Configurations of Points and Lines

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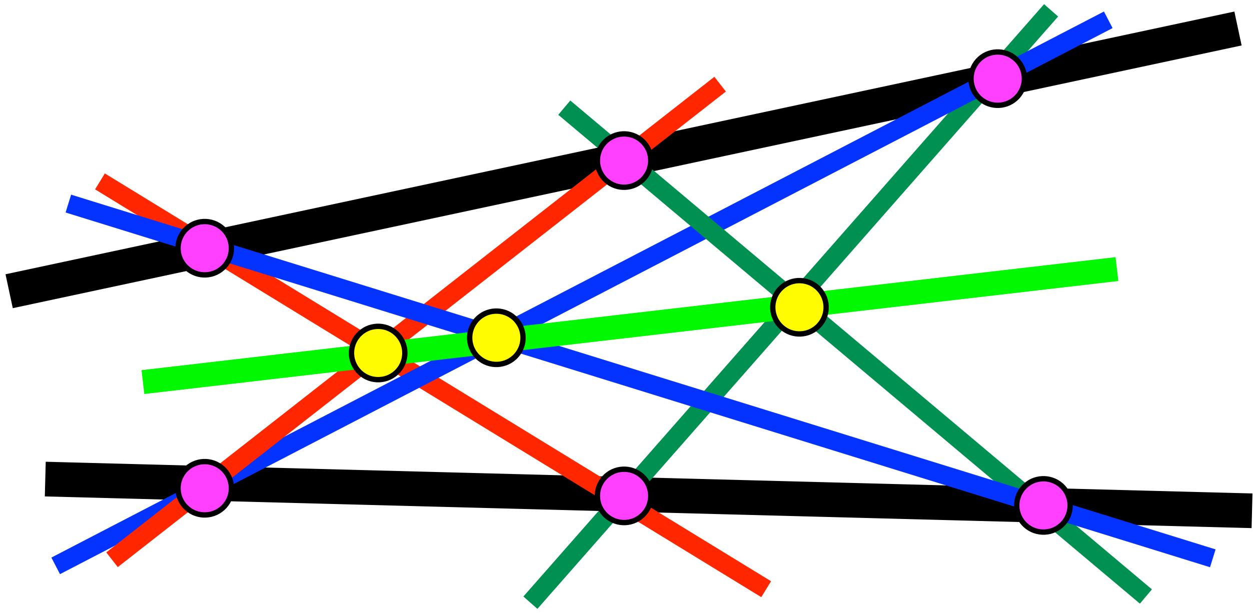
Algebraic Graph Theory International Webinar

What is a (p_q, n_k) configuration?

- p points
- n lines
- q points on each line
- k lines through each point

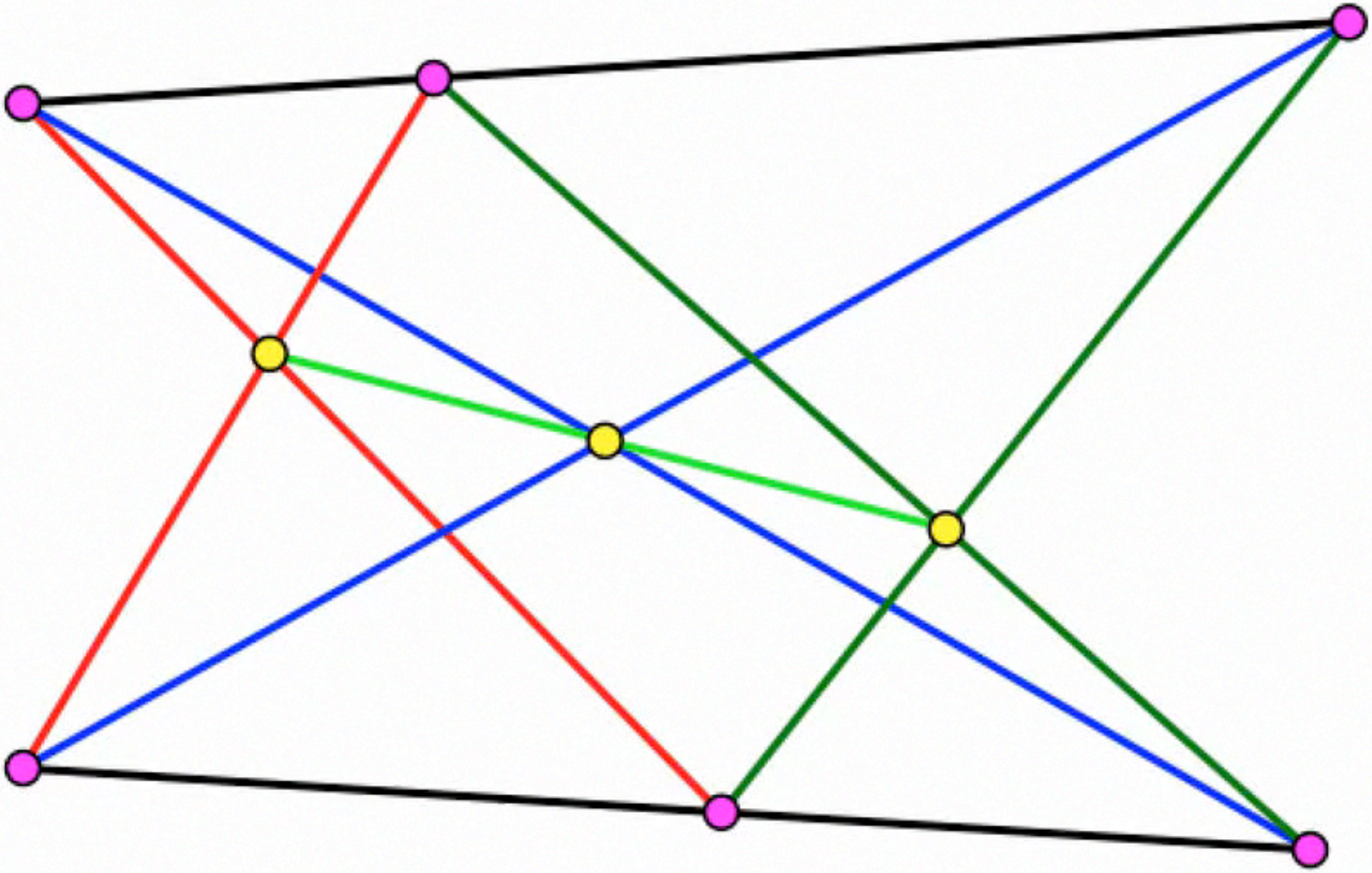


Pappus Configuration:

(9_3, 9_3) configuration, (9_3) configuration

3-configuration

Balanced configuration



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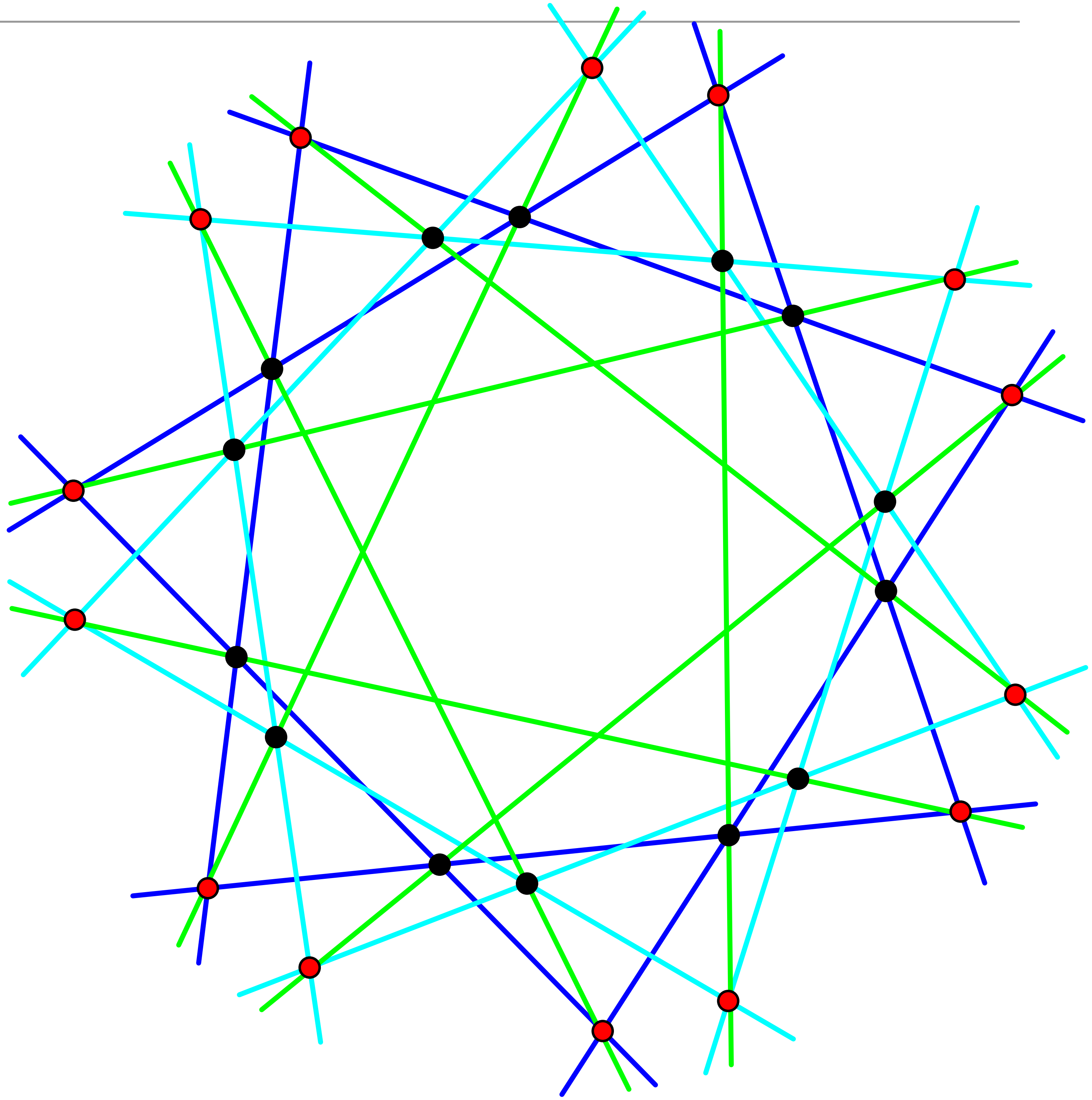


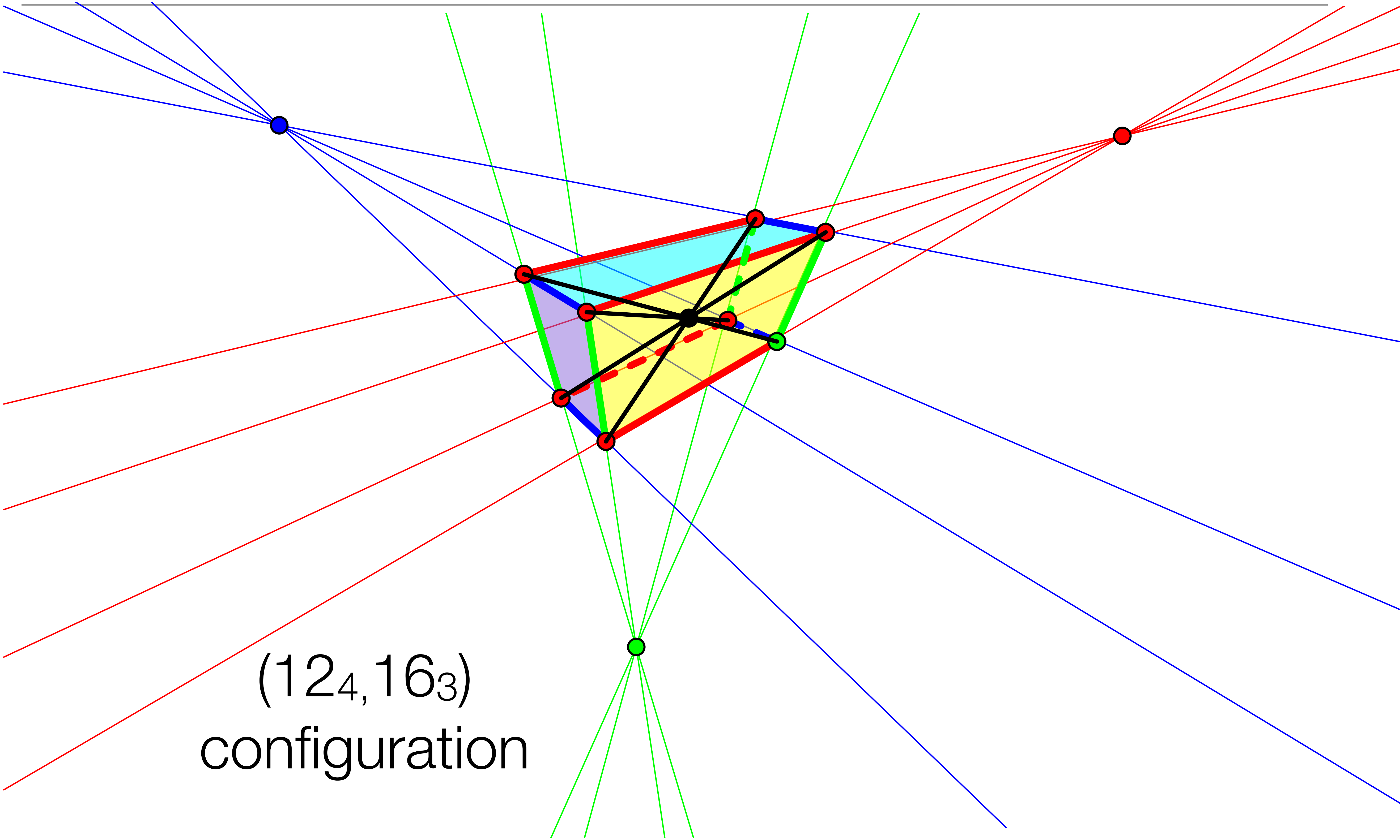
$(28_3, 21_4)$ configuration

(3,4)-configuration

Unbalanced configuration

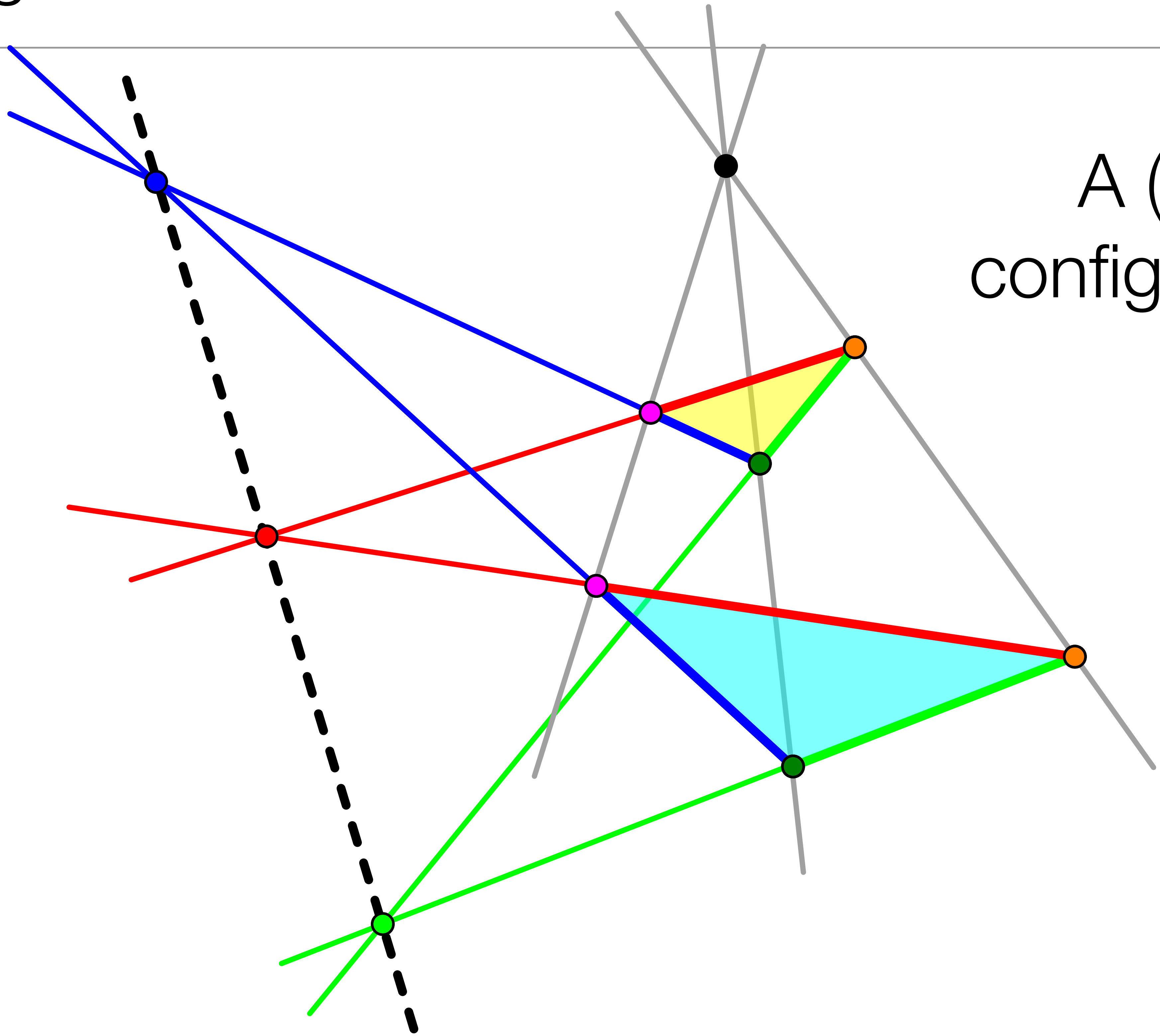
Where to find more examples?

Reye Configuration



$(12_4, 16_3)$
configuration

Desargues' Theorem



A (10_3)
configuration!

