Faithful transitive permutation representations of regular polytopes

Maria Elisa and Claudio Piedade
maria.elisa@ua.pt ‖ claudio.a.piedade@ua.pt

CIDMA Departamento de Matemática
Universidade de Aveiro   Portugal

The degree of a polytope

We commonly use permutation representations to represente groups.
Michael I. Hartley's Atlas, Dimitri Leemans' Atlas.

The automorphism group G of a regular polytope is a string C-group.

$$G = \langle \rho_0, \ldots, \rho_{r-1} \rangle$$

- Given a group G and a core-free subgroup H of G. The action of G on G/H gives a faithful transitive permutation representation of degree $|G : H|$. Moreover, H is the stabilizer of a point.
- On the other hand, the stabilizer of a point in a faithful transitive permutation representation is core-free.

This gives a one-to-one correspondence between core-free subgroups and faithful transitive permutation representations.

Degree of a polytope $\equiv$ number of vertices of a Schreier coset graph of G.

Finite universal regular polytopes $\{\{4,4\}_{(t_1,t_2)},\{4,4\}_{(s_1,s_2)}\}$

(t_1,t_2)	(s_1,s_2)	$\|G\|$	G
(2,0)	(s,s), $s \geq 2$	$64s^2$	$(C_2 \times C_2) \rtimes [4,4]_{(s,s)}$
(2,0)	$(2s,0)$, $s \geq 1$	$128s^2$	$(C_2 \times C_2) \rtimes [4,4]_{(2,0)}$, $s = 1$ $((C_2 \times C_2) \rtimes [4,4]_{(s,s)}) \times C_2$, $s \geq 2$
(3,0)	(3,0)	1440	$S_6 \times C_2$
(3,0)	(4,0)	36864	$C_2 \wr [4,4]_{(3,0)}$
(3,0)	(2,2)	2304	$(S_4 \times S_4) \rtimes (C_2 \times C_2)$
(2,2)	(2,2)	1024	$C_2^4 \rtimes [4,4]_{(2,2)}$
(2,2)	(3,3)	9216	$C_2^6 \rtimes [4,4]_{(3,3)}$
(3,0)	(5,0)	3916800	$Sp_4(4) \times C_2 \times C_2$

What are the degrees of these regular polytopes?

Computational Results

(t_1, t_2)	(s_1, s_2)	Set of Possible Degrees
(3,0)	(3,0)	$\{m \mid m$ a divisor of $1440 \wedge m \geq 60\wedge$ $\wedge m \neq 96\} \cup \{40, 30, 24, 20, 12\}$
(3,0)	(4,0)	$\{m \mid m$ a divisor of $36864 \wedge m \geq 72\} \cup \{18, 36, 48\}$
(3,0)	(2,2)	$\{m \mid m$ a divisor of $2304 \wedge m \geq 12\}$
(2,2)	(2,2)	$\{m \mid m$ a divisor of $1024 \wedge m \geq 16\}$
(2,2)	(3,3)	$\{m \mid m$ a divisor of $9216 \wedge m \geq 24\}$
(3,0)	(5,0)	$\{2^i \cdot 255,\ 2^i \cdot 1275, 2^i \cdot 3825,\ 2^i \cdot 425 \mid 2 \leq i \leq 10\} \cup$ $\cup \{2^i \cdot 765 \mid 3 \leq i \leq 10\} \cup \{2^i \cdot 15,\ 2^i \cdot 17 \mid 5 \leq i \leq 6\} \cup$ $\cup \{2^i \cdot 85 \mid i \in \{2, 6, 7, 8\}\} \cup \{2^i \cdot 225 \mid 8 \leq i \leq 10\} \cup$ $\cup \{2^i \cdot 153 \mid 7 \leq i \leq 10\} \cup \{2^i \cdot 51 \mid 7 \leq i \leq 8\}$

We also found the degrees of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,0)}\}$ and $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$ up to $s = 79$ and $s = 47$, respectively.

The normal abelian subgroup T of G of order s^2

Consider a faithful transitive permutation representation of G of degree n and T be a normal abelian subgroup of G generated by two elements of order s.

