many are chiral.

Two chiral maps of type $\{6,3\}$ on the torus





These maps are chiral, and mirror images of each other (and their duals are orientably-regular embeddings of K_7)

Regular maps of hyperbolic type

If M is an orientably-regular map of type $\{m, k\}$ on a surface of genus $g > 1$, with $|V|$ vertices, $|E|$ edges and $|F|$ faces, and orientation-preserving automorphism group G, then by arc-transitivity and 2-edge-connectivity(*), we have

$$|G| = |\text{Aut}^{\text{o}} M| = k|V| = 2|E| = m|F|$$

and so by the Euler-Poincaré formula, we have

$$2 - 2g = \chi = |V| - |E| + |F| = |G|\left(\tfrac{1}{k} - \tfrac{1}{2} + \tfrac{1}{m}\right).$$

As the LHS is < 0, this requires $\frac{1}{k} + \frac{1}{m} < \frac{1}{2}$, or equivalently,

- $k = 3$ and $m \geq 7$, or (dually) $m = 3$ and $k \geq 7$,
- $k = 4$ and $m \geq 5$, or (dually) $m = 4$ and $k \geq 5$,
- $m \geq k \geq 5$, or (dually) $k \geq m \geq 5$.

In these cases the map M is said to have hyperbolic type.

Digression: 2-edge-connectivity of VT graphs

The identity $2|E| = m|F|$ on the previous slide may result from counting the number of incident edge-face pairs in two different ways: on one hand, each face has m edges, but on the other hand, why does every edge lie in 2 faces?

This is frequently assumed, but seldom proved!

[**Note after talk**: Steve Wilson pointed out that we can get around this by counting in a different way! If we trace the edges around each face, then every edge is counted twice.]

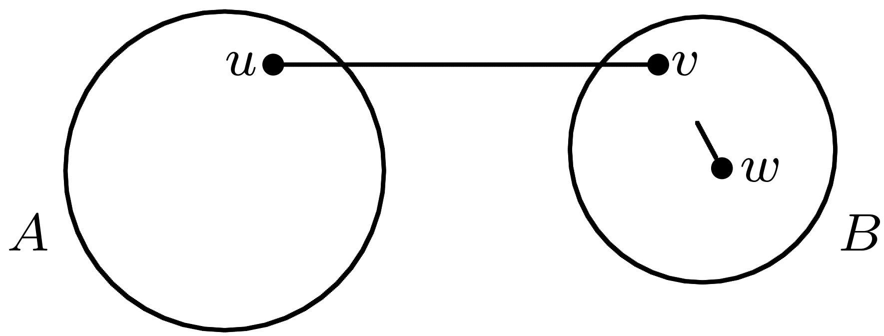
Can you prove that every edge lie in exactly two faces?

Better still, can you prove that if X is any connected finite vertex-transitive graph other than K_2, then no edge of X is a 'bridge', and hence X is 2-edge-connected?

What follows is one of at least three different short proofs.

(This one results from a recent brief email discussion with Brian Alspach, whose 1994 Masters student Tai-Yu Chen (at Simon Fraser University) proved something stronger, namely that if the connected simple graph X is vertex-transitive, then X is k-edge-connected if and only if X has valency k.)

Proof. Assume the contrary, and let $e = \{u, v\}$ be a bridge. Then removal of e leaves two connected components, say A and B, containing u and v respectively, and without loss of generality, we may suppose that $|A| \geq |B|$.



Now because X has valency at least 2, we know $|B| \geq 2$, and so B contains another vertex w. Then by vertex-transitivity w is incident with a bridge edge whose removal from X leaves two components of sizes $|A|$ and $|B|$. But clearly one of those components contains $A \cup \{v\}$, so its size is greater than both $|A|$ and $|B|$ ... contradiction. $\square$

Connection with triangle groups

If M is an orientably-regular map of type $\{m, k\}$, and (v, e, f) is any incident vertex-edge-face triple, then there exist two orientation-preserving automorphisms R and S such that

- R acts locally like a single-step rotation of the face f,
- S acts locally like a single-step rotation around vertex v,
- RS acts locally like a 180-degree rotation of the edge e.