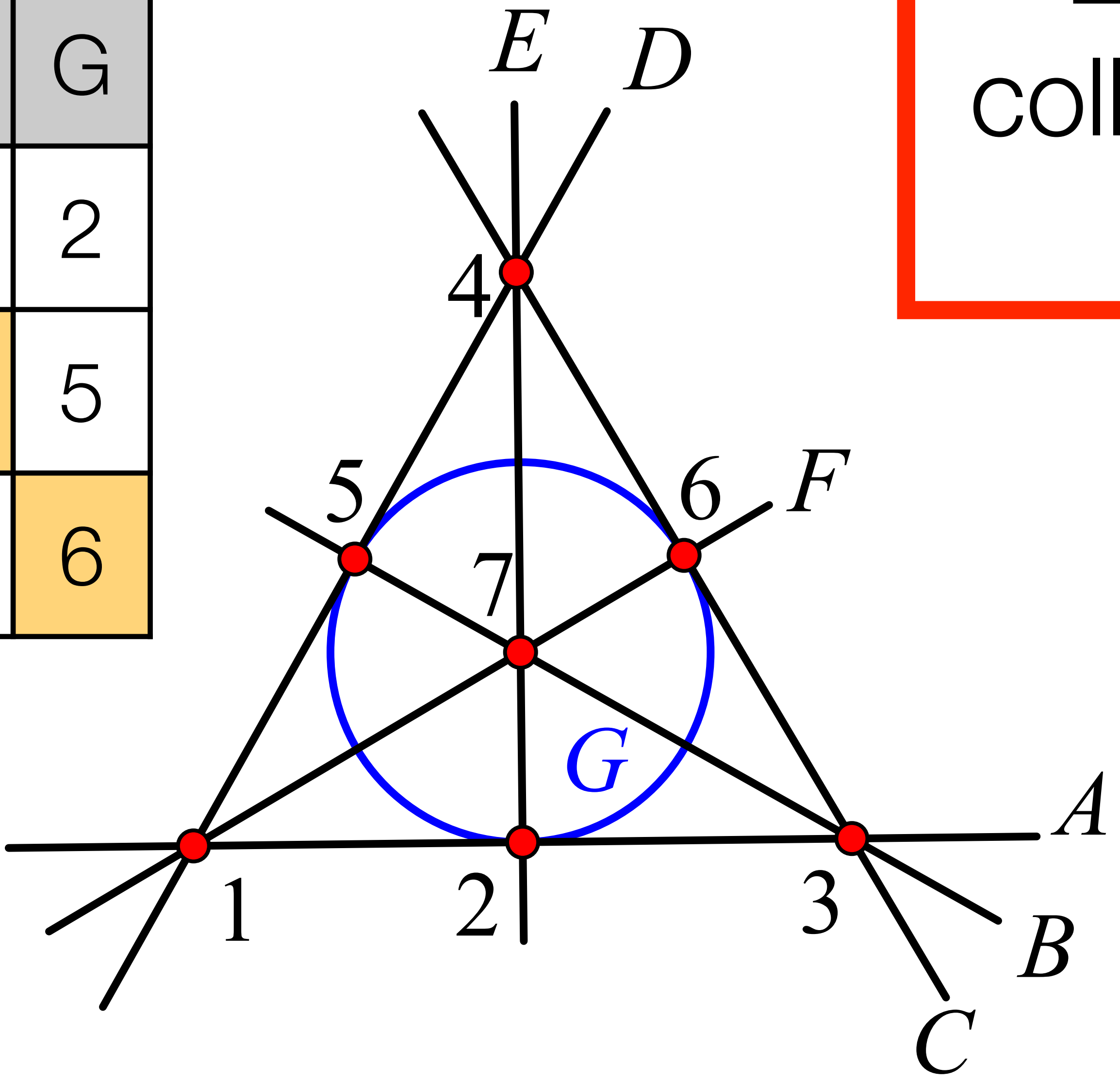
Kinds of configurations

Combinatorial

A	B	C	D	E	F	G
1	3	3	1	2	1	2
2	5	4	4	4	6	5
3	7	6	5	7	7	6

Each number is in 3 columns

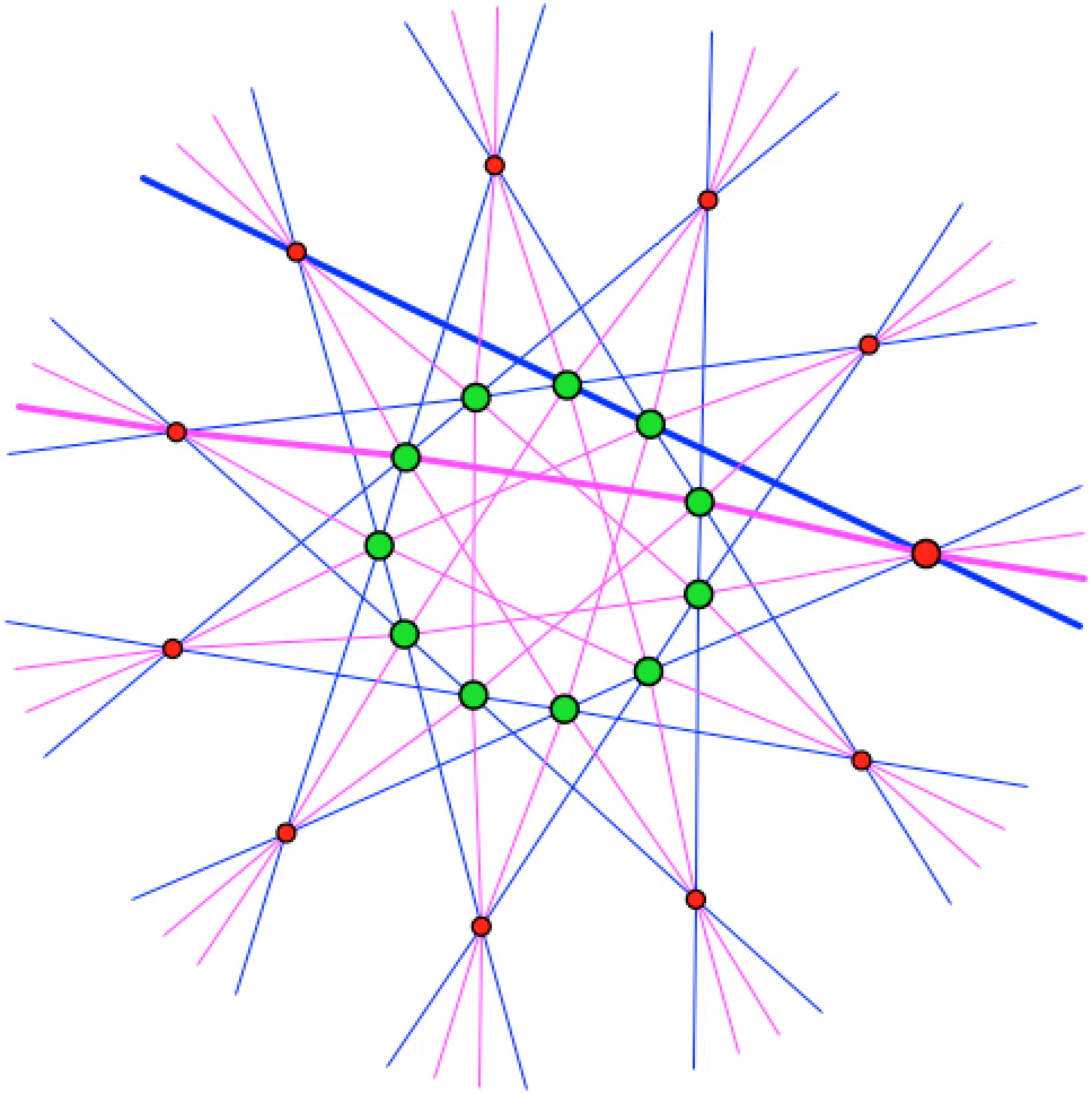
“Lines” are collections of points



(7$_3$) Fano Configuration

Kinds of configurations

Topological

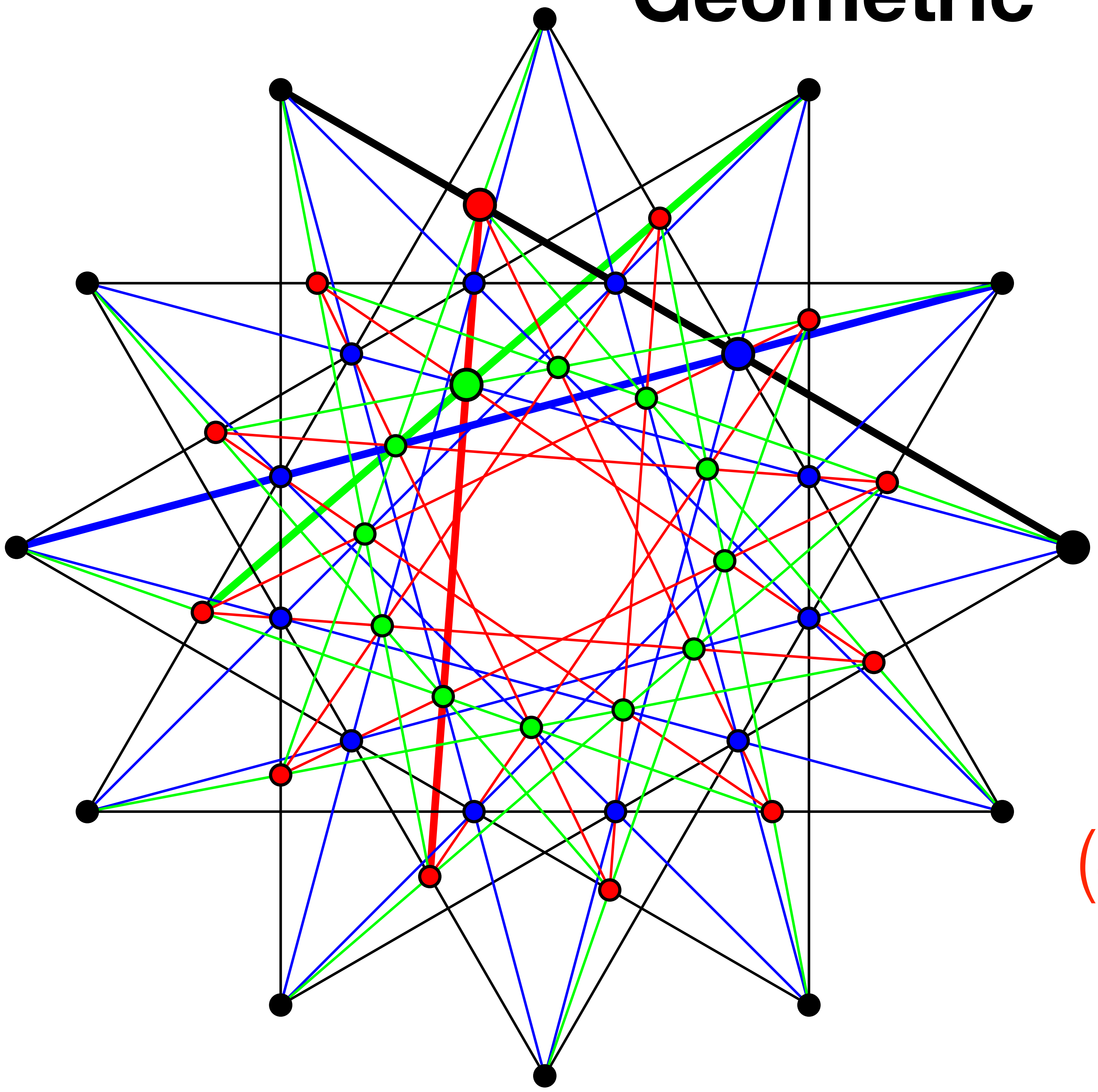


“Lines” can curve, but they can’t intersect twice

(22_4) topological configuration

Kinds of configurations

Geometric



“Lines” are Actual Straight Lines!

(48_5) geometric configuration

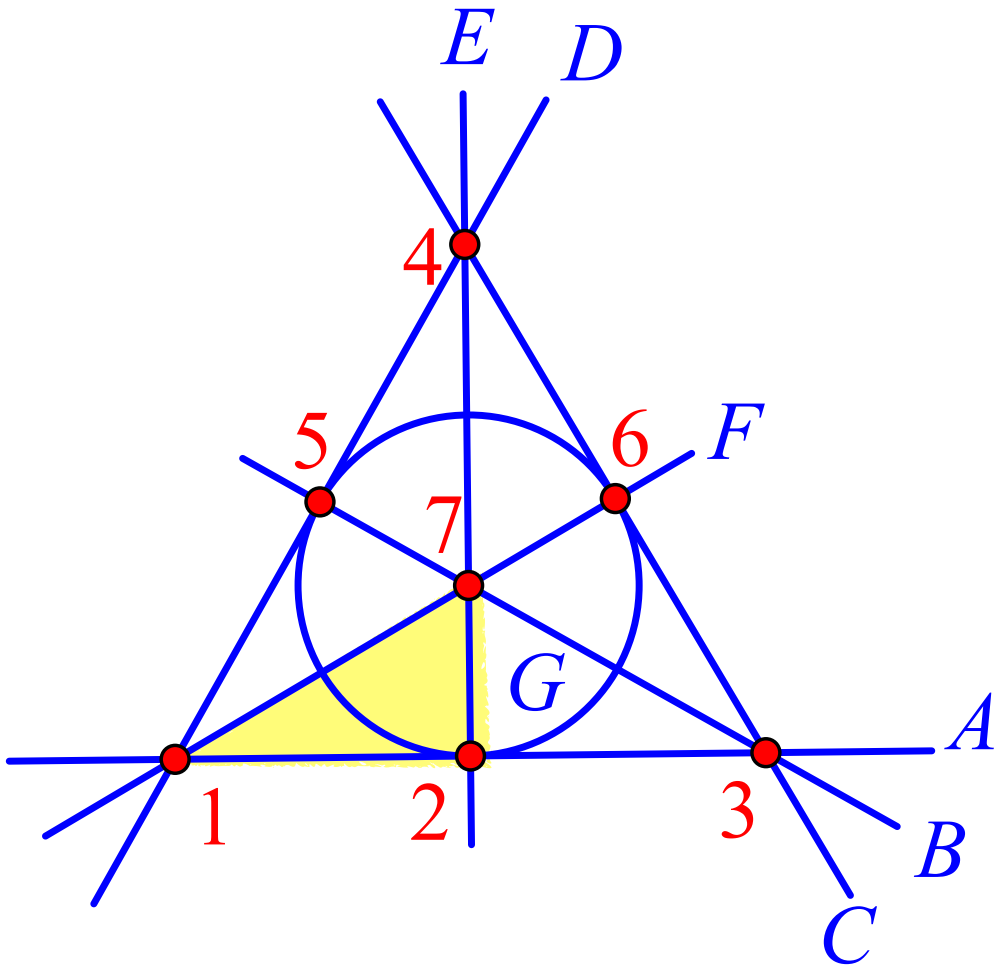
Questions about configurations

- **Existence**: *are there any….?*
- **Identification:** *can we find some…?*
- **Classification**: *what features…?*
- **Application:** *how can we use…?*

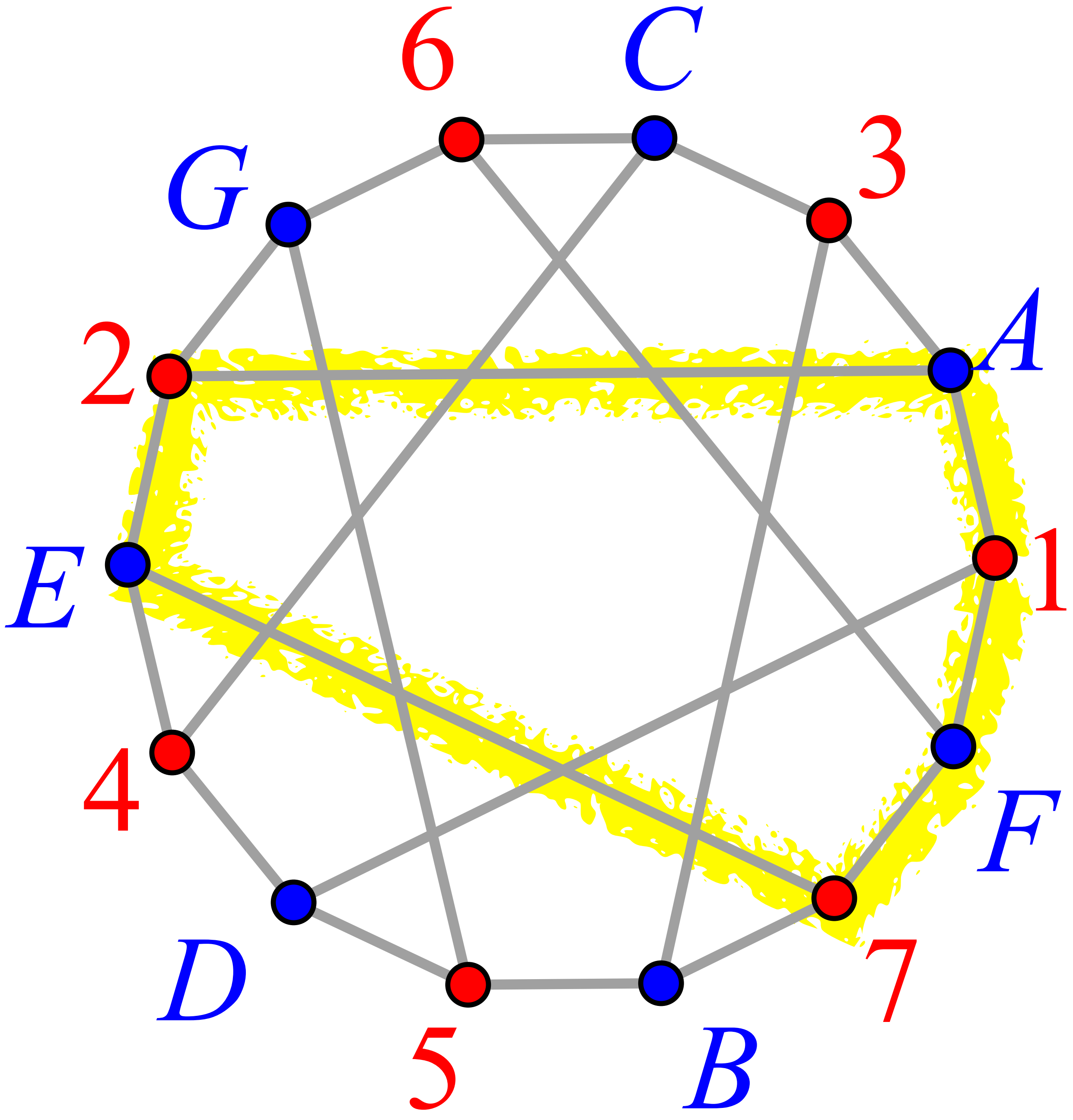
Existence:

For which n do there exist (n_k) configurations? For a fixed n, how many are there?

Combinatorial Configurations and Graphs

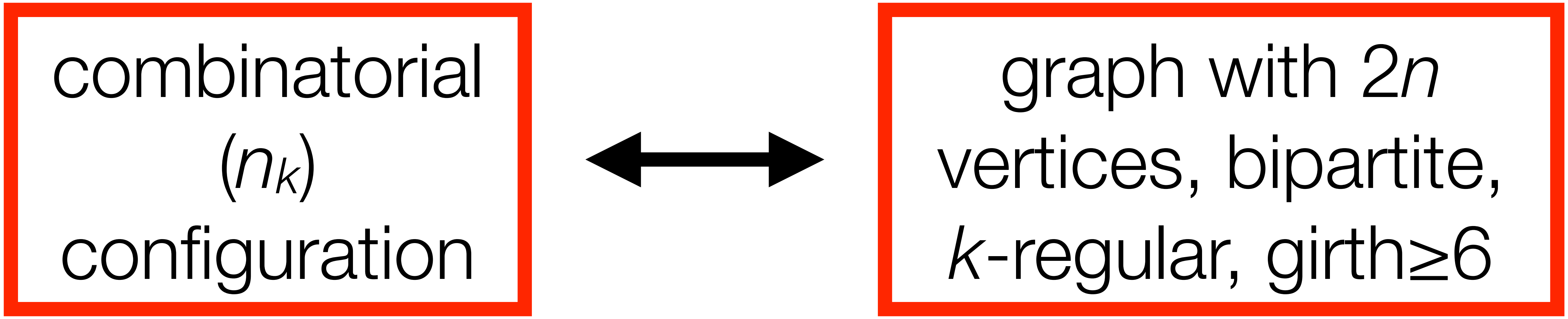


Combinatorial Configuration



Incidence (“Levi”) Graph

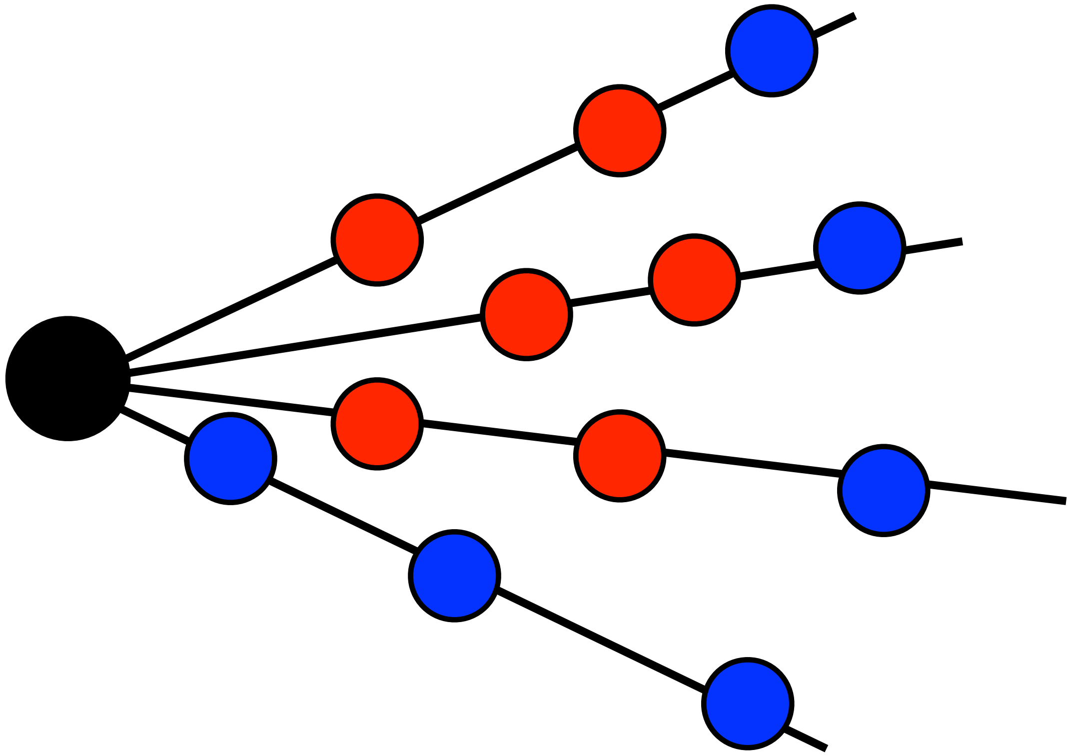
Combinatorial Configurations and Graphs



combinatorial (n_k) configuration

graph with $2n$ vertices, bipartite, k-regular, girth$\geq$6

Combinatorial Configurations (n_k)

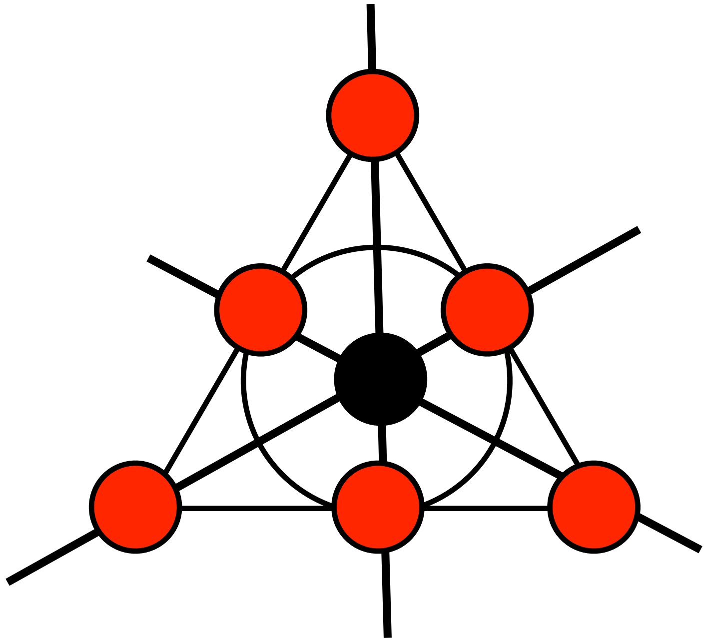


(n_3): $n \geq 7$

(n_4): $n \geq 13$

(n_k): $n \geq k(k-1)+1$

Minimal: finite projective planes of order *k*-1…or *(k,6)-cages* in general



Smallest combinatorial (n_k) configuration?