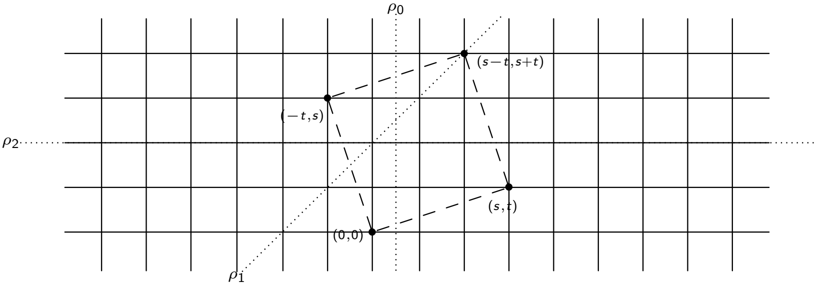
1. If T is transitive, then $n = s^2$.
2. If T is intransitive then $n = mk$ with
   (i) $k = ab$ where $s = lcm(a, b)$ and,
   (ii) m is a divisor of $\frac{|G|}{s^2}$.

The degrees of types $[4,4]$ and $[4,4,4]$

Regular Polytope	$\frac{\lvert G \rvert}{s^2}$	Set of Degrees
$\{4,4\}_{(s,0)}$	8	s^2, $2ab$, $4ab$, $8ab$
$\{4,4\}_{(s,s)}$	16	$2s^2$, $4ab$, $8ab$, $16ab$
$\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$	64	$4s^2, 8ab, 16ab, 32ab, 64ab$ or $n = 4ab$ if a and b are both even
$\{\{4,4\}_{(2,0)}, \{4,4\}_{(2s,0)}\}$	128	$8s^2, 16ab, 32ab, 64ab, 128ab$ or $n = 8ab$ if a and b are both even

- Let $G \leq K$ and $|K : G| = \kappa$. If H is a core-free subgroup of G, then H is core-free in K. Hence, if n is a degree of G, then κn is a degree of K.

The toroidal regular maps of type $[4,4]$



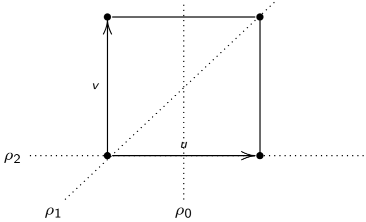
The automorphim groups of the regular toroidal maps $\{4,4\}_{(s,0)}$ and $\{4,4\}_{(s,s)}$ are factorizations of the Coxeter group $[4,4]$, by

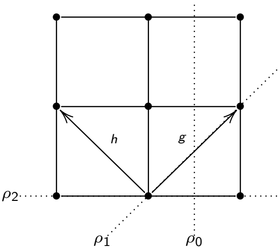
$$(\rho_0\rho_1\rho_2\rho_1)^s = 1 \text{ or } (\rho_0\rho_1\rho_2)^{2s} = 1,$$

respectively. Their sizes are $8s^2$ and $16s^2$, respectively.

The toroidal regular maps of type $[4,4]$

1. For the map $\{4,4\}_{(s,0)}$ let $T = \langle u, v\rangle$ where $u = \rho_0\rho_1\rho_2\rho_1$ and $v = u^{\rho_1}$ corresponding to the unitary translations on the left;
2. For the map $\{4,4\}_{(s,s)}$ let $T = \langle g, h\rangle$ where $g := uv = (\rho_0\rho_1\rho_2)^2$ and $h := u^{-1}v = g^{\rho_0}$ corresponding to unitary translations on the right.





We have the following equalities

1. $u^{\rho_0} = u^{-1}$, $u^{\rho_2} = u$, $v^{\rho_0} = v$ and $v^{\rho_2} = v^{-1}$.
2. $g^{\rho_1} = g$, $g^{\rho_2} = h^{-1}$ and $h^{\rho_1} = h^{-1}$.

The existence of degrees s^2, $2ab$, $4ab$, $8ab$ for the map $\{4,4\}_{(s,0)}$

The groups of $\{4,4\}_{(s,0)}$ ($s > 2$) act faithfully on the sets of vertices, faces, edges, darts and flags.

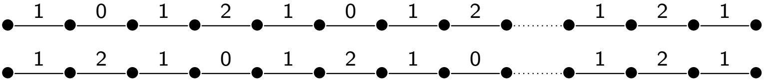
If a and b are nonnegative integers and $s = lcm(a,b)$ then

1. $H = \langle u^a, v^b \rangle$ is core-free and $|G : H| = 8ab$,
2. $H = \langle u^a, v^b \rangle \rtimes \langle \rho_0 \rangle$ is core-free and $|G : H| = 4ab$,
3. if $ab \neq s$ then $H = \langle u^a, v^b \rangle \rtimes \langle \rho_0, \rho_2 \rangle$ is core-free and $|G : H| = 2ab$, and
4. $H = \langle u \rangle \rtimes \langle \rho_0, \rho_2 \rangle$ is core-free and $|G : H| = 2s$.