By connectedness, and the fact that any orientation-preserving automorphism is uniquely determined by its effect on any arc, it follows that R and S generate $\mathrm{Aut}^{\mathrm{O}}(M)$ and satisfy the $\Delta(2, k, m)$ triangle group relations $R^m = S^k = (RS)^2 = 1$.

Conversely, every such map M of type $\{m, k\}$ can be constructed algebraically from elements R and S of orders m and k generating a finite group G such that RS has order 2.

Orientably-regular maps of given type

Spherical type: If the map has genus 0, then $\frac{1}{k} - \frac{1}{2} + \frac{1}{m} > 0$, so $k = 2$ or $m = 2$ or $\{m, k\} = \{3, 3\}$, $\{3, 5\}$, $\{5, 3\}$, $\{3, 4\}$ or $\{4, 3\}$, and all possibilities are achievable, by 'equatorial' maps, 'polar maps', and the 'Platonic' maps on the sphere.

Euclidean type: If the map has genus 1, then $\frac{1}{k} - \frac{1}{2} + \frac{1}{m} = 0$, so $\{m, k\} = \{3, 6\}$, $\{4, 4\}$ or $\{6, 3\}$, and all possibilities are achievable, by uniform triangulations, quadrangulations and 'honeycombs' of the torus.

For the rest of this talk, we'll focus on **hyperbolic type**, with $\frac{1}{k} - \frac{1}{2} + \frac{1}{m} < 0$.

Some early history

Murray Macbeath (1969) used a 2×2 matrix construction and trace argument to show that for every pair (m, k) of positive integers s.t. $1/k + 1/m < 1/2$, there are infinitely many primes p for which the group $\mathrm{PSL}(2, p)$ is a 'smooth' quotient of the $\Delta(2, k, m)$ triangle group.

(**Note**: 'Smooth' means that the orders 2, k and m of the relevant elements of $\Delta(2, k, m)$ are preserved. The proof finds suitable elements of $\mathrm{SL}(2, p)$ that cannot generate a proper subgroup, and then projects those to $\mathrm{PSL}(2, p)$.)

All of the resulting maps are reflexible [Singerman (1974)] and hence fully regular, and so Macbeath's theorem proves the following:

> For every hyperbolic pair $\{m, k\}$, there exist infinitely many fully regular orientable maps of type $\{m, k\}$.

A further construction: the 'Macbeath trick'

Let $G = \langle R, S \rangle$ be the rotation group of an orientably-regular map M of hyperbolic type $\{m, k\}$, and genus $g \geq 2$.

Then there exists a homomorphism from $\Delta(2, k, m)$ onto G, the kernel K of which is torsion-free and isomorphic to the fundamental group of the carrier surface of M. In particular, K is generated by $2g$ elements $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ subject to the single defining relation $[a_1, b_1][a_2, b_2] \ldots [a_g, b_g] = 1$.

Now for any positive integer s, the subgroup $L_s = K'K^{(s)}$ of K generated by all commutators $[x, y]$ and the sth powers x^s of all elements of K is characteristic in K, and therefore normal in $\Delta(2, k, m)$, with $K/L_n \cong (\mathbb{Z}_s)^{2g}$, and hence we get infinitely many new quotients $\Delta(2, k, m)/L_s$ and infinitely many 'covering' maps of M, all of the same type $\{m, k\}$.

Note: A related method was used by Biggs and Conway and (independently) Djoković in order to construct infinitely many connected finite 5-arc-transitive 3-valent graphs from a given example.

Coset graph constructions

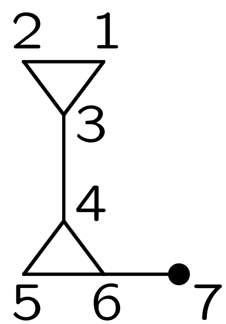
A (Schreier) coset graph is a graph that depicts the effect of a finitely-generated permutation group G on a set X. The vertices are the points of X, and for every generator g of G, an arc labelled g joins each vertex x to its image x^g under g.