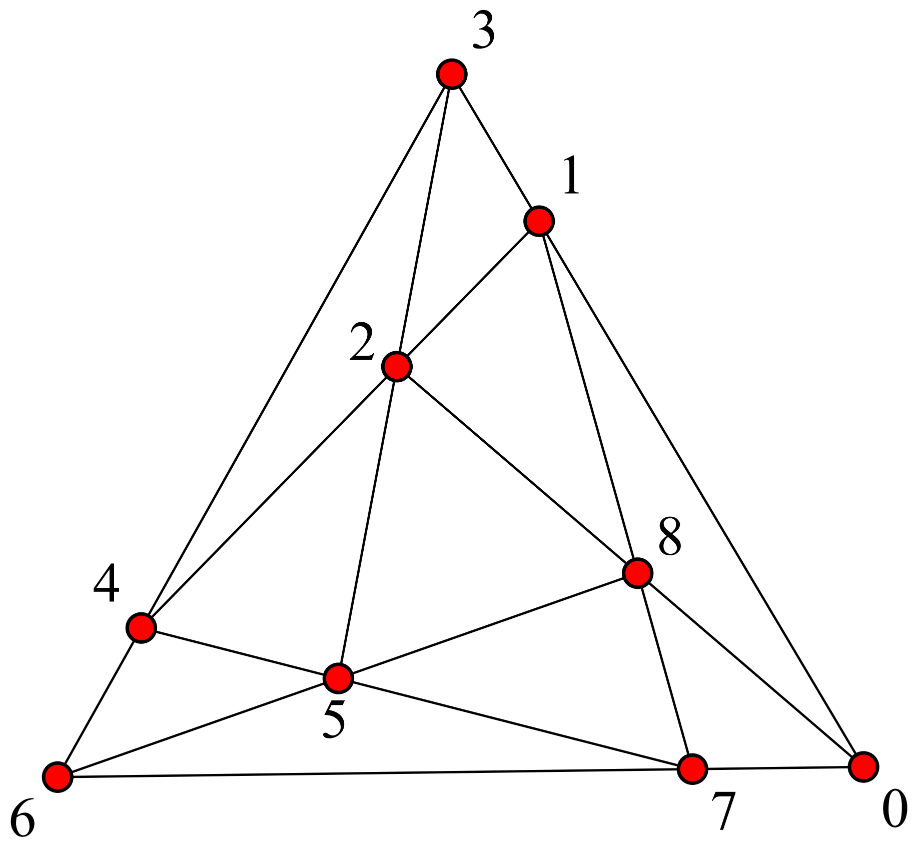
k	smallest	FPP?
3	(7_3)	yes
4	(13_4)	yes
5	(21_5)	yes
6	(31_6)	yes
7	(45_7)	no!
8	(57_8)	yes
9	(73_9)	yes
10	(91_{10})	yes
11	$112 \leq n \leq 120$	no
12	(133_{12})	yes
13	$157 \leq n \leq 168$	no

Combinatorial (n_k) configurations

- Some are easy to find: cyclic configurations
  - k=3: $\mathrm{Cyc}_n[0, 1, 3]$
  - k=4: $\mathrm{Cyc}_n[0,1, 3, 6]$
  - k=5: $\mathrm{Cyc}_n[0, 1, 4, 14, 16]$

A	B	C	D	E	F	G	H	I
0	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	0
3	4	5	6	7	8	0	1	2

Open Question: what can we say about geometric embedding of cyclic 4-configurations?

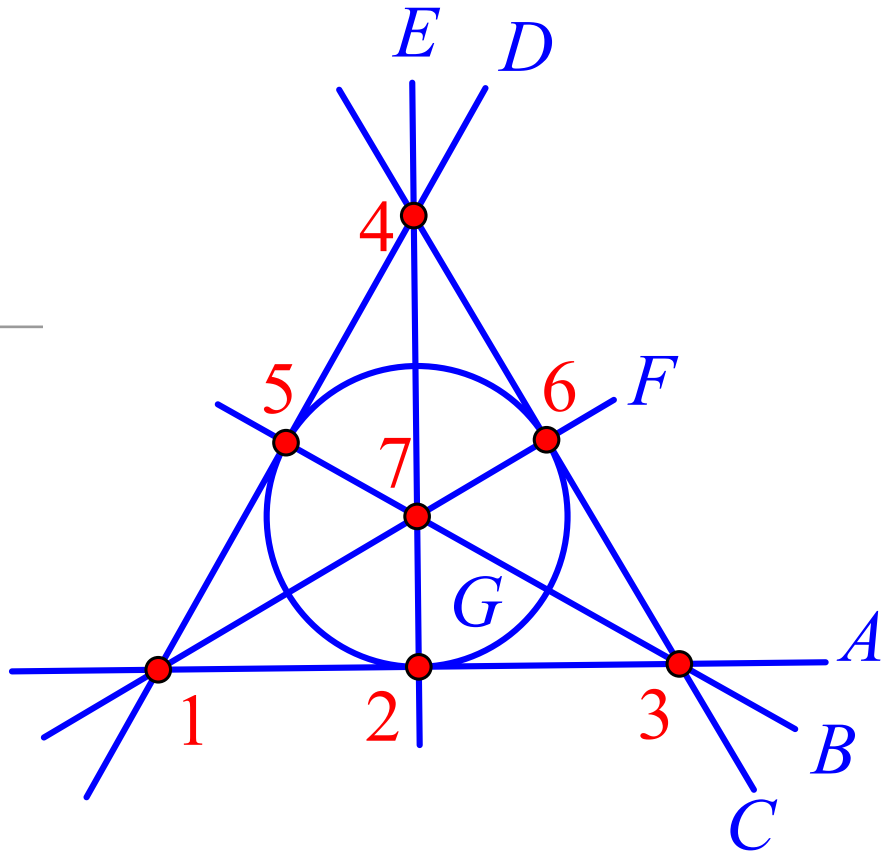


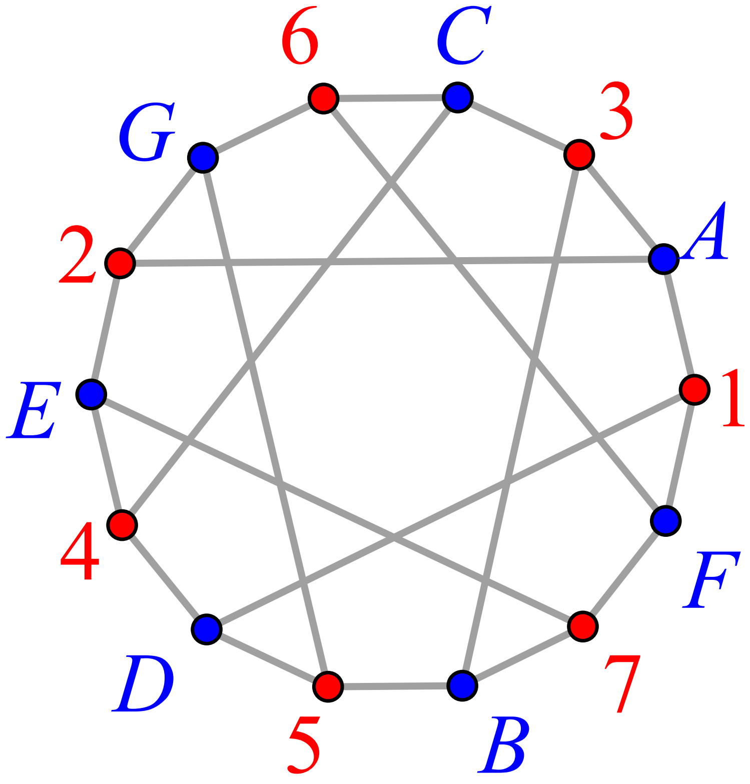
Existence of 3-configurations

Combinatorial 3-configurations

- Exist for all $n \geq 7$
- Enumerated for $7 \leq n \leq 19$

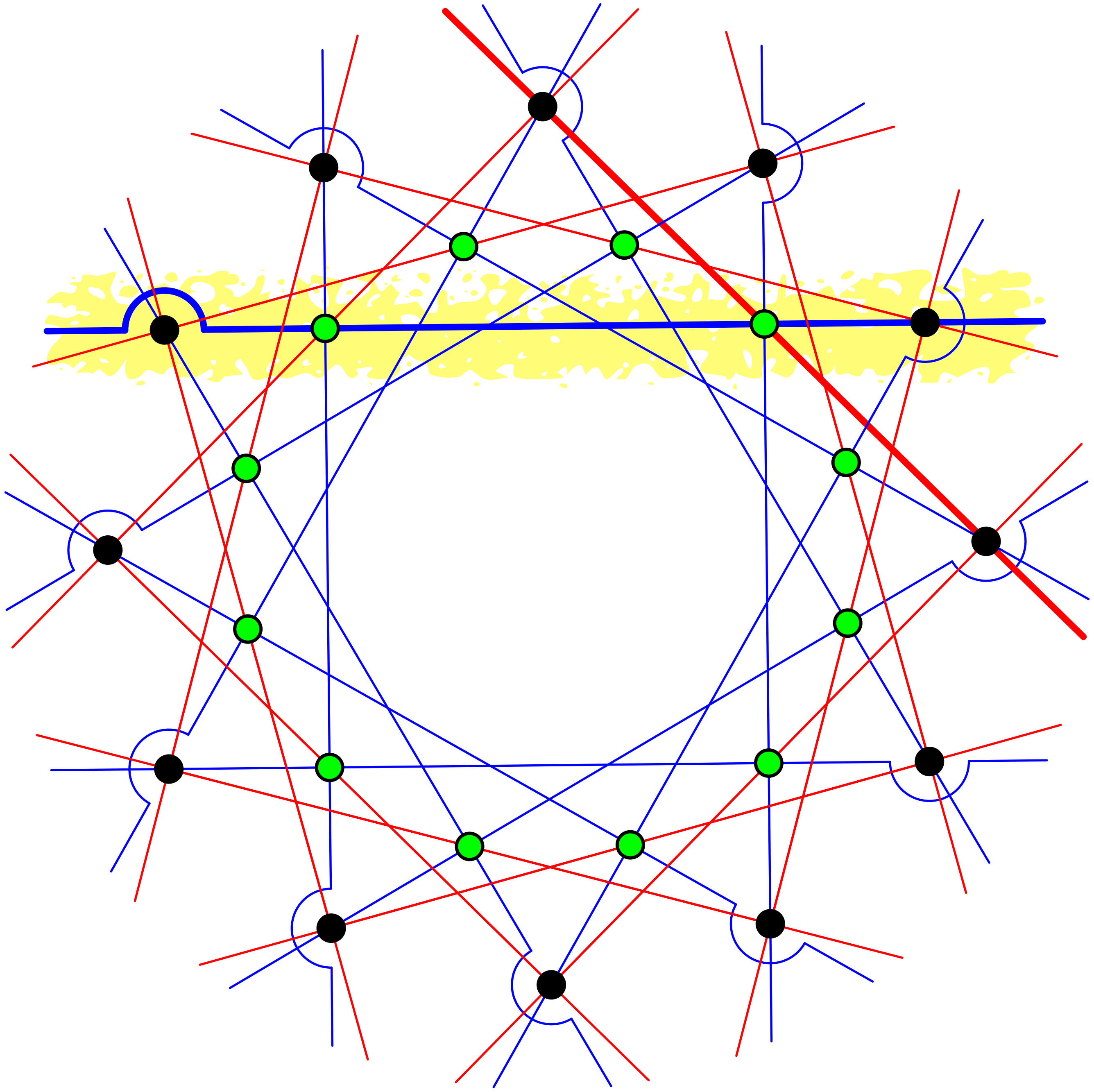
  (Gropp 1990, Betten & Betten 1999, Betten, Brinkmann, Pisanski 2000)
- combinatorial 3-configurations ⟷ cubic bipartite graphs of girth at least 6: Levi Graphs





Topological 3-configurations

- Exist for all n ≥ 9
- Easy to construct:
    - Steinitz (1894) Every (n_3) can be realized with at most one curve, *but might have extra incidences…*
- Avoid with pseudolines!



Geometric 3-configurations

- none for n=7,8; at least one (cyclic) for all n≥9
- #(9_3) = 3, including Pappus & Cyc_9(0,1,3)
- #(10_3) = 9, including Desargues; one combinatorial (10_3) configuration is non-realizable!
- Daublebsky [1894]: #(11_3) = 31, #(12_3)=228*
  - *no, 229 (missed one, overcounted one!) (Gropp 1997)
- All realizable with rational points! (Sturmfels & White 2000)

4-configurations

Combinatorial 4-configurations

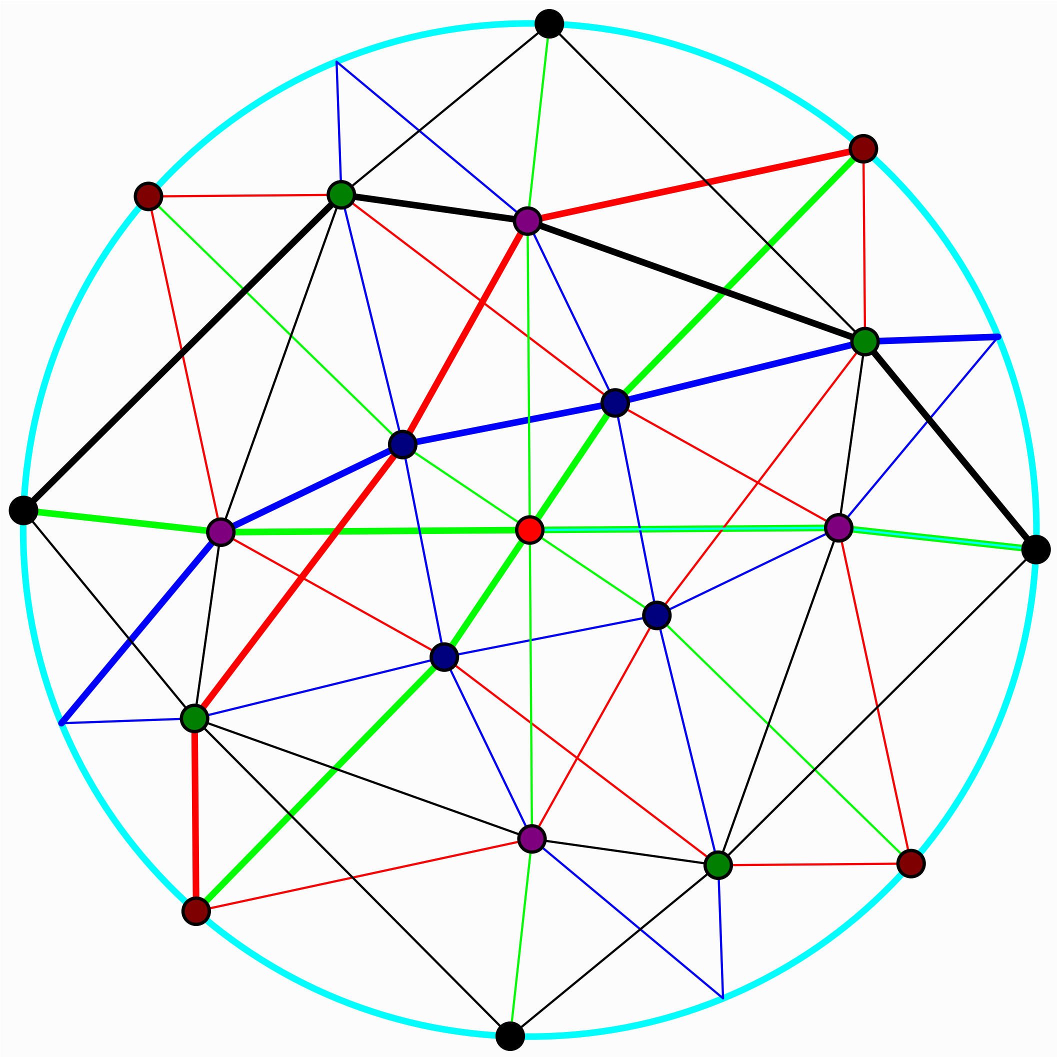
- Completely enumerated for $13 \leq n \leq 19$
    - **n = 13 …,18:** Betten & Betten 1999
    - **n = 19:** San Augustín Chi & Páez Osuna 2012
- At least one exists for all n (e.g., cyclic)

n	*# combinatorial (n_4)*
13	*1*
14	*1*
15	*4*
16	*19*
17	*1972*
18	*971 171*
19	*269 224 652*

Combinatorial Explosion!

Topological 4-configurations

- none for $13 \leq n \leq 16$ (Bokowski & Schewe 2005)
- $n = 15, 16$ hard: oriented matroids!
- exist for $n \geq 17$ (Bokowski, Grünbaum, Schewe 2009)



Topological (17_4) (non-stretchable)

Geometric 4-configurations

- None for $n \leq 17$ (Bokowski & Schewe 2005)
- Exactly two for $n = 18$ (Bokowski & Schewe 2009; Bokowski & Pilaud 2011)
- **None for n = 19** (Bokowski & Pilaud 2012)
- At least one for all other n except...
  - **Unknown** for $n = 23$
- Recently closed: $n = 19, 37, 41, 43$ (Bokowski & Pilaud), 22, 26 (Cuntz 2018)

(n_k) configurations for $n > 4$?