Minimal degree Schreier Coset Graphs for $\{4,4\}_{(s,0)}$(of degree $2s$):

s odd:

s even:



The degrees of $\{4,4\}_{(s,s)}$

Let G be the group of $\{4,4\}_{(s,s)}$ acting faithfully on n points and suppose that $T = \langle g, h \rangle$.

T is intransitive:

Assume that T is transitive. Let $u = \rho_0\rho_1\rho_2\rho_1$.
As T is regular we may see the n points as elements of the group T, one of which is 1 (the identity), therefore $1u = g^{\alpha}h^{\beta}$.
As u commutes with both g and h,

$$g^i h^j u = g^{i+\alpha}h^{j+\beta} \text{ and } g^i h^j u^s = g^{i+s\alpha}h^{j+s\beta} = g^i h^j.$$

Hence the order of u is at most s, a contradiction.

$\rightarrow$ Hence has observed before G, the group of $\{4,4\}_{(s,s)}$, is embedded into $S_k \wr S_m$ with $m \in \{2,4,8,16\}$.

The degrees of $\{4,4\}_{(s,s)}$

If $m = 2$ then $n = 2s^2$:

Consider the group G' of automorphisms of $\{4,4\}_{(2s,0)}$ and $H = \langle u, v \rangle < G'$ its translation group of size $(2s)^2$.

We have that G' acts faithfully on two copies of the set of n points as follows:

x is a point on one of the copies, $x(uv)^s$ is on the other copy.

As T is a proper subgroup of H, H must be transitive on $2n$ points. Hence,

$$2n = (2s)^2.$$

Conclusion: The degrees of $\{4,4\}_{(s,s)}$ are the degrees of $\{4,4\}_{(s,0)}$ multiplied by 2.

$$\{4,4\}_{(s,0)} \underset{\times 2}{\overset{:2}{\leftrightarrows}} \{4,4\}_{(s,s)}$$

Relations between the degrees of types $[4,4]$ and $[4,4,4]$

If n is a degree of $\{4,4\}_{(s,s)}$, then $4n$ is a degree of $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$.

This guarantees that $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$ has degrees

$$8s^2,\ 16ab,\ 32ab \text{ and } 64ab,$$

with $s = lcm(a,b)$; while the degrees of $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$ are twice bigger.

But this list is incomplete...

Let G be the group of degree n of $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$.

Input: n a degree of G

Consider the central involution $\delta := (\rho_0\rho_1)^2$ in G. Let $f : G \to S_{\frac{n}{2}}$ be the embedding of G into $S_2 \wr S_{\frac{n}{2}}$ determined by the $\langle\delta\rangle$-orbits.

Claim! $Ker(f) = \langle\delta\rangle$.

> Suppose that $\langle g, h\rangle \cap Ker(f)$ is nontrivial.
>
> As $Ker(f)$ is embedded into $C_2^{\frac{n}{2}}$.
> The involutions of $\langle g, h\rangle$ are $g^{s/2}$, $h^{s/2}$ or $(gh)^{s/2}$ (s must be even).
> Any case implies that $(gh)^{s/2} \in Ker(f)$.
> As $(gh)^{s/2}$ is a central involution we get $(gh)^{s/2} = \delta$, a contradiction.

Output: $n/2$ is a degree of $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}/\langle(\rho_0\rho_1)^2\rangle$.

Correspondence between $\{4,4\}_{(s,s)}$ and $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$

Let G be the group of $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$.

Input: n a degree of G; ρ_0 fixed-point-free.

The orbits of $\langle\rho_0,(\rho_0\rho_1)^2\rangle$, a normal subgroup of G, are 4-sets

$$\{x,\, x\rho_0,\, x(\rho_0\rho_1)^2,\, x\rho_1\rho_0\rho_1\}.$$

$G/\langle\delta\rangle$ is isomorphic to $\langle\alpha\rangle\times H$ of degree $\frac{n}{2}$.

- α is an involution (ρ_0 acting on 2-sets $\{x, x(\rho_0\rho_1)^2\}$).
- H is the group of $\{4,4\}_{(s,s)}$.

Now $\langle\alpha\rangle\times H$ is embedded into $S_2 \wr S_{\frac{n}{4}}$.

Again... the kernel is $\langle\alpha\rangle$.