When the action is transitive, this is equivalent to the graph whose vertices are the right cosets Hx in G of a point-stabiliser H, with an arc labelled g joining each vertex Hx to its image Hxg under right multiplication by g.

(Also some people working on abstract polytopes recently have called a special case of this a ‘C-group permutation representation graph’ (or ‘CPR’ graph), but it’s really the same thing and so doesn’t need a new term.)

Example

Below is a coset graph for an action of the $(2, 3, 7)$ triangle group $\Delta = \langle\, x, y, z \mid x^2 = y^3 = z^7 = xyz = 1 \,\rangle$ on 7 points:



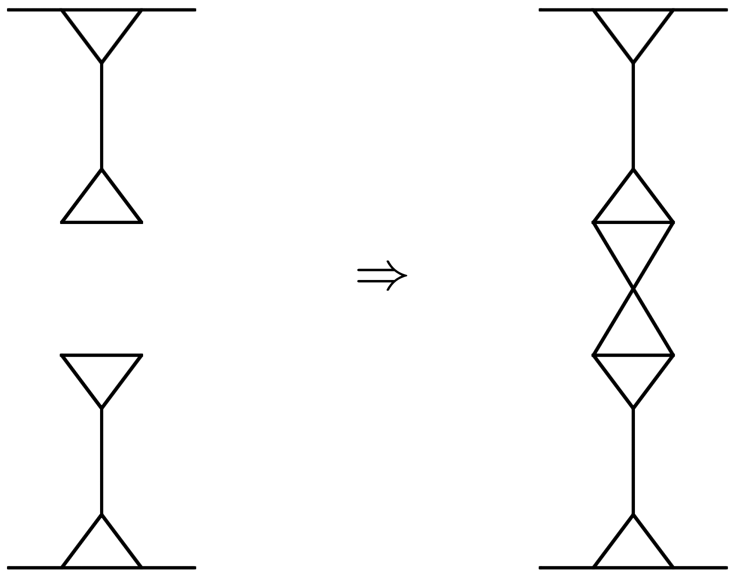
$x \mapsto (3, 4)(6, 7)$
$y \mapsto (1, 2, 3)(4, 5, 6)$
$z \mapsto (1, 4, 7, 6, 5, 3, 2)$

This gives a homomorphism from Δ to $\mathrm{Aut}^{\mathrm{o}}(M) \cong \mathrm{PSL}(2, 7)$ taking $(x, y, z) \mapsto (RS, R, S)$, for Klein's quartic map M of type $\{3, 7\}$ and genus 3.

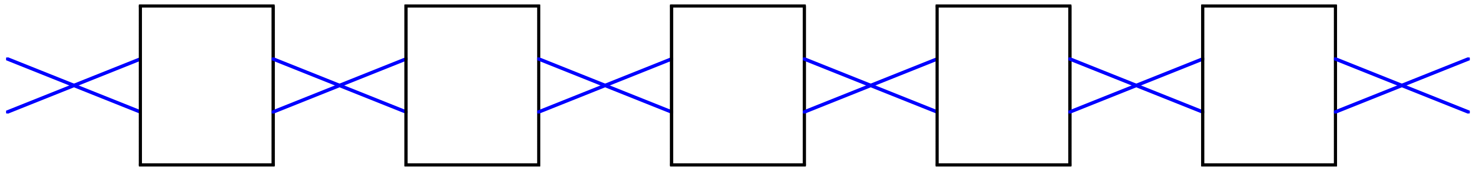
Composition of coset graphs [Graham Higman/MC]

Sometimes two coset graphs for the same group G on (say) n_1 and n_2 points can be composed to produce a transitive permutation representation of larger degree $n_1 + n_2$

– e.g.