Known **lower** bounds

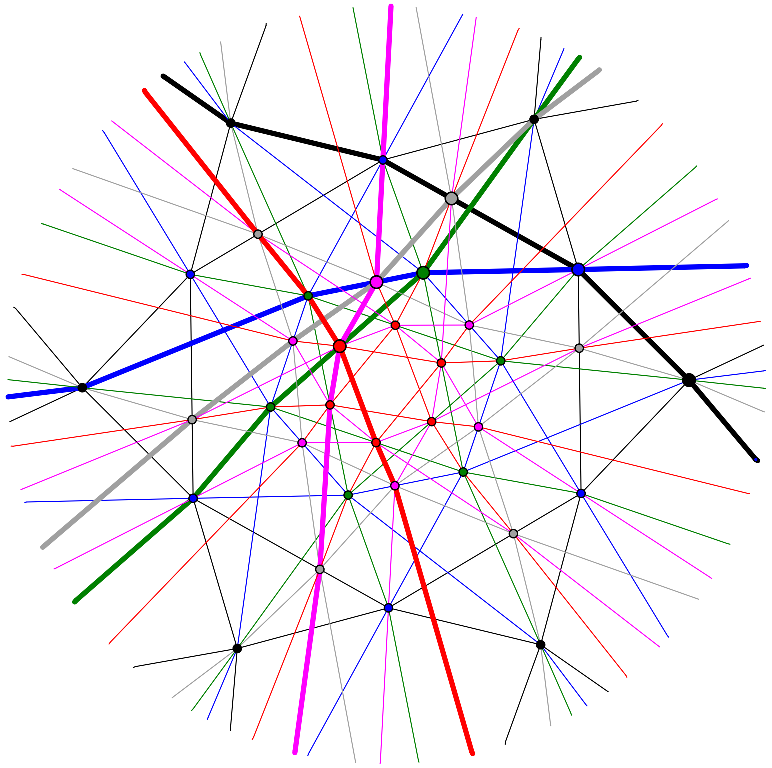
(n_k)	combinatorial	topological	geometric
(n_3)	(7_3)	(9_3)	(9_3)
(n_4)	(13_4)	(17_4)	(18_4)
(n_5)	(21_5)	$n \geq 27$*; (36_5)	(48_5)
(n_6)	(31_6)	$n \geq 42$*; (88_6)	(96_6)
(n_7)	(45_7)	$n \geq 57$*	(288_7)**
(n_8)	(57_8)	$n \geq 75$*	(525_8)***

*(Bokowski personal communication, 2016)

(Smallest I know)

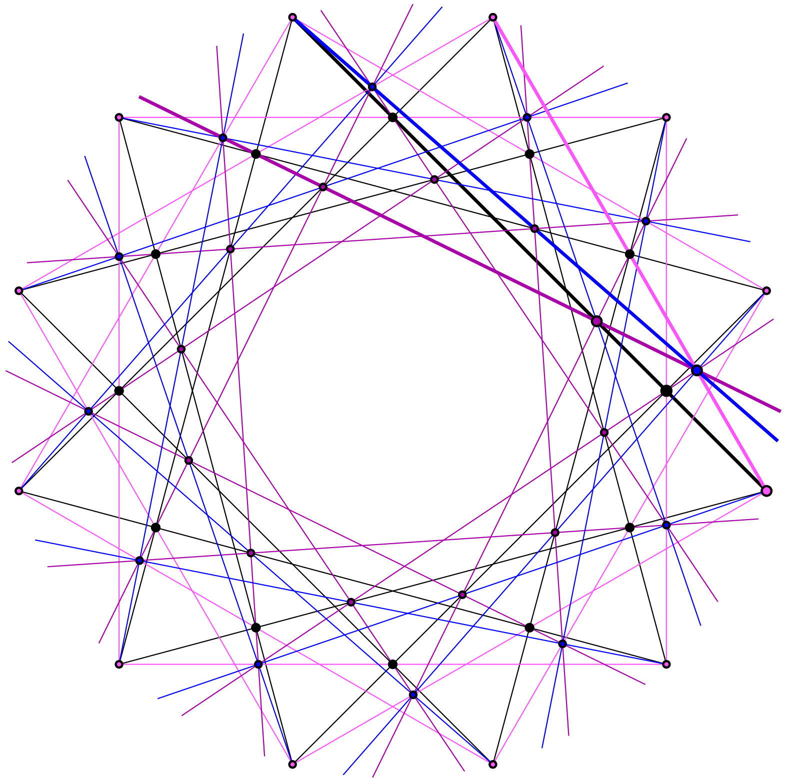
**A(9; 5,5; 1,2,3,4,6) B. & J. Faudree 2013; ***multicelestial 15#(3,2,1);(7,6,5,4)

We don't know very much about lower bounds!



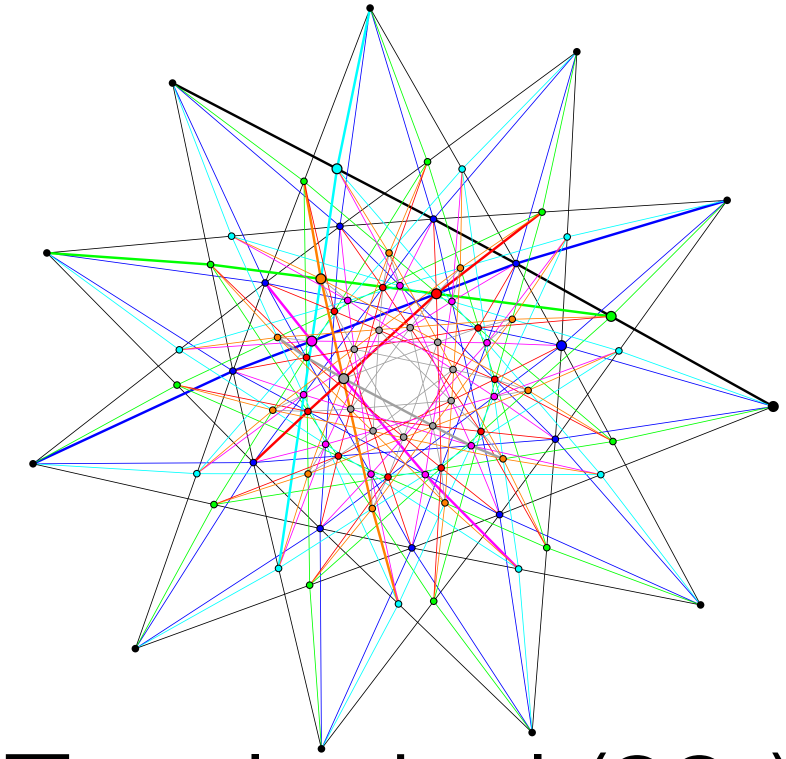
Topological (36_5)

B. & Bokowski, unpublished



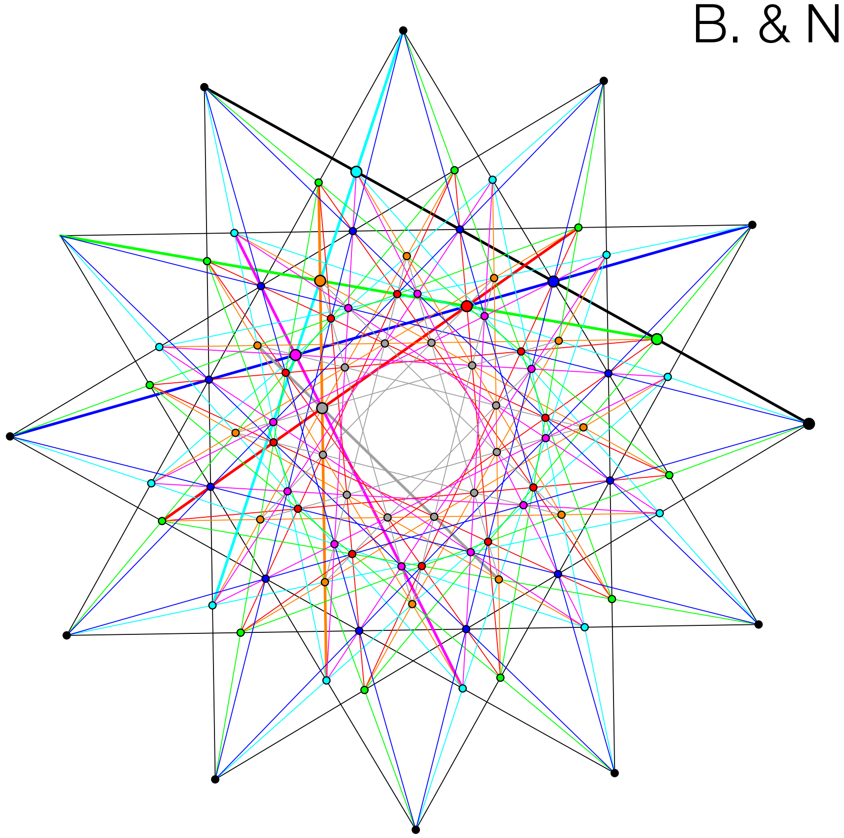
Geometric (48_5)

B. & Ng, 2010



Topological (88_6)

B., 2008



Geometric (96_6)

B. 2014

Theorem: Geometric (n_k) configurations exist for all large enough n !

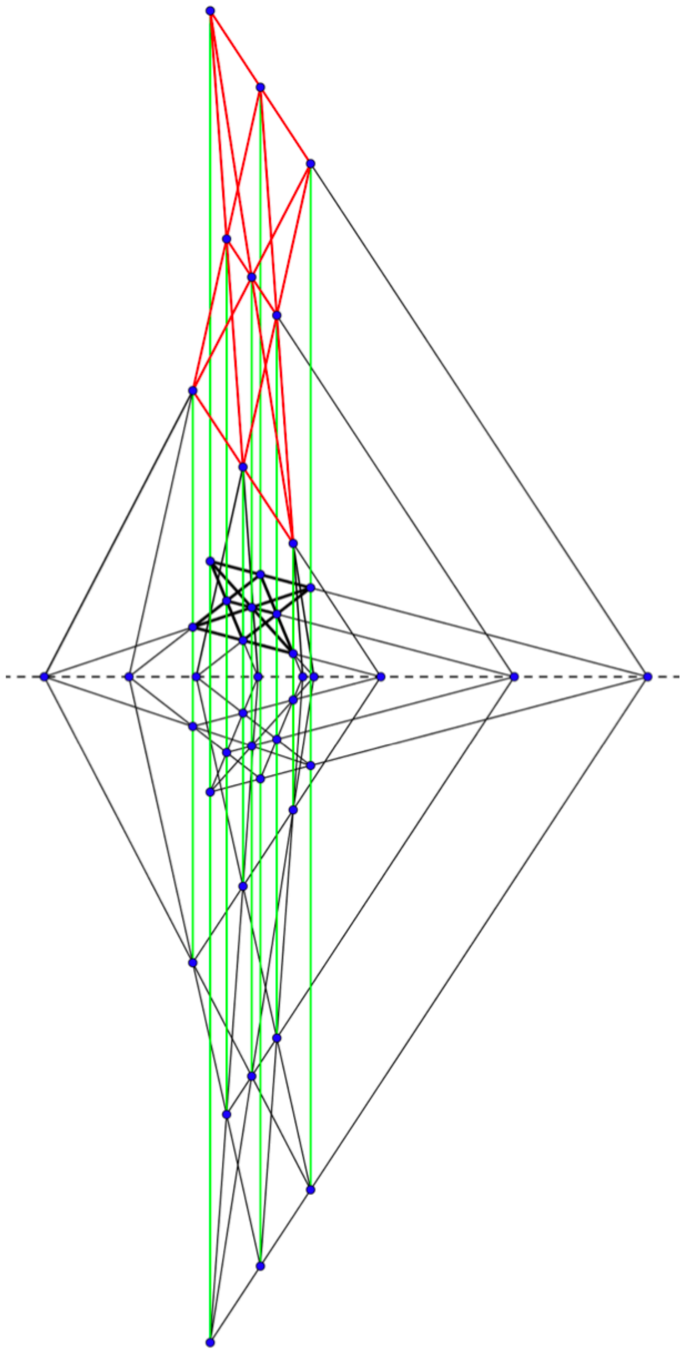
For each $k \geq 2$ there exists N_k so that for $n \geq N_k$, there exists at least one (n_k) configuration.

(B., Gévay, Pisanski, in review)

k	$N_k \leq \ldots$	k	$N_k \leq \ldots$
4	24	8	1333584
5	576	9	19353600
6	7350	10	287400960
7	96768	11	3832012800

Affine Replication

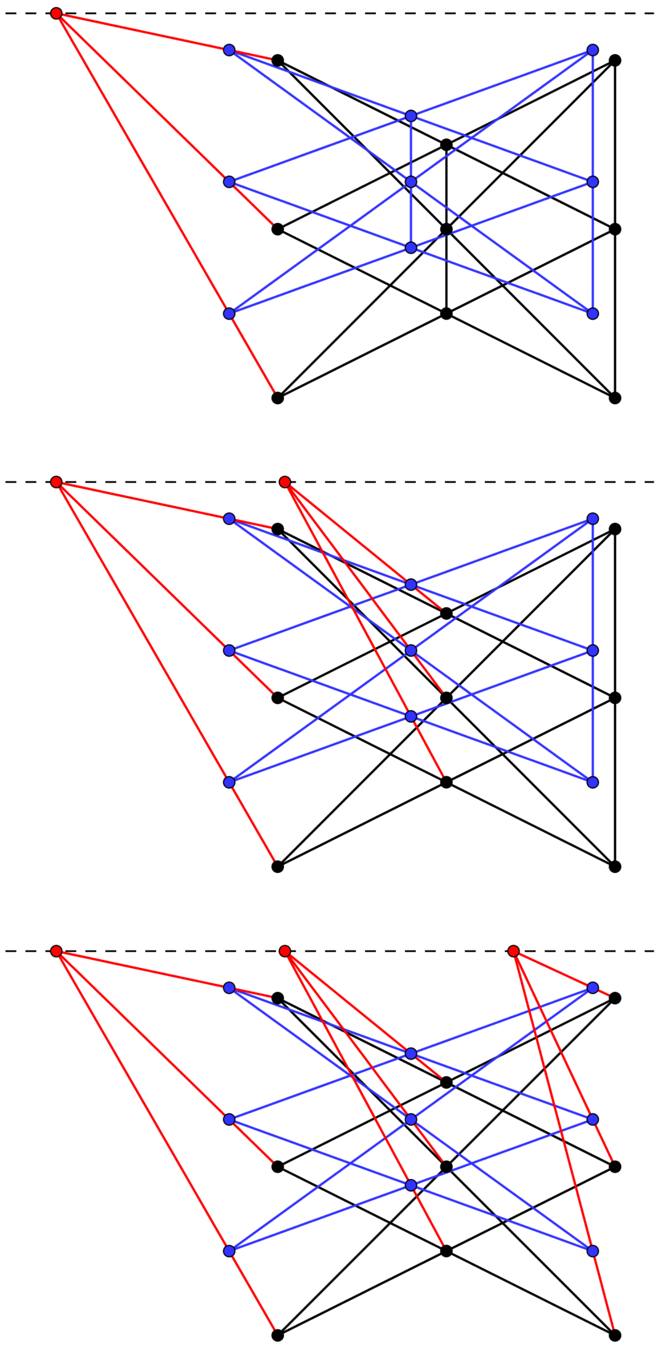
$(m_{k-1}) \rightarrow ((k+1)m_k)$



$(9_3) \rightarrow (45_4)$

Affine Switch

$(m_k) \rightarrow (((k-1)m+1)_k), \ldots (((k-1)m+p)_k)$



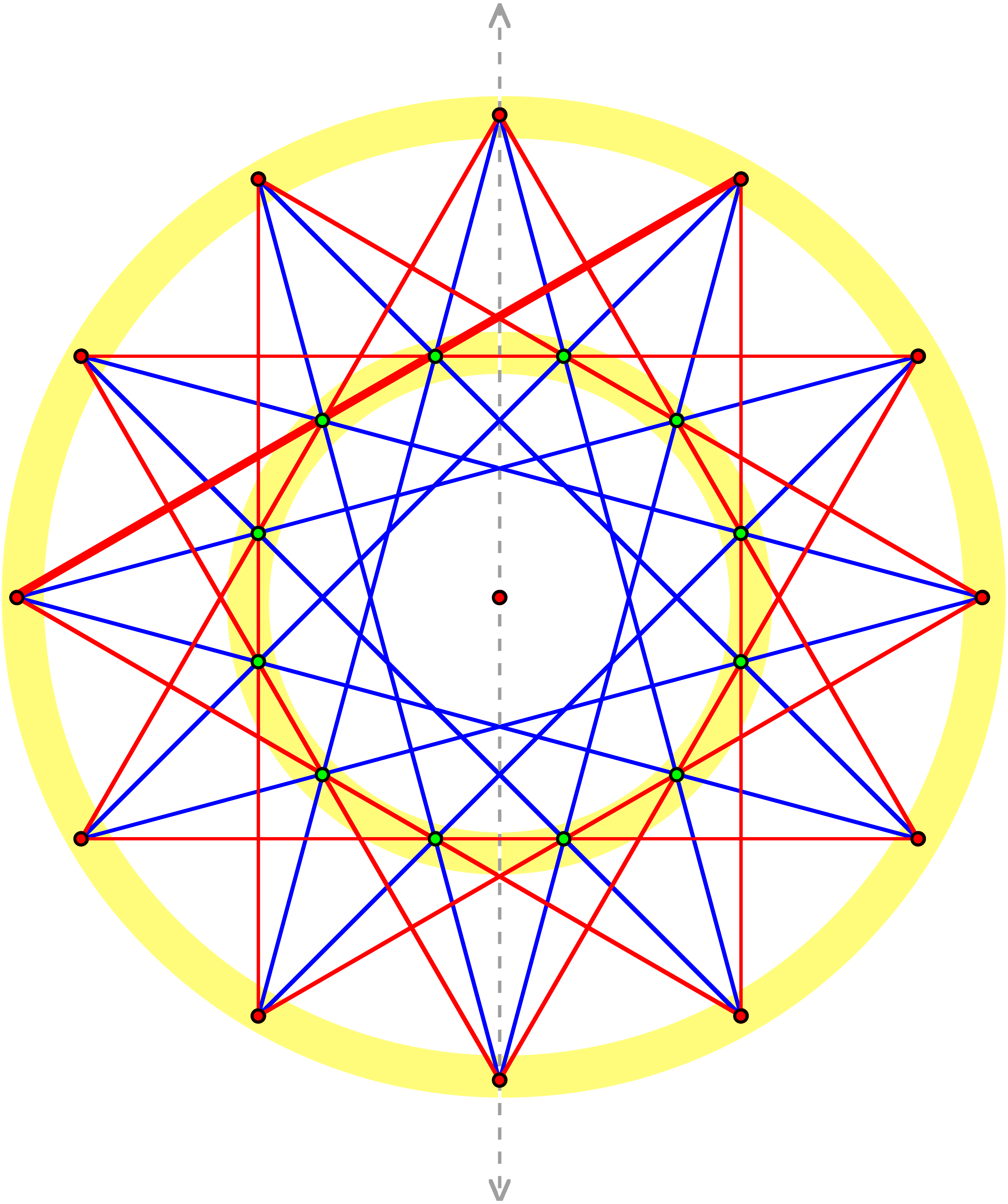
$(9_3) \rightarrow (19_3), (20_3), (21_3)$

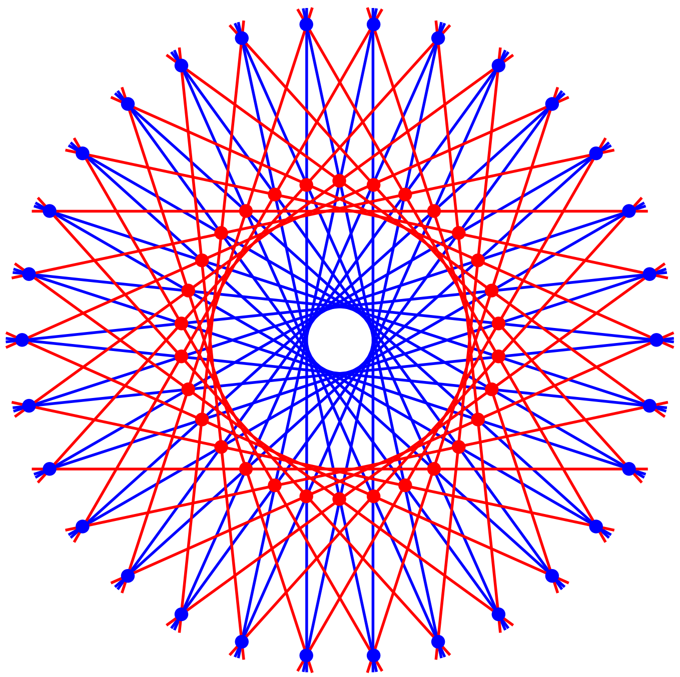
Identification:

How can we find new (infinite classes of) configurations?

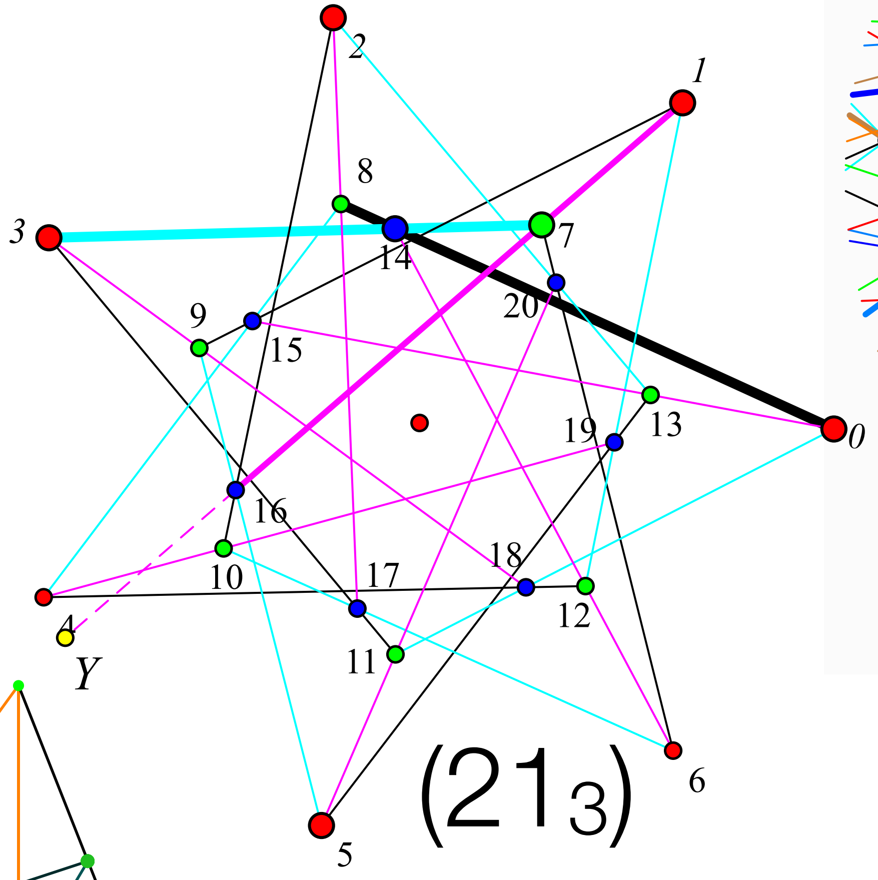
Symmetry

- Non-trivial geometric symmetry: rotations and reflections
- Symmetry classes? *k*-astral.
- “small” number of symmetry classes
    - symmetric vs. balanced

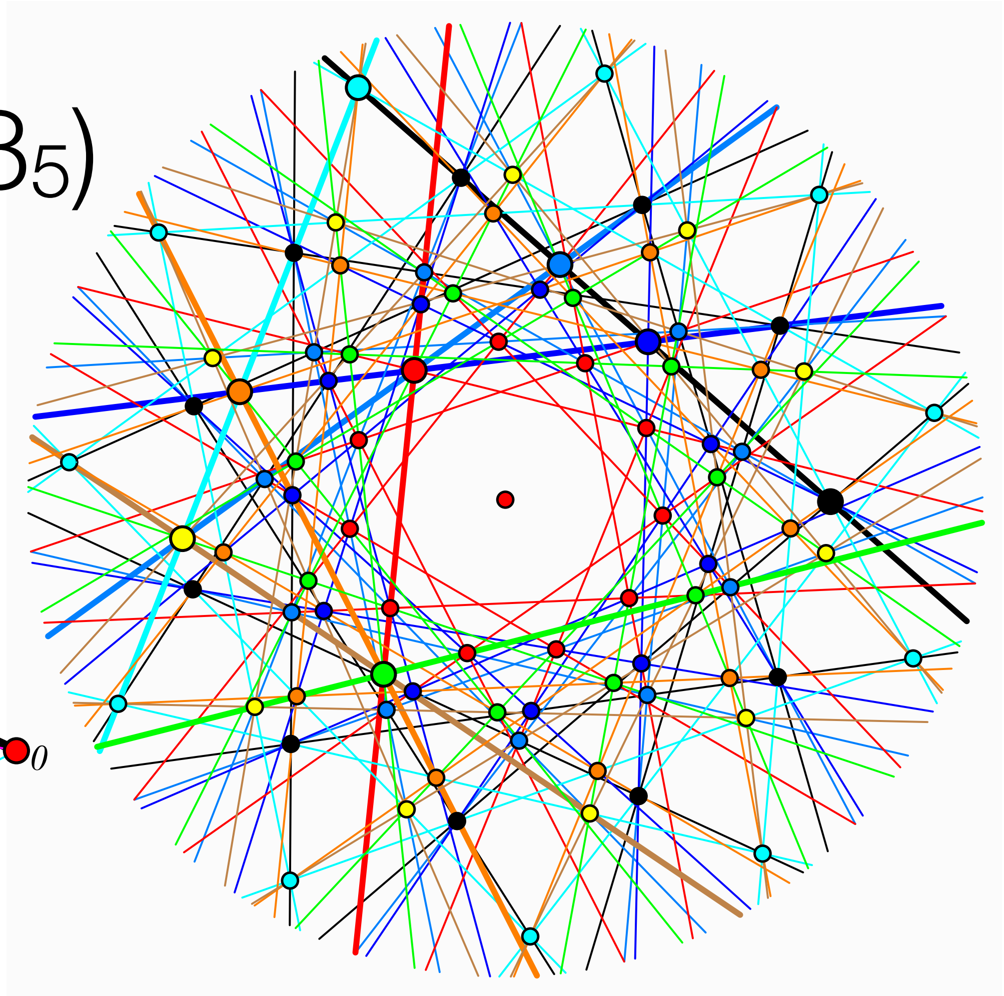




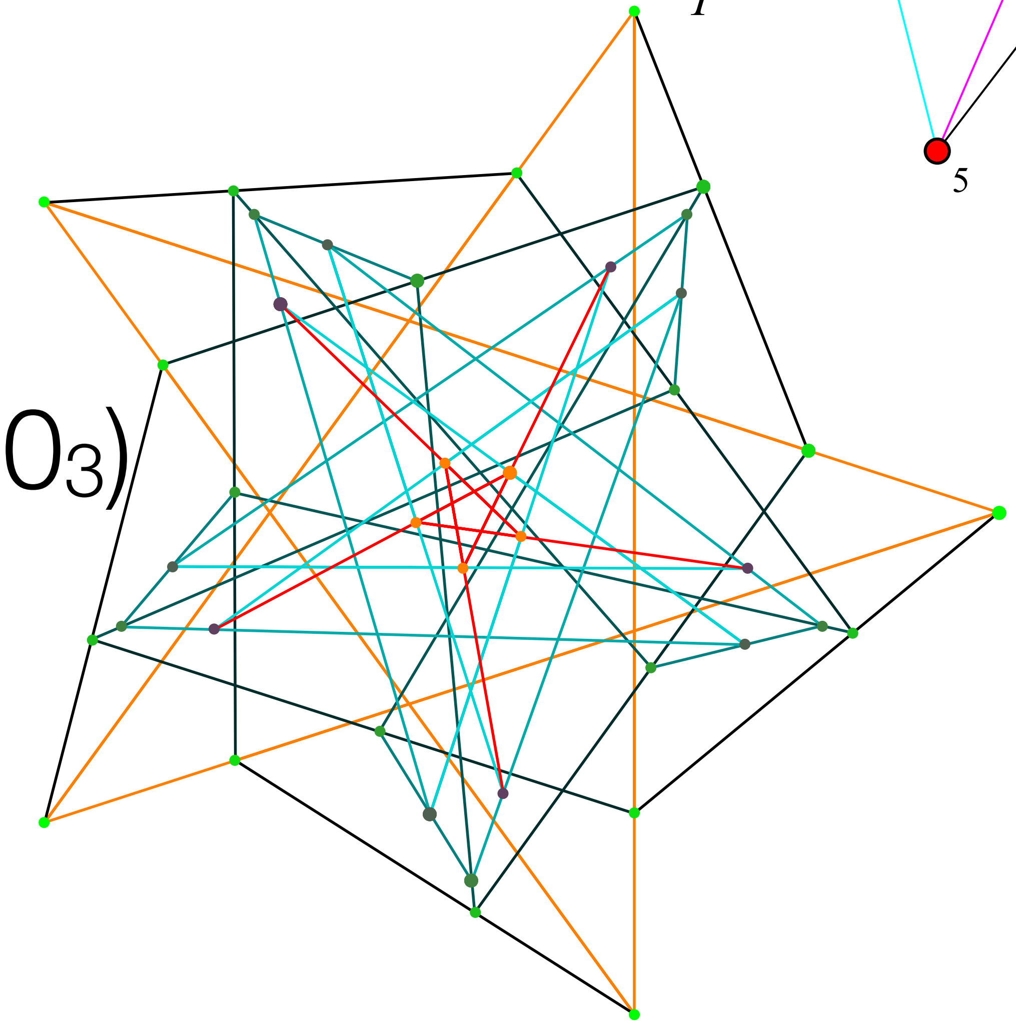
(60_4)



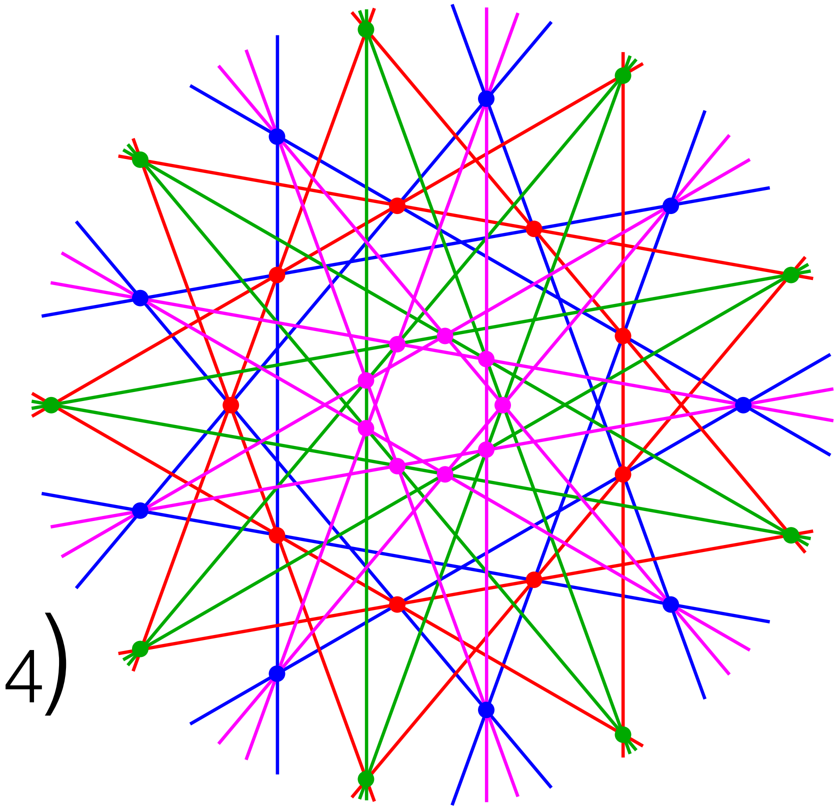
(21_3)



(88_5)

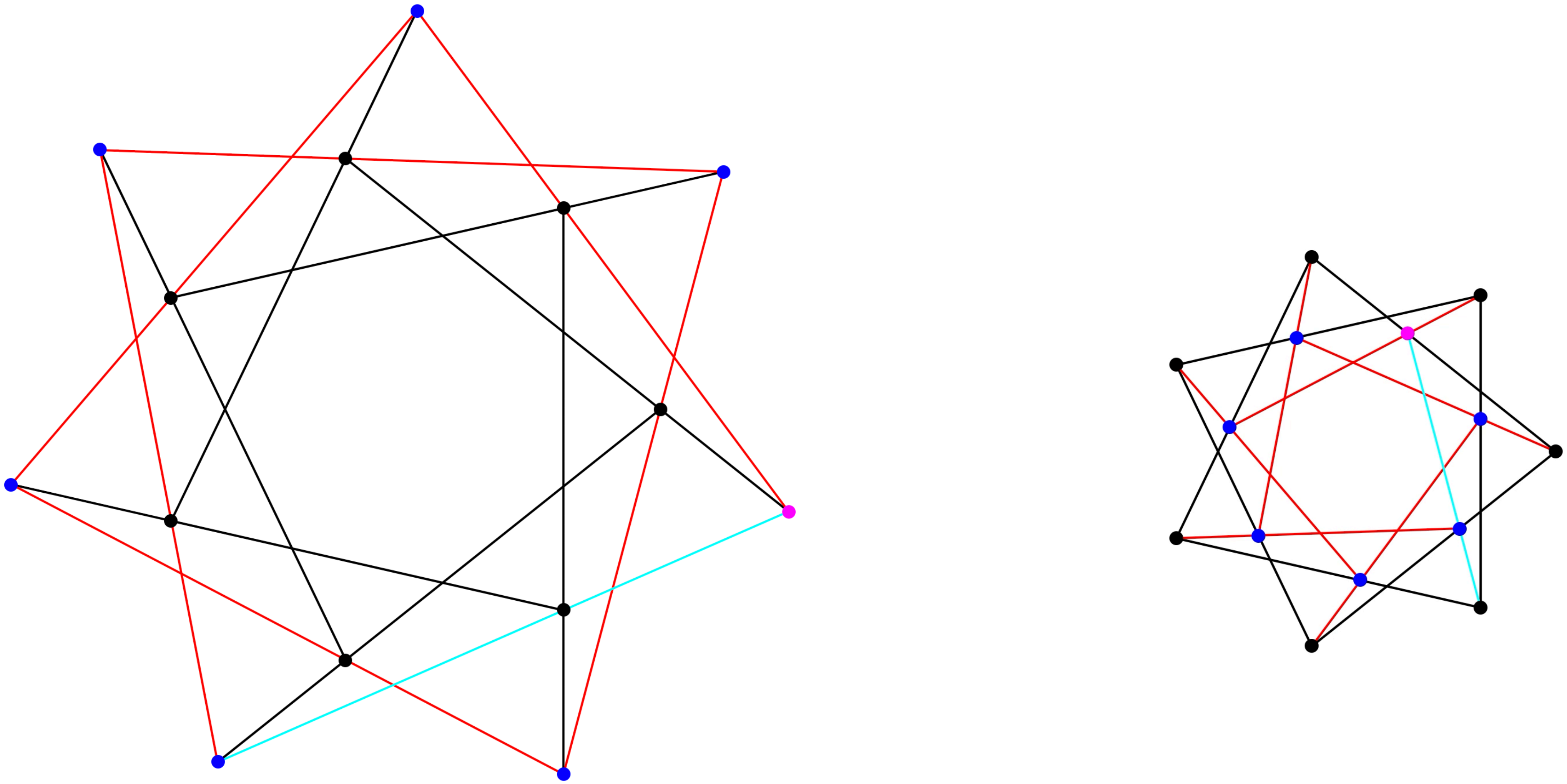


(30_3)



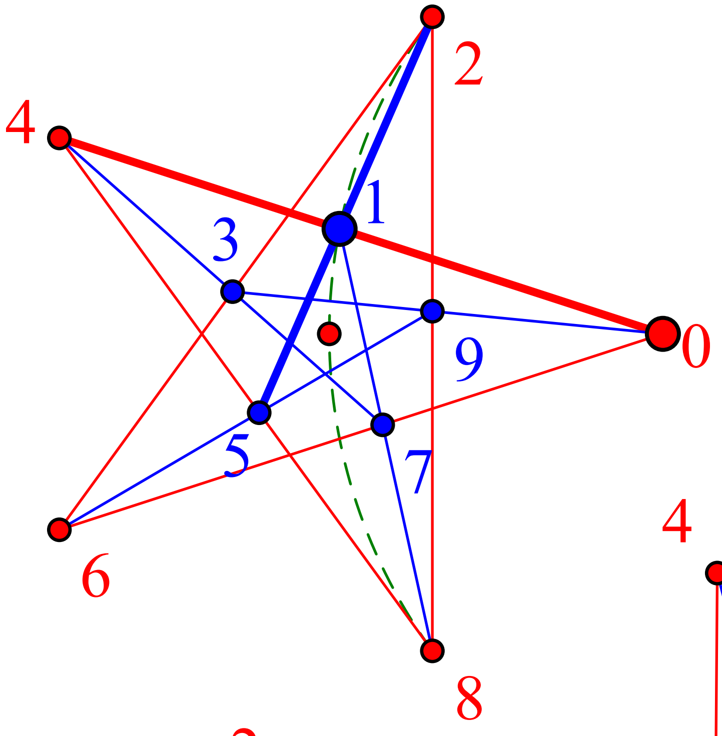
(36_4)

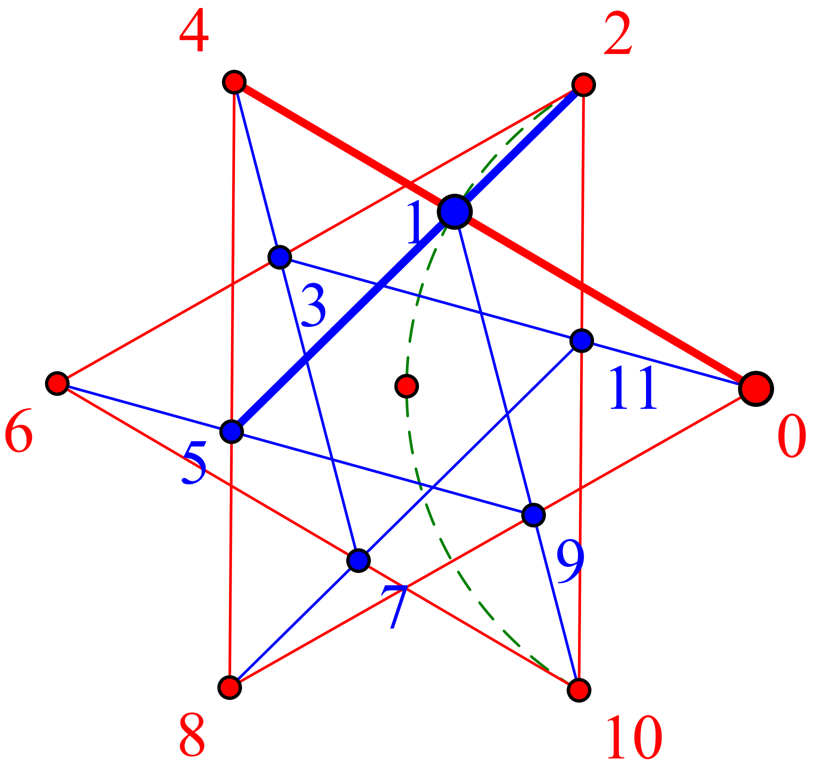
Symmetry helps find examples!



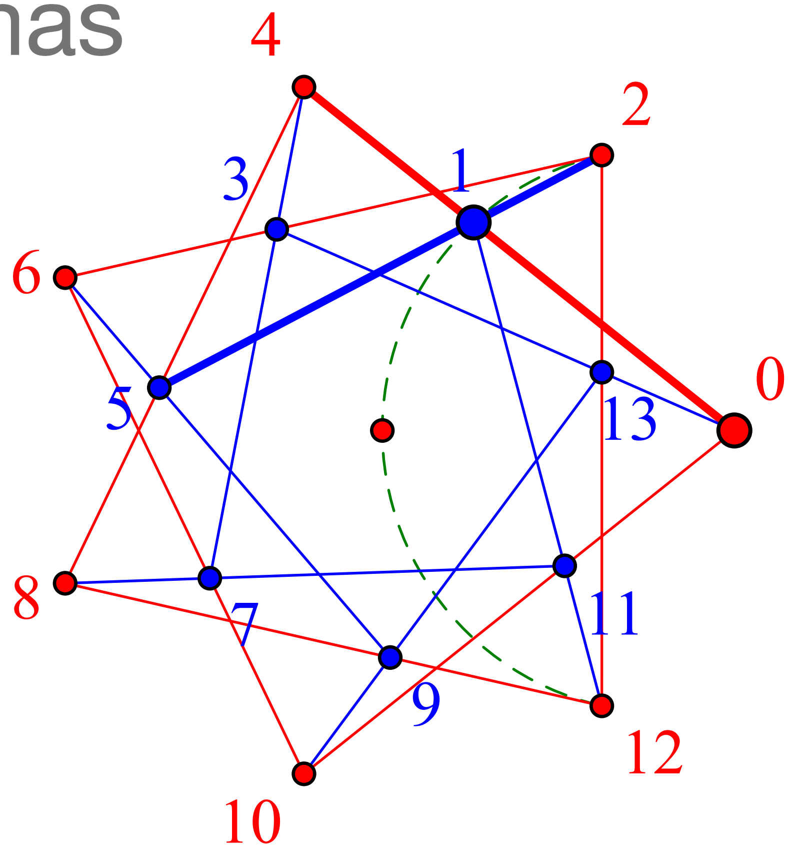
Main Tools

- Reduced Levi Graph: *represent families of configurations compactly*

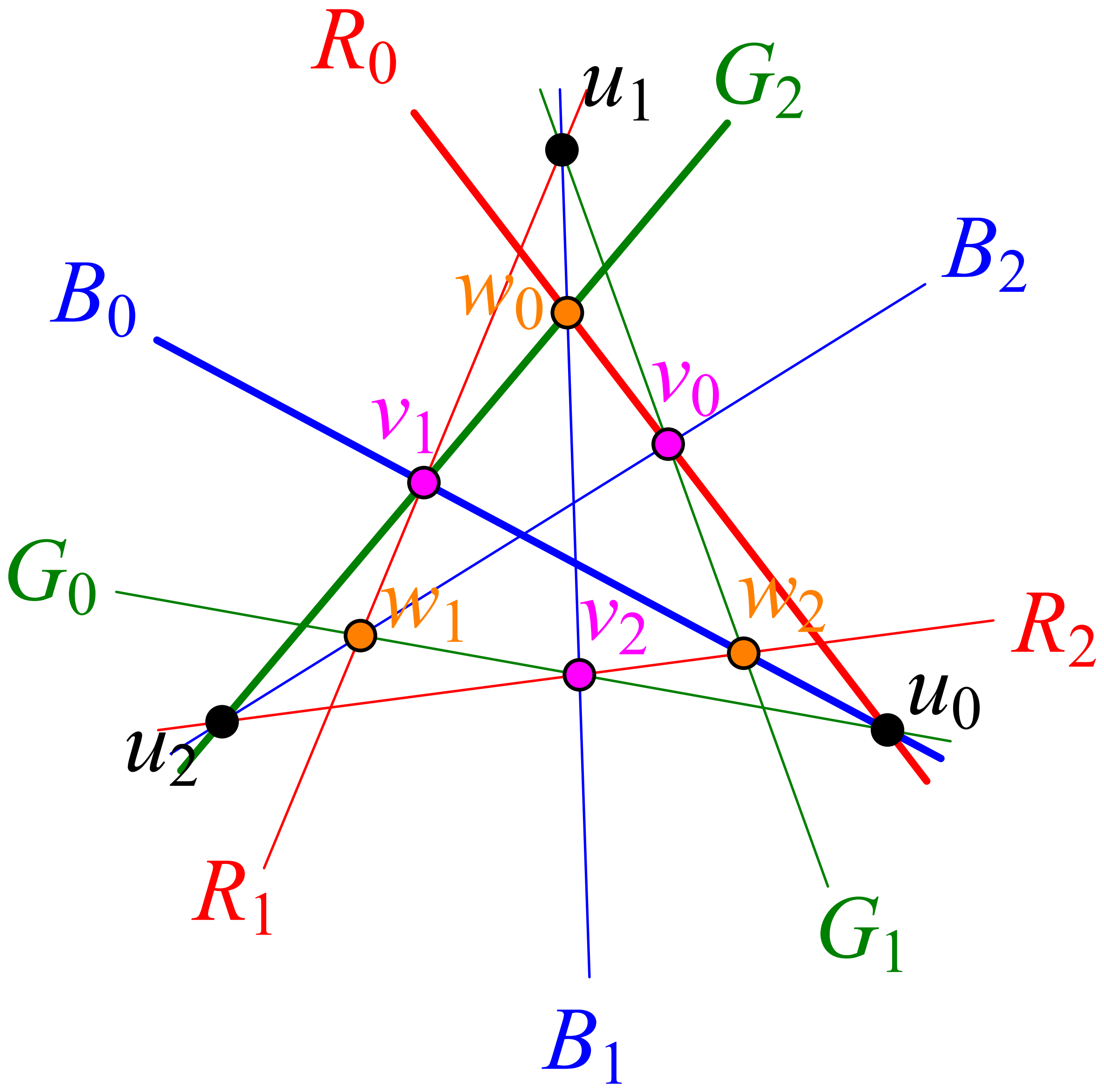




- Geometric Lemmas

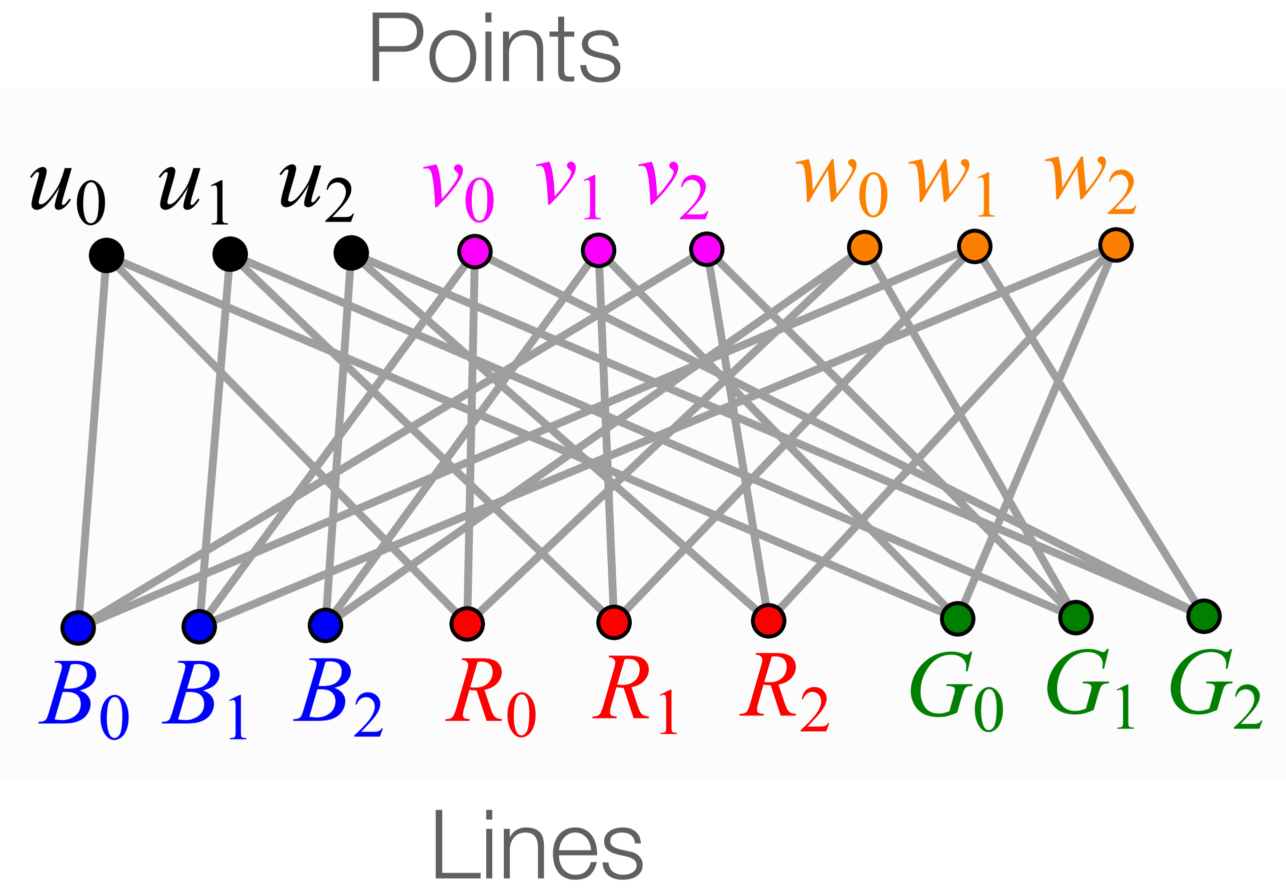


Levi Graphs

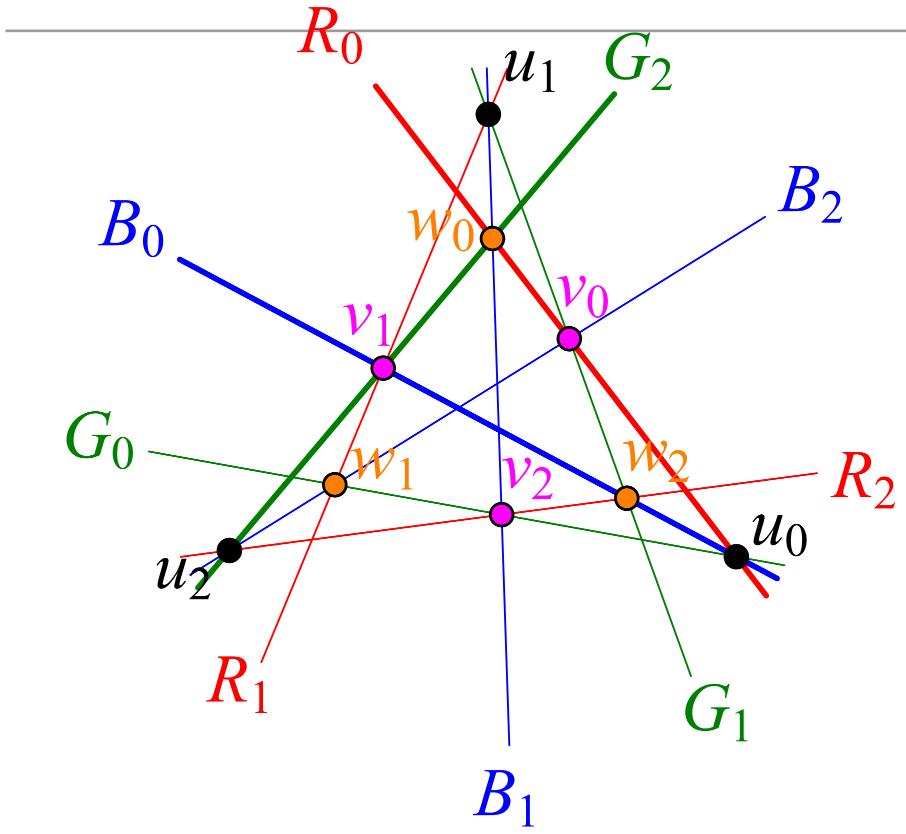


Pappus configuration
(9_3)

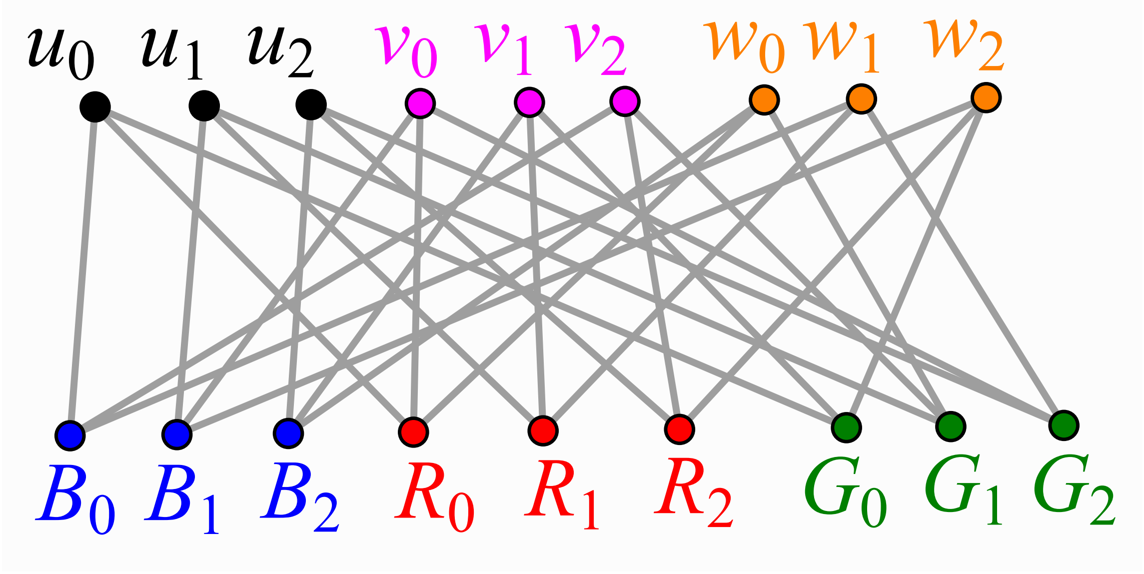
Associated Levi Graph



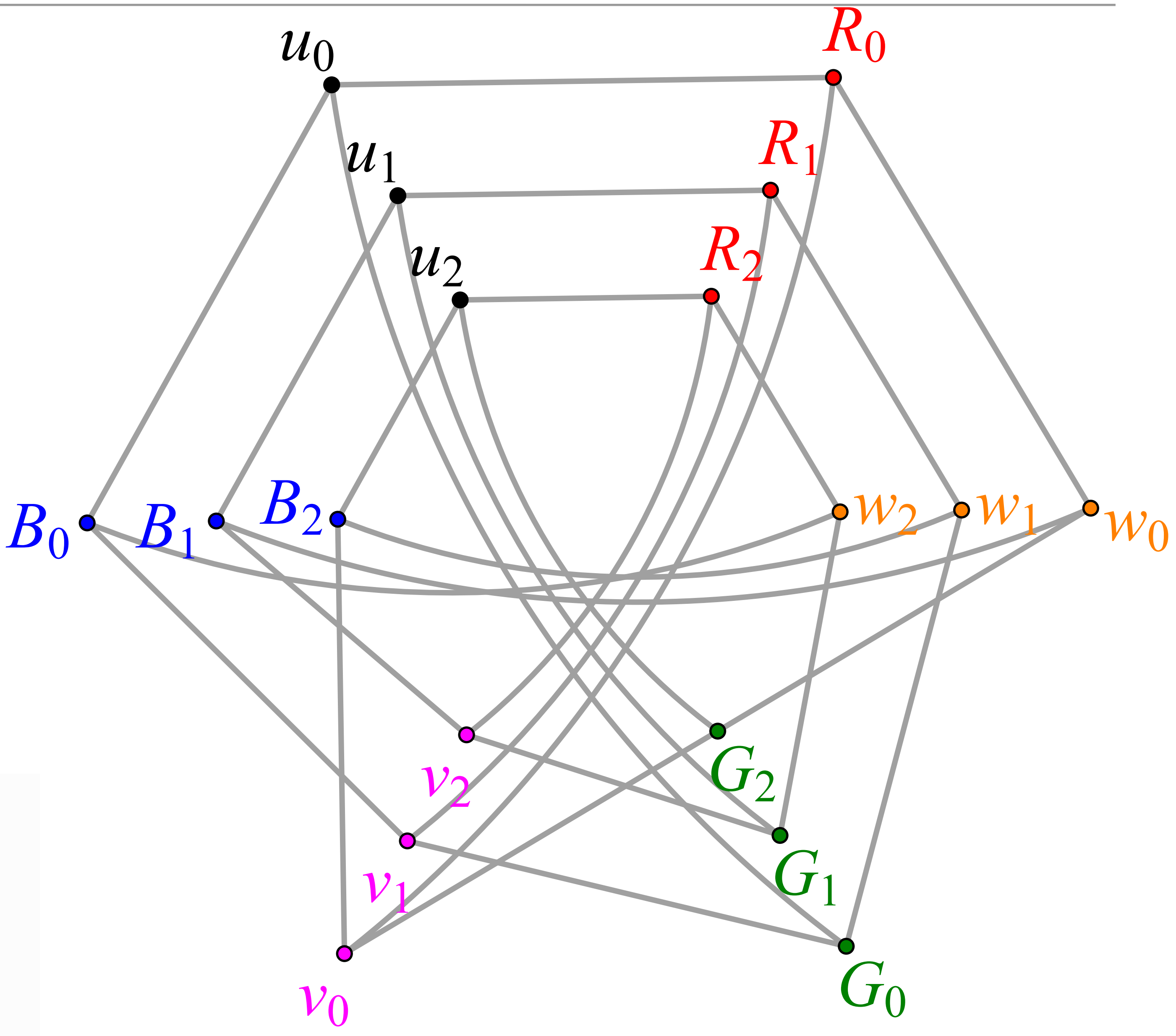
Levi Graphs



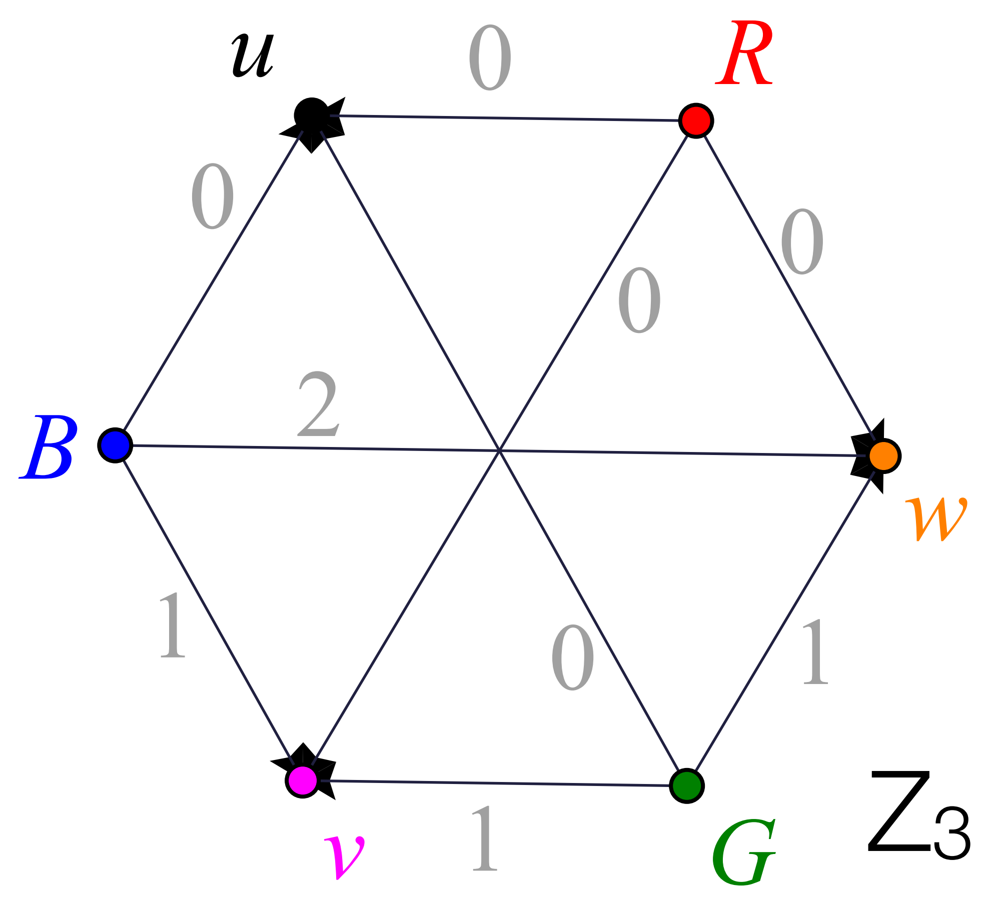
Points

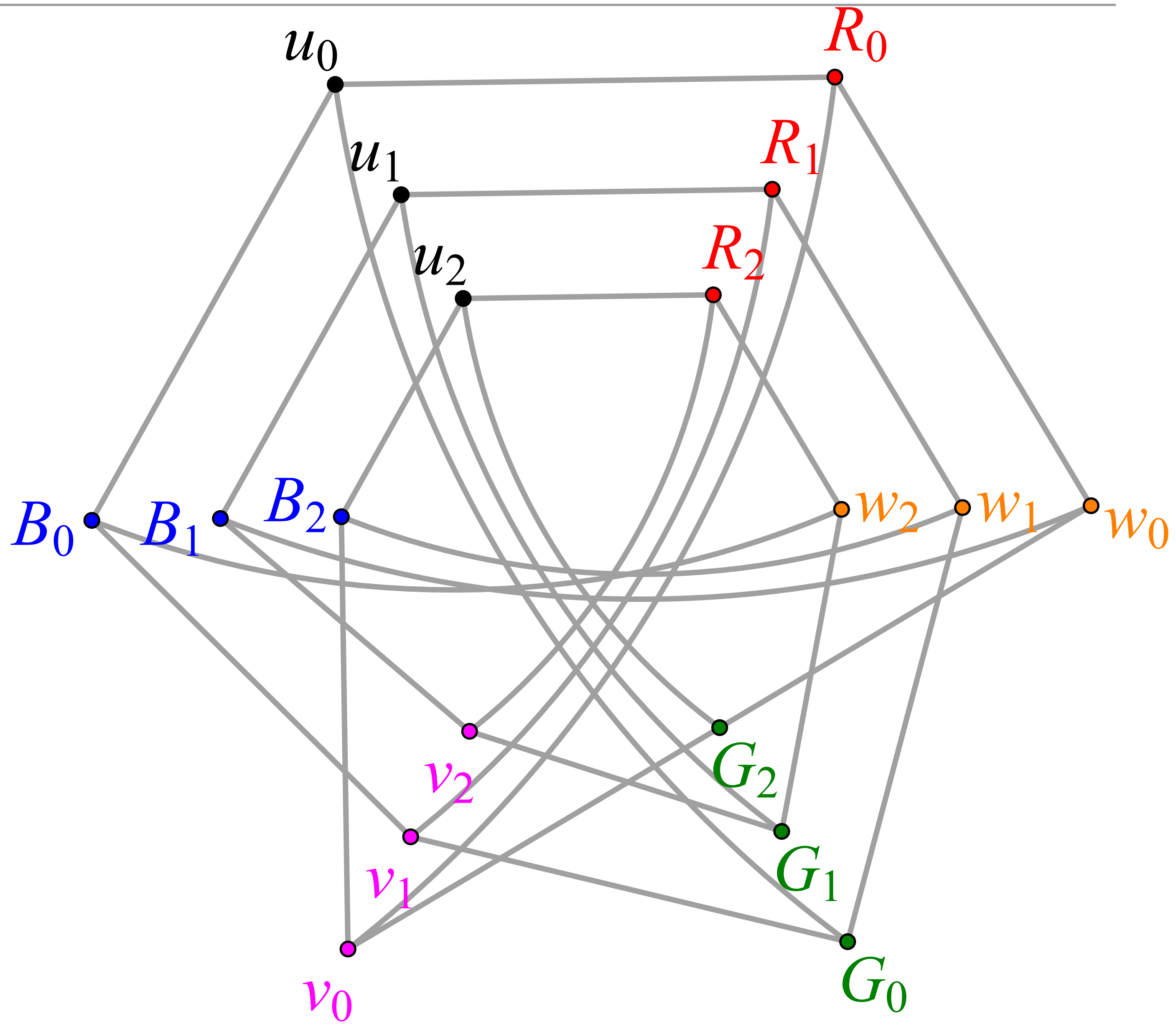


Lines



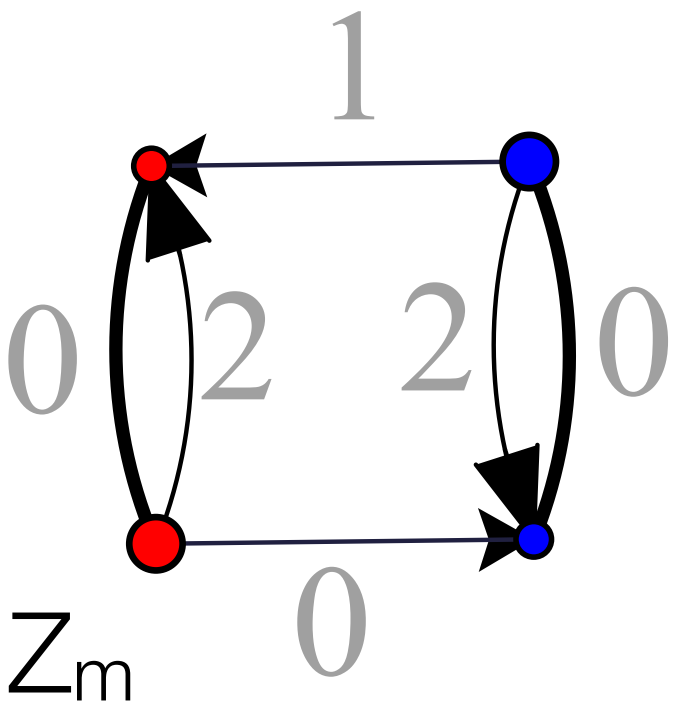
Reduced Levi Graphs

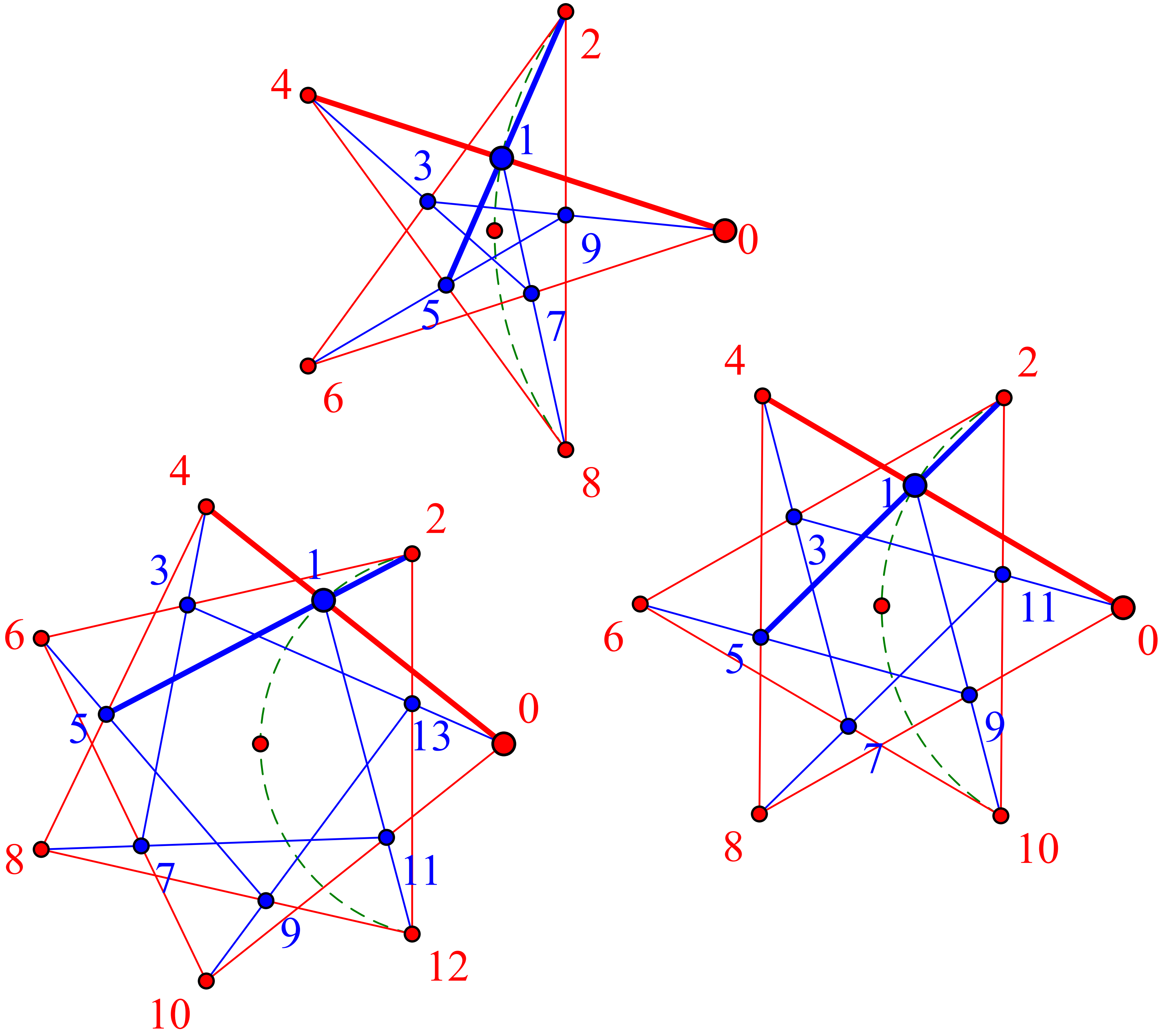




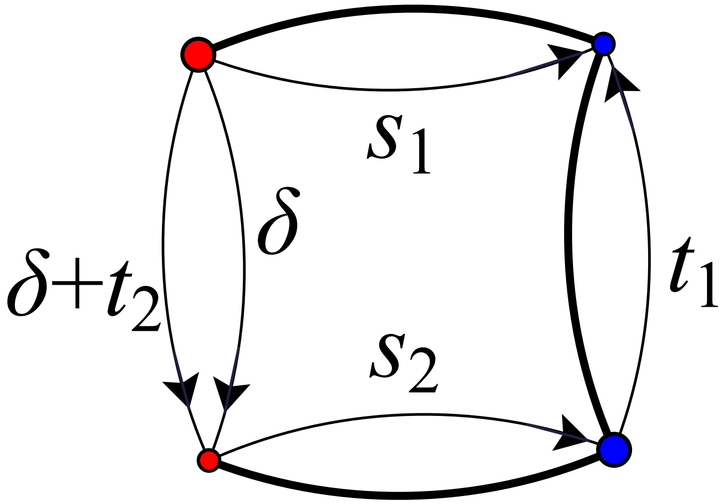
● $\xrightarrow{a}$ ● means Point $●_{i+a}$ lies on Line $●_{i}$ mod m

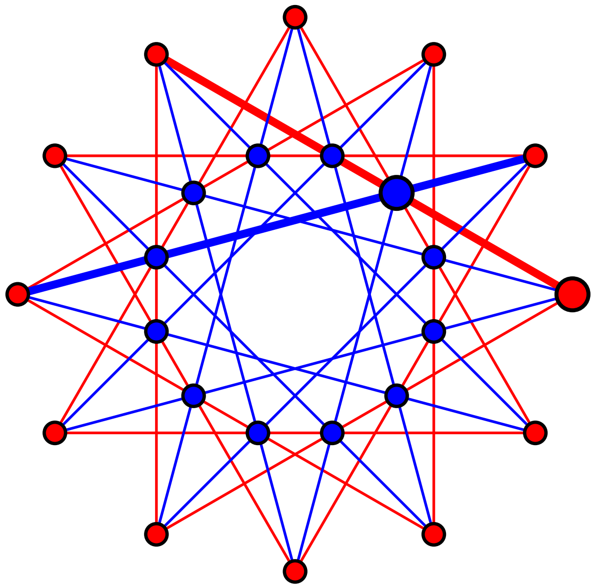
Infinite Families of Configurations

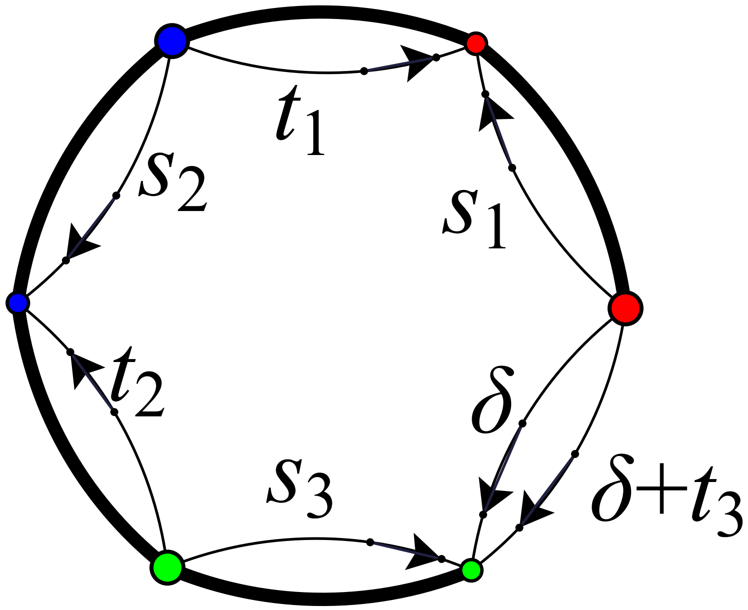


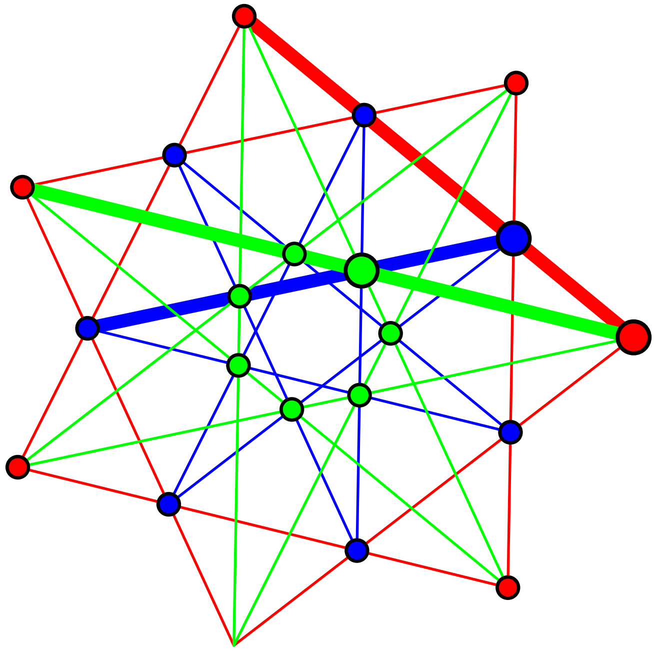


Infinite Families of Configurations





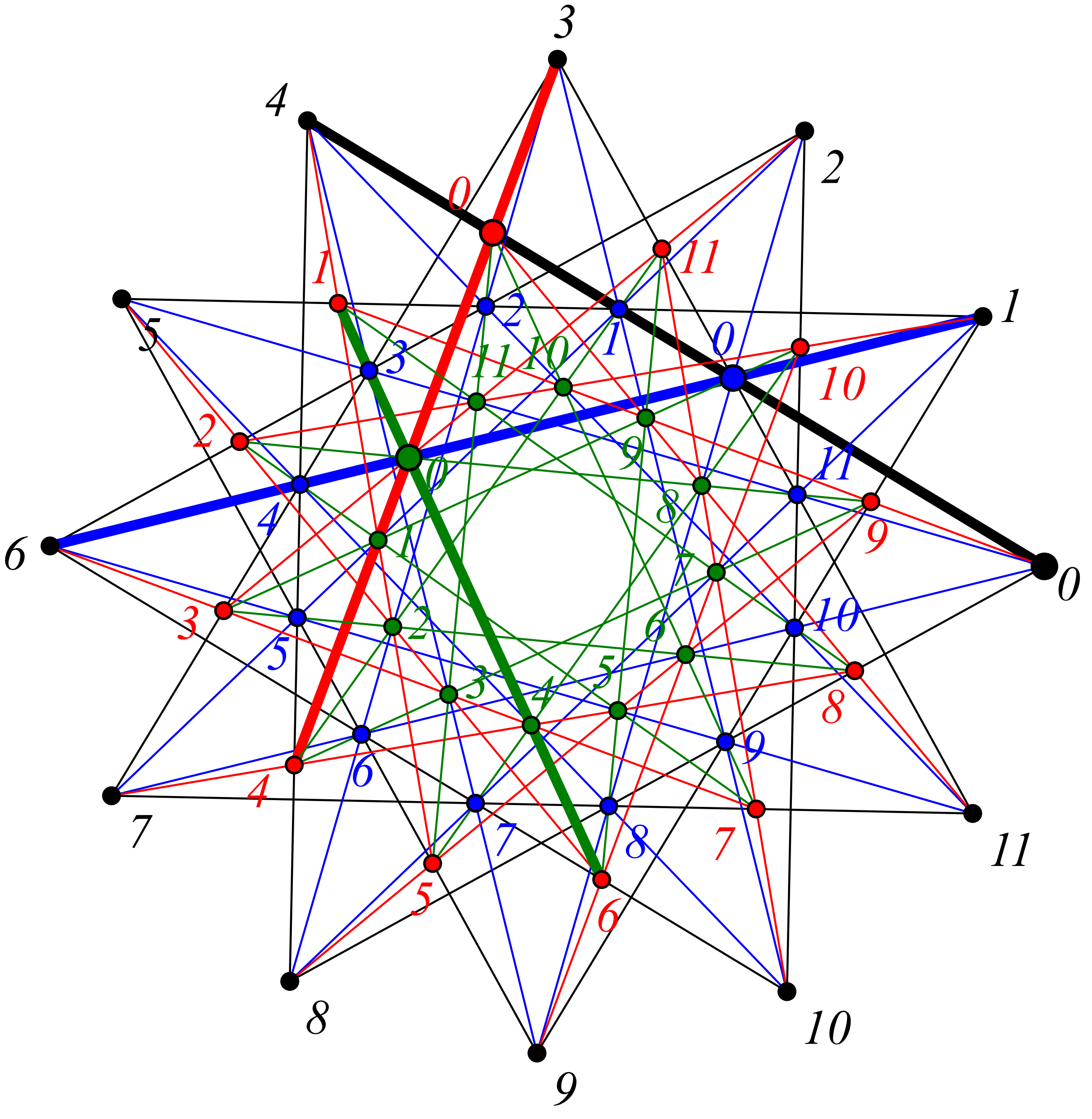


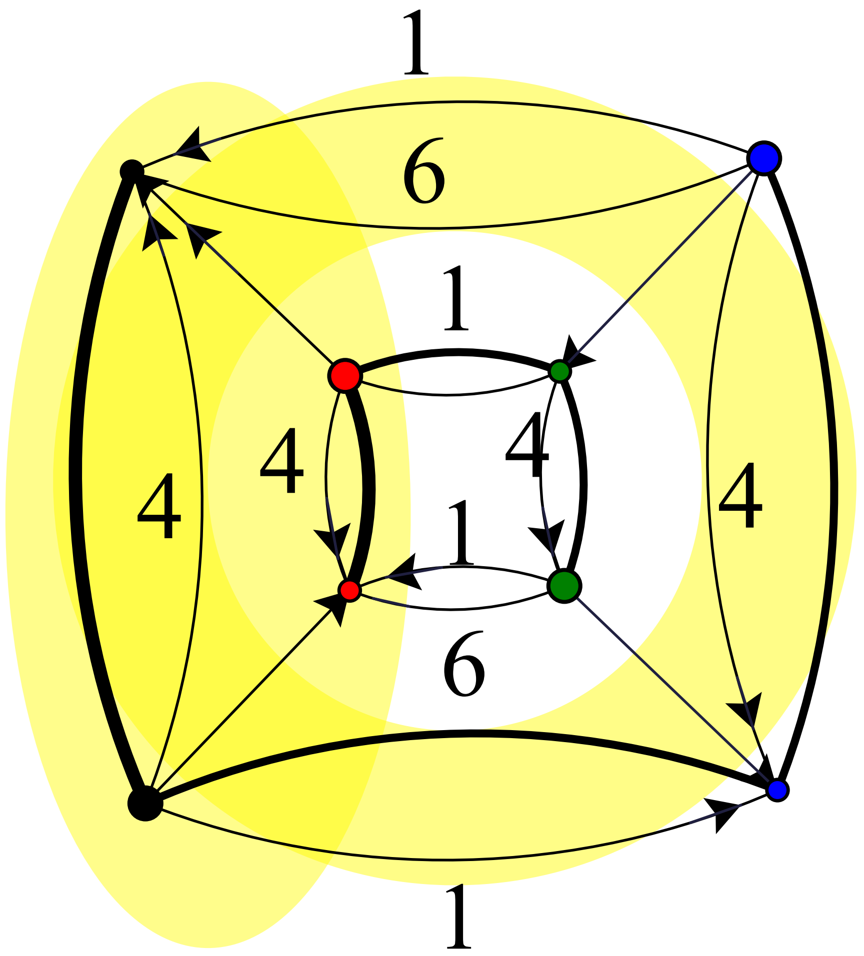


$\prod \text{Cos}(s_i \pi/m) = \prod \text{Cos}(t_i \pi/m)$

$\sum(s_i - t_i)$ is even

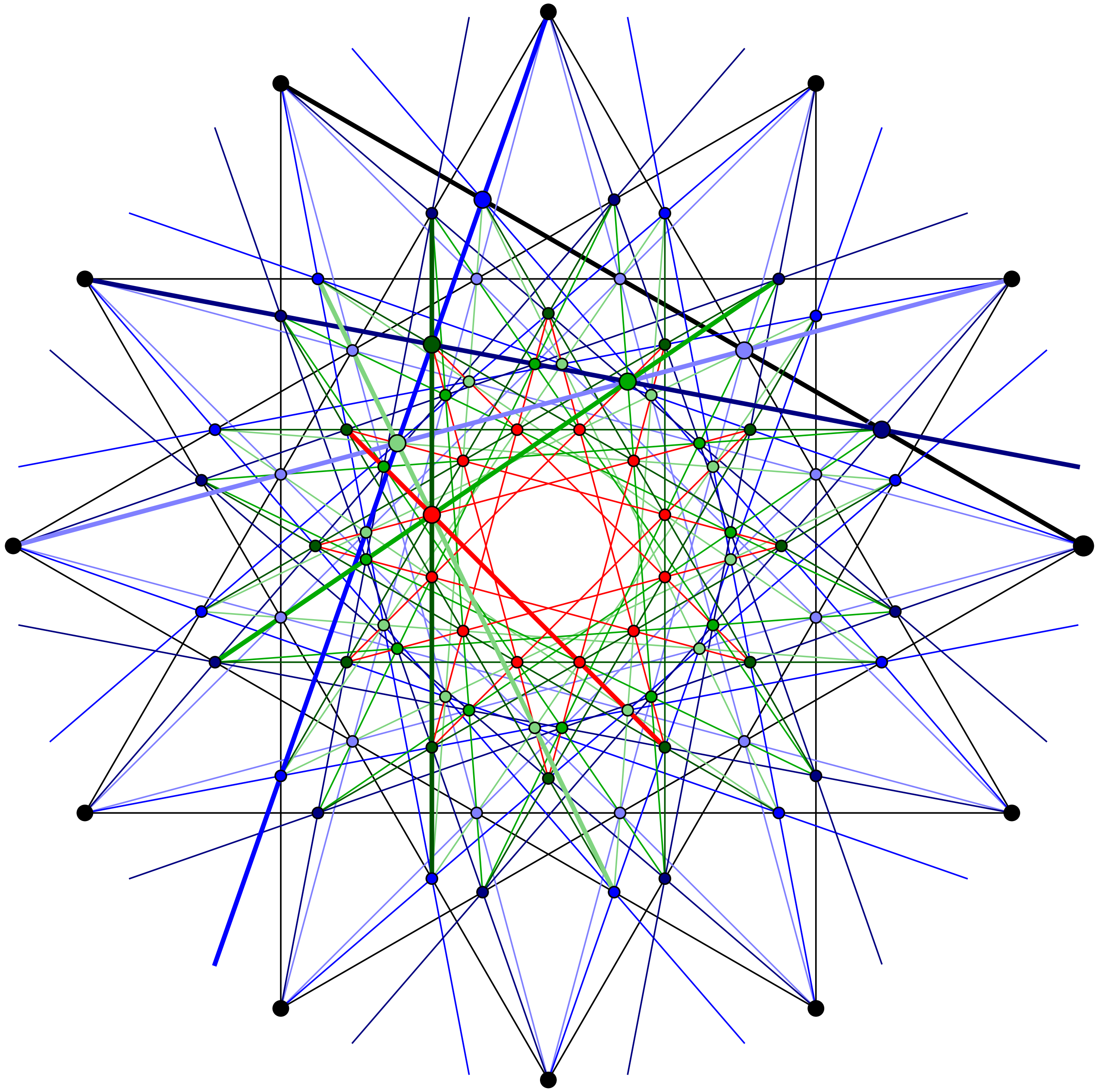
Combine for new examples!

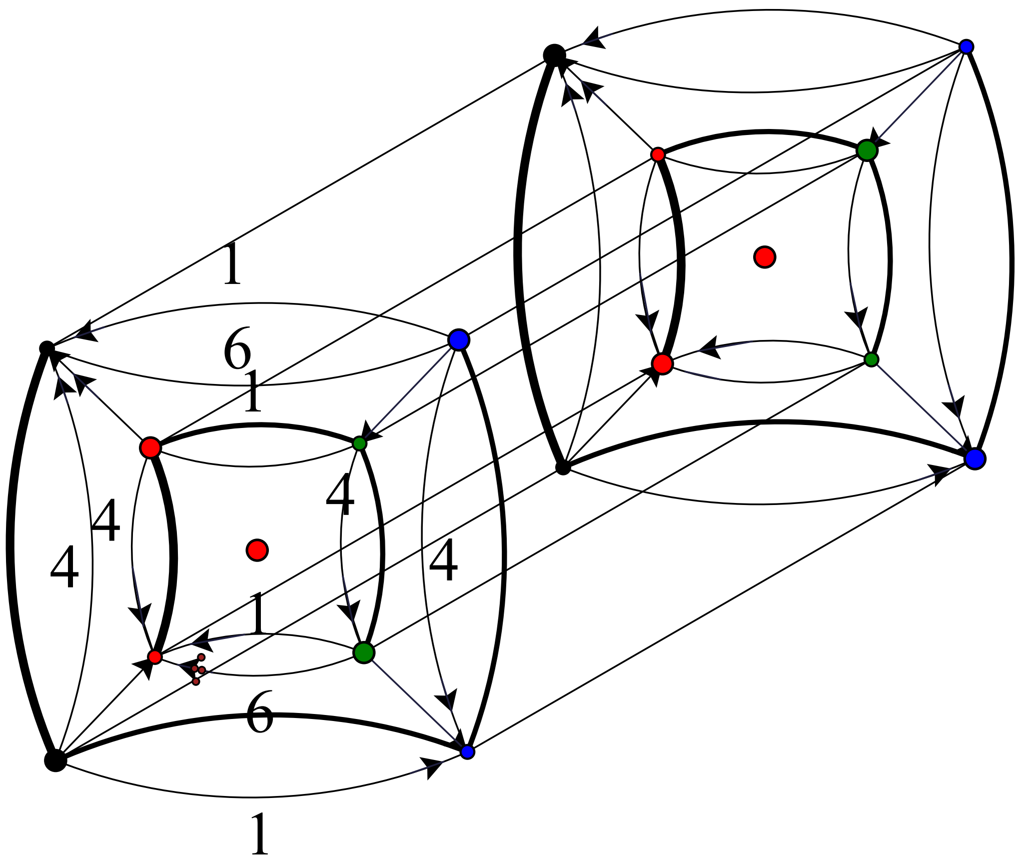




Smallest known 5-configuration: (48_5)

Combine for new examples!

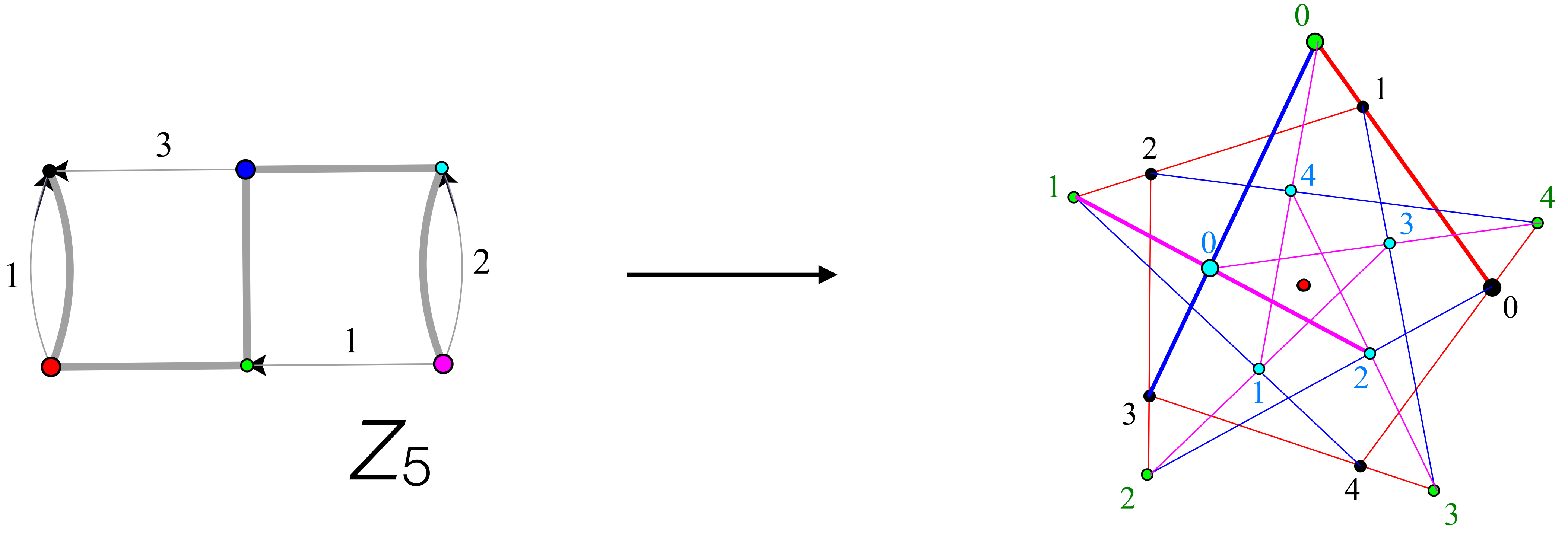




Smallest known 6-configuration: (96_6)

Important open question

Given a reduced Levi graph, can we find a corresponding geometric configuration?

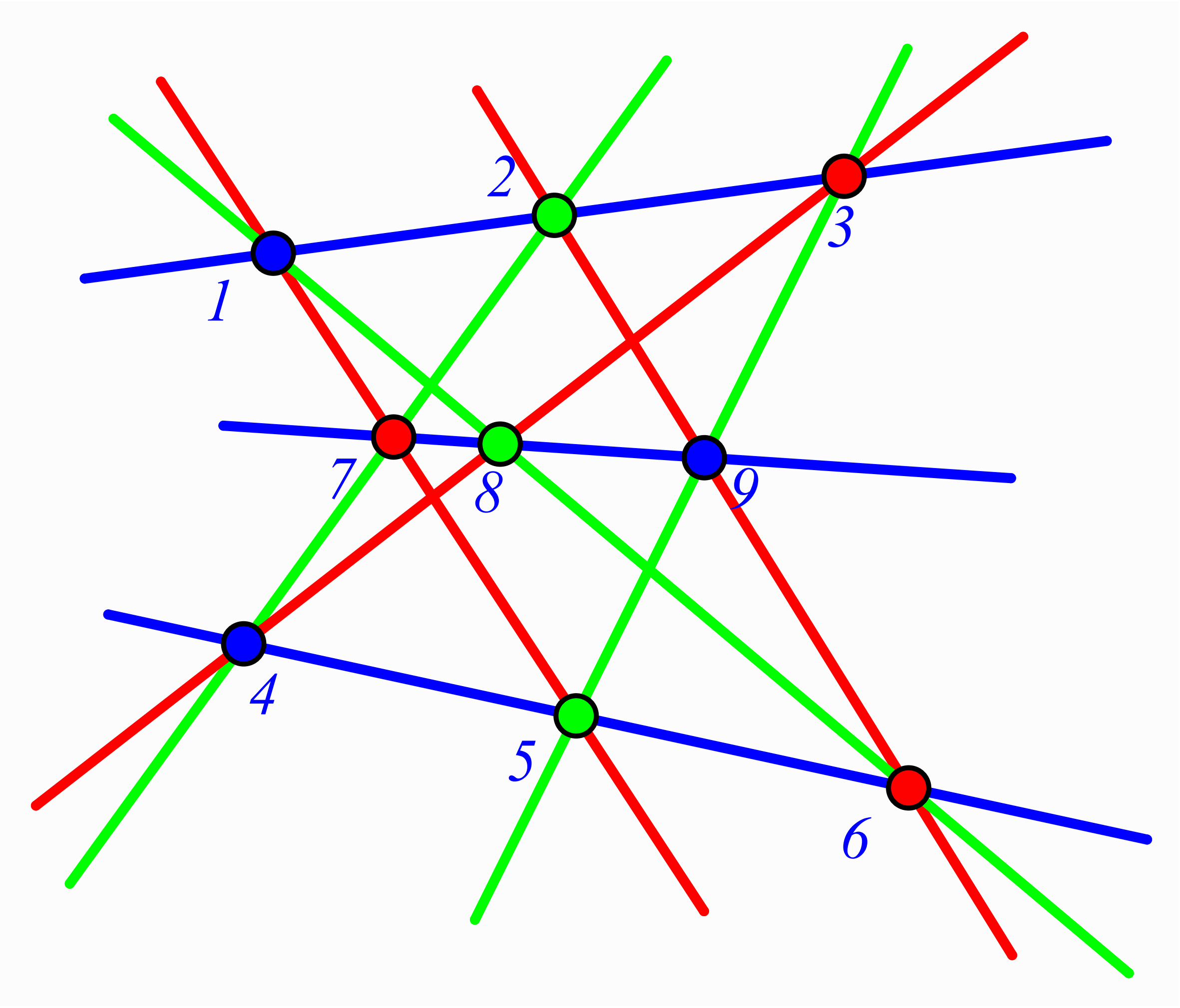


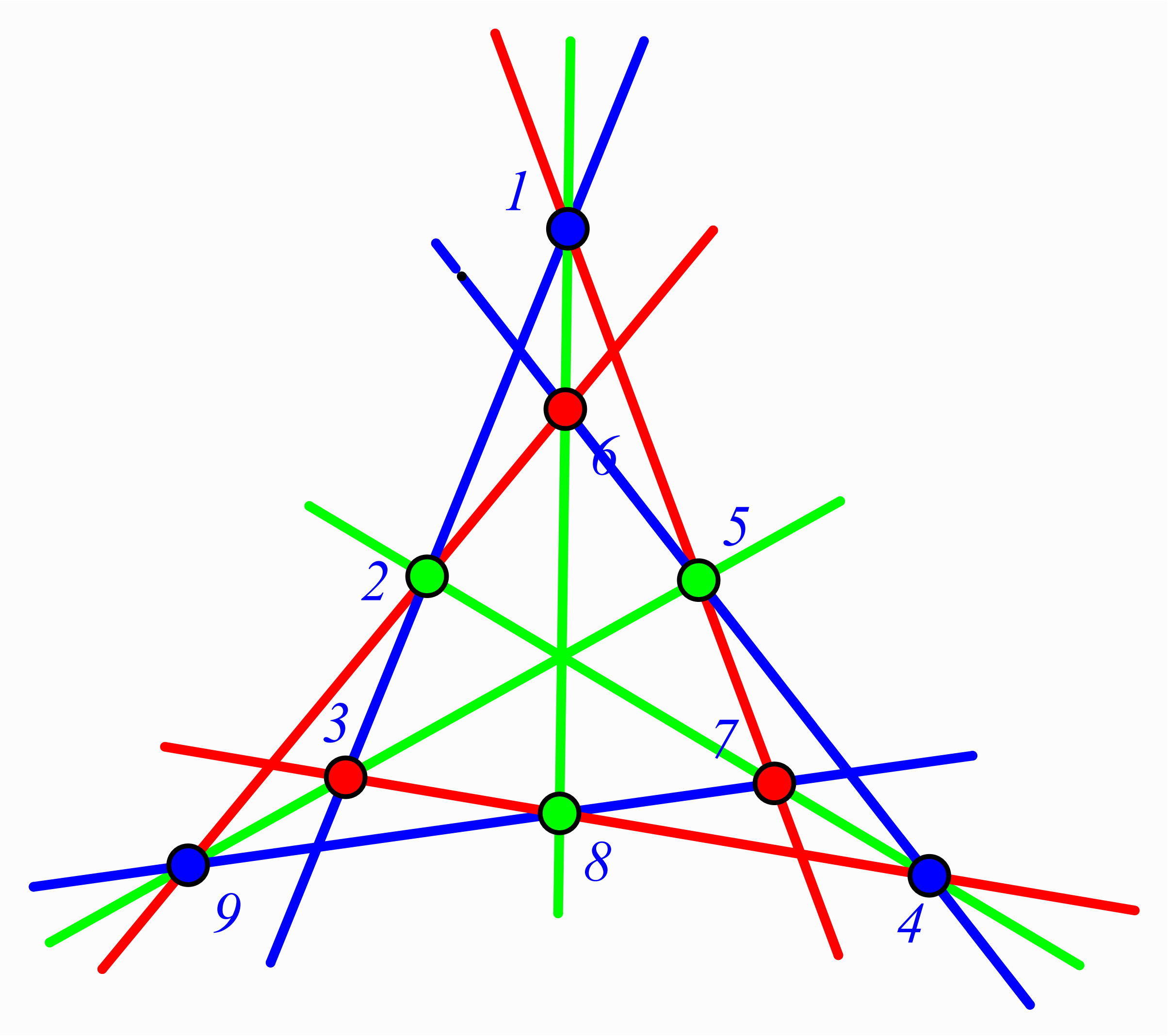
Techniques for 3-configurations…wide open in general!

Classification:

Given an infinite class of (geometric) configurations, how are the elements related?

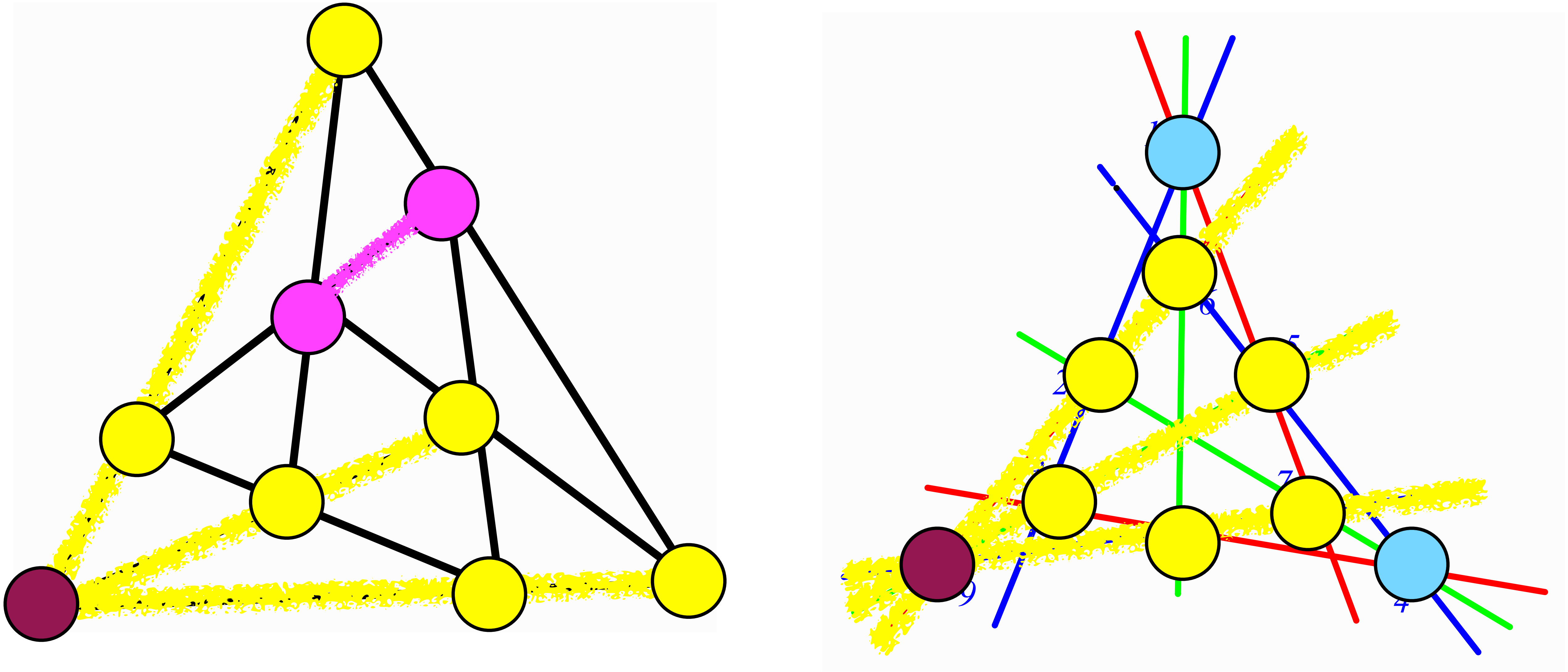
Configuration Isomorphism