$\{4,4\}_{(s,s)}$ acts faithfully on the 4-sets $\{x,\, x(\rho_0\rho_1)^2,\, x\rho_0,\, x\rho_0(\rho_0\rho_1)^2\}$.

Output: $\{4,4\}_{(s,s)} \underset{\times 4}{\overset{:4}{\leftrightarrows}} \{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$.

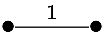
Correspondence between $\{4,4\}_{(s,s)}$ and $\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\}$

What happens when if ρ_0 has a fixed point?

$m \neq 1$: Suppose that T is transitive. If ρ_0 has a fixed point then, as ρ_0 commutes with g and h, ρ_0 is trivial, a contradiction.

$m \neq 2$:

Suppose that $m = 2$.



- $\rightarrow$ ρ_0 fix an entire block point-wise.
- $\rightarrow$ ρ_1 must swap the two blocks, otherwise ρ_0 would be trivial.
- $\rightarrow$ Neither ρ_2 nor ρ_3 can swap the blocks.
- $\rightarrow$ $(\rho_0\rho_1)^2$ fixes the blocks and therefore, s is even.

We then consider the actions of g and h inside the blocks. Always leading to a contradiction.

$m = 4$: Many more possibilities for the block actions!!!!

If $k \neq s^2$, then $k = ab$, with a and b being even divisors of s such that $s = lcm(a,b)$.

Sporadic core-free subgroups

Let a and b be positive integers such that $s = lcm(a, b)$. The subgroups

1. $\langle \rho_0 \rangle \times \langle \rho_2, \rho_3 \rangle$;
2. $\big(\langle \rho_0 \rangle \times \langle g^{a/2}, h^b \rangle\big) \rtimes \langle \rho_2, \rho_1\rho_2\rho_1 \rangle$ if a is even and $lcm(a/2, b) = s$ and $\big(\langle \rho_0 \rangle \times \langle g^{a}, h^{b/2} \rangle\big) \rtimes \langle \rho_2, \rho_1\rho_2\rho_1 \rangle$ if b is even and $lcm(a, b/2) = s$;
3. $\big(\langle \rho_0 \rangle \times \langle g^{a}, h^b \rangle\big) \rtimes \langle \rho_2, \rho_1\rho_2\rho_1 \rangle$

are core-free subgroups of G with indexes $4s^2$, $4ab$ and $8ab$, respectively. This completes the list of degrees of degree of $\{\{4,4\}_{(2,0)}, \{4,4\}_{(s,s)}\}$.

$\{\{4,4\}_{(2,0)},\{4,4\}_{(s,s)}\} \underset{\times 2}{\overset{:2}{\leftrightarrows}} \{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$

For $s = 2$ ✓ Let $s \geq 3$ and assume by induction that it holds for $s' < s$.

Let G be the group of $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s,0)}\}$ and $T := \langle u^2, v^2 \rangle \triangleleft G$.

$\delta := (uv)^s$ determines an embedding of G into $S_2 \wr S_{\frac{n}{2}}$

Let f denote the homomorphism $G \to S_{\frac{n}{2}}$ determined by this embedding.

When $Ker(f) = \langle \delta \rangle$ ✓ But that is not always the case!

As the elements of $Ker(f)$ are involutions,

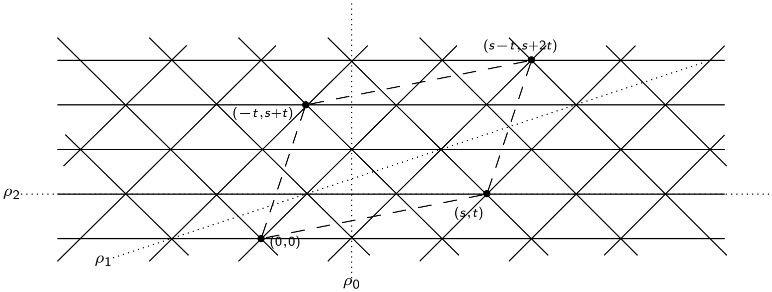
$$Ker(f) \neq \langle \delta \rangle \Rightarrow s \text{ is even.}$$

$$s \text{ even and } Ker(f) \neq \langle \delta \rangle \Rightarrow \langle u^s, v^s \rangle \leq Ker(f)$$

Then $G/Ker(f)$ is isomorphic to the group of $\{\{4,4\}_{(2,0)},\{4,4\}_{(2s',0)}\}$ where $s' := s/2$.