This lets us string together multiple copies of coset graphs:



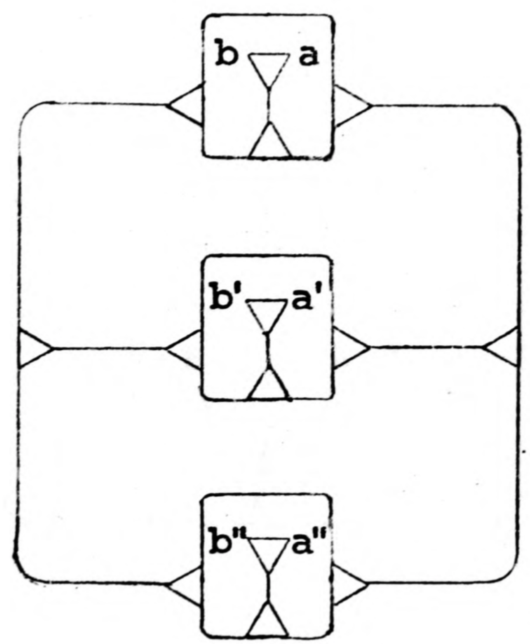
We can often use this method to do all sorts of things, such as prove that certain finitely-presented groups are infinite.

If suitable graphs A and B have a points and b points, then we string together p copies of A and q copies of B and get a new one on $n = pa + qb$ points, and if $\gcd(a, b) = 1$, then $n = pa + qb$ can be any sufficiently large positive integer.

Then add a single copy of an extra graph C (with c points) to disturb the cycle structure of particular elements, and make the permutations from the new graph generate the alternating group A_{n+c} or the symmetric group S_{n+c}.

Example

For $\Delta(2,3,7)$ we could take this 42-point coset graph as A:



and a 113-point coset graph as B, and so on ...

Theorem [MC (doctoral thesis), 1980]

(a) For every $k \geq 7$, all but finitely many alternating groups A_n occur as smooth quotients of $\Delta(2, 3, k)$

(b) For every even $k \geq 8$, all but finitely many symmetric groups S_n occur as smooth quotients of $\Delta(2, 3, k)$.

For each $k \geq 7$ this gives orientably-regular maps of types $\{3, k\}$ and $\{k, 3\}$ with rotation group A_n for all but finitely many n, and others with rotation group S_n for all but finitely many n when k is even.

Moreover, the proofs of (a) and (b) make almost all of the above maps reflexible, and also produce non-orientable regular maps of each type with full automorphism group A_n for all but finitely many n, and others for each type with full automorphism group S_n when k is even.

Theorem [Brent Everitt, 2000]

If $\mathcal{F}$ is a finitely-generated, non-elementary Fuchsian group (that is, a universal group for conformal group actions on compact Riemann surfaces of genus $g > 1$), then all but finitely many alternating groups A_n occur as quotients of $\mathcal{F}$.

Note: those quotients are not claimed to be 'smooth' ...

Corollary [proved in steps by BE & some others (1990s)]

For every hyperbolic pair $\{m, k\}$, all but finitely many alternating groups A_n occur as the rotation group of an orientable-regular map of type $\{m', k'\}$ for some m' and k' dividing m and k respectively.

What about chiral maps?

This question was asked by David Singerman in 1992.

Some answers have arisen more recently.

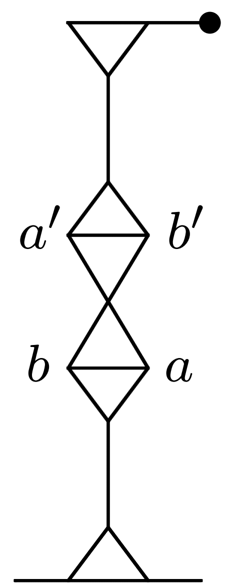
Theorem [Emilio Bujalance, MC & Antonio Costa (2010)]

For every $k \geq 7$, all but finitely many alternating groups A_n are the automorphism group of an orientably-regular but chiral map of type $\{3, k\}$.

This was proved as part of the construction of infinitely many 'pseudo-real' surfaces with automorphism group of largest possible order, namely $12(g-1)$ where g is the genus.

The proof used coset graphs as earlier, but composed with a single copy of one extra graph that breaks reflectional symmetry.

For example, in the (2,3,7) case, we can adjoin an irreflexible 7-point coset graph as follows:



The same kind of construction also shows the following:

For every <u>even</u> $k \geq 8$, all but finitely many symmetric groups S_n occur as the automorphism group of an orientably-regular but chiral map of type $\{3, k\}$.

An important recent theorem (2016):

For every hyperbolic pair (k, m), there exist infinitely many orientably-regular but chiral maps of type $\{m, k\}$.

One 'base' example for each type can be found by using

- permutation representations of the group $\Delta(2, k, m)$ [MC, Veronika Hucíková, Roman Nedela & Jozef Širáň],

or

- group representations and the theory of differentials on Riemann surfaces [Gareth Jones].

Indeed in the former case, a base example can be found with an alternating or symmetric group as automorphism group.

Then infinitely many of each such type can be found using the 'Macbeath trick' to construct covers.

Easy Corollary:

For every given integer $k \geq 3$, there exist infinitely many orientably-regular but chiral maps with valency k.

Proof. This holds for such maps of type $\{m, k\}$ for some m (indeed for infinitely many m).

Stronger version:

For every given integer $k \geq 3$, all but finitely many alternating groups A_n occur as the automorphism group of an orientably-regular but chiral map with valency k.

Proof. We know that all but finitely many A_n occur for chiral maps of type $\{7, 3\}$ or type $\{3, k\}$ for any given $k \geq 7$, and then valency $k \in \{4, 5, 6\}$ can be dealt with using the same approach with the groups $\Delta(2, 4, 5)$ and $\Delta(2, 6, 6)$.

Similarly:

For every given integer $k \geq 3$, all but finitely many symmetric groups S_n occur as the automorphism group of an orientably-regular but chiral map with valency k.

Proof. First, we know that all but finitely many S_n occur for chiral maps of type $\{8, 3\}$ or $\{3, k\}$ for any even $k \geq 8$.

Next, valency $k \in \{4, 5, 6\}$ can be dealt with using the same approach with $\Delta(2, 4, 5)$ and $\Delta(2, 6, 6)$.

Finally, for odd $k \geq 7$ we can adapt the construction for irreflexible alternating quotients of $\Delta(2, 3, k)$ into one that produces irreflexible symmetric quotients of $\Delta(2, 12, k)$, by using 4-cycles in place of 3-cycles in a few places, and hence obtain chiral maps of type $\{12, k\}$ with S_n as automorphism group for all but finitely many n.

Challenge question:

Is it true that for every hyperbolic pair (k, m), all but finitely many alternating groups occur as the full automorphism group of an orientably-regular but chiral map of type $\{m, k\}$?

And finally ...

A long-standing important question:

Among orientably-regular maps on hyperbolic surfaces, how prevalent are the chiral ones?

Specifically, for every integer $g > 1$, let $n_o(g)$ be the number of non-isomorphic orientably-regular maps on orientable surfaces of genus 2 to g, and let $n_r(g)$ and $n_c(g)$ be the numbers of those that are reflexible or chiral, respectively.

For $2 \leq g \leq 300$, the ratios $n_r(g)/n_o(g)$ and $n_c(g)/n_o(g)$ are greater than 1/2 and less than 1/2, but the latter increases while the former decreases.

What happens to $n_r(g)/n_o(g)$ and $n_c(g)/n_o(g)$ as $g \to \infty$?

Thank you – from the antipodes!

Regular and chiral maps with given valency or given type

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Abstract:

A map is 2-cell embedding of a connected graph in a closed surface, breaking up the surface to simply-connected regions called faces. The map is called ‘regular’ if its automorphism group has a single orbit on flags (which are like incident vertex-edge-face triples), or ‘orientably-regular’ if the surface is orientable and the automorphism group of the map has a single orbit on arcs (incident vertex-edge pairs). If a map of the latter kind admits no reflections (e.g. fixing an arc but swapping the two faces incident with it), then the map is called ‘chiral’. In all such maps with a high degree

of symmetry, all vertices have the same valency, say k, and all faces have the same size, say m, and then we call the ordered pair $\{m, k\}$ the ‘type’ of the map.

Writing a section of a forthcoming book on such maps (with Gareth Jones, Jozef Siran and Tom Tucker) has prompted me to review and extend what is known about regular and chiral maps with given valency k, or with given type $\{m, k\}$, including what happens in the special case where the automorphism group is isomorphic to an alternating or symmetric group. I will summarise findings in this talk.