Then induction works perfectly...

The toroidal maps of type $[3,6]$



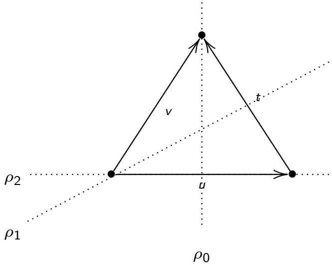
The group of symmetries of $\{3,6\}_{(s,0)}$ and $\{3,6\}_{(s,s)}$ are factorizations of the Coxeter group $[3,6]$ by

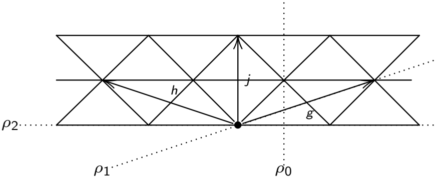
$$(\rho_0\rho_1\rho_2)^{2s} = 1 \text{ and } ((\rho_2\rho_1)^2\rho_0)^{2s} = 1,$$

respectively. The number of flags of $\{3,6\}_{(s,0)}$ is $12s^2$ while the number of flags of $\{3,6\}_{(s,s)}$ is $36s^2$.

The toroidal maps $\{3,6\}$

For the map $\{3,6\}_{(s,0)}$ consider the translations $u = \rho_0(\rho_1\rho_2)^2\rho_1$, $v = u^{\rho_1} = (\rho_0\rho_1\rho_2)^2$ and $t = u^{-1}v$. In the case of the map $\{3,6\}_{(s,s)}$, consider $g := uv = (\rho_0(\rho_1\rho_2)^2)^2$, $h := u^{-2}v = g^{\rho_0}$ and $j := hg$.





We have the following equalities

1. $u^{\rho_0} = u^{-1}$, $u^{\rho_2} = u$, $v^{\rho_0} = t$ and $v^{\rho_2} = t^{-1}$.
2. $g^{\rho_1} = g$, $g^{\rho_2} = h^{-1}$ and $h^{\rho_1} = j^{-1}$.

The degrees of the maps of type $[3,6]$

Let G be the group of $\{3,6\}_{(s,0)}$ and $T = \langle u, v\rangle$. If T is intransitive then $G \leq S_k \wr S_m$ with

- $k = ds$ where d a divisor of s. (which is equivalent to write $k = ab$ with $s = lcm(a,b)$)
- m is a divisor of 12.

Let $s \geq 2$. The degrees of $\{3,6\}_{(s,0)}$ are

- s^2,
- $3ds$, $6ds$ or $12ds$ for any divisor d of s,
- $2ds$ and $4ds$ if and only if d is a divisor of s and all prime divisors of s/d are equal to 1 mod 6.

For $m = 2$ and $m = 4$ the degrees are sporadic!!!!!

$\rightarrow$ The degrees of $\{3,6\}_{(s,s)}$ are the above one multiplied by 3.

The degrees of the toroidal hypermaps

Despite that $(3,3,3)_{\mathsf{s}}$ is an index two subgroup of $\{6,3\}_{\mathsf{s}}$ for $\mathsf{s} \in \{(s,0),(s,s)\}$, it is not true in general that if n is a degree of $\{6,3\}_{\mathsf{s}}$ then $n/2$ is the degree of a toroidal hypermap $(3,3,3)_{\mathsf{s}}$.

For the toroidal regular hypermaps, the results can be summarized as follows. Let $s \geq 2$.

- for the hypermap $(3,3,3)_{(s,0)}$, the possible degrees are s^2, $2ds$, $3ds$ and $6ds$ where d is a divisor of s. Moreover, the degree $2ds$ exists if and only if all prime divisors of s/d are congruent to 1 modulo 6;
- for the hypermap $(3,3,3)_{(s,s)}$, the possible degrees are those of the hypermap $(3,3,3)_{(s,0)}$ multiplied by 3.

References

1. *The degrees of regular polytopes of type* $[4, 4, 4]$*.* SIAM Journal on Discrete Mathematics, to appear, 14 pages.
2. *The degrees of toroidal regular proper hypermaps.* Art Discrete Appl. Math. 4 (2021), #P3.13.
3. *Correction to "Faithful permutation representations of toroidal regular maps".* Journal of Algebraic Combinatorics, 733–738 (2021)
4. *Faithful permutation representations of toroidal regular maps.* Journal of Algebraic Combinatorics, 52 (2020), 317–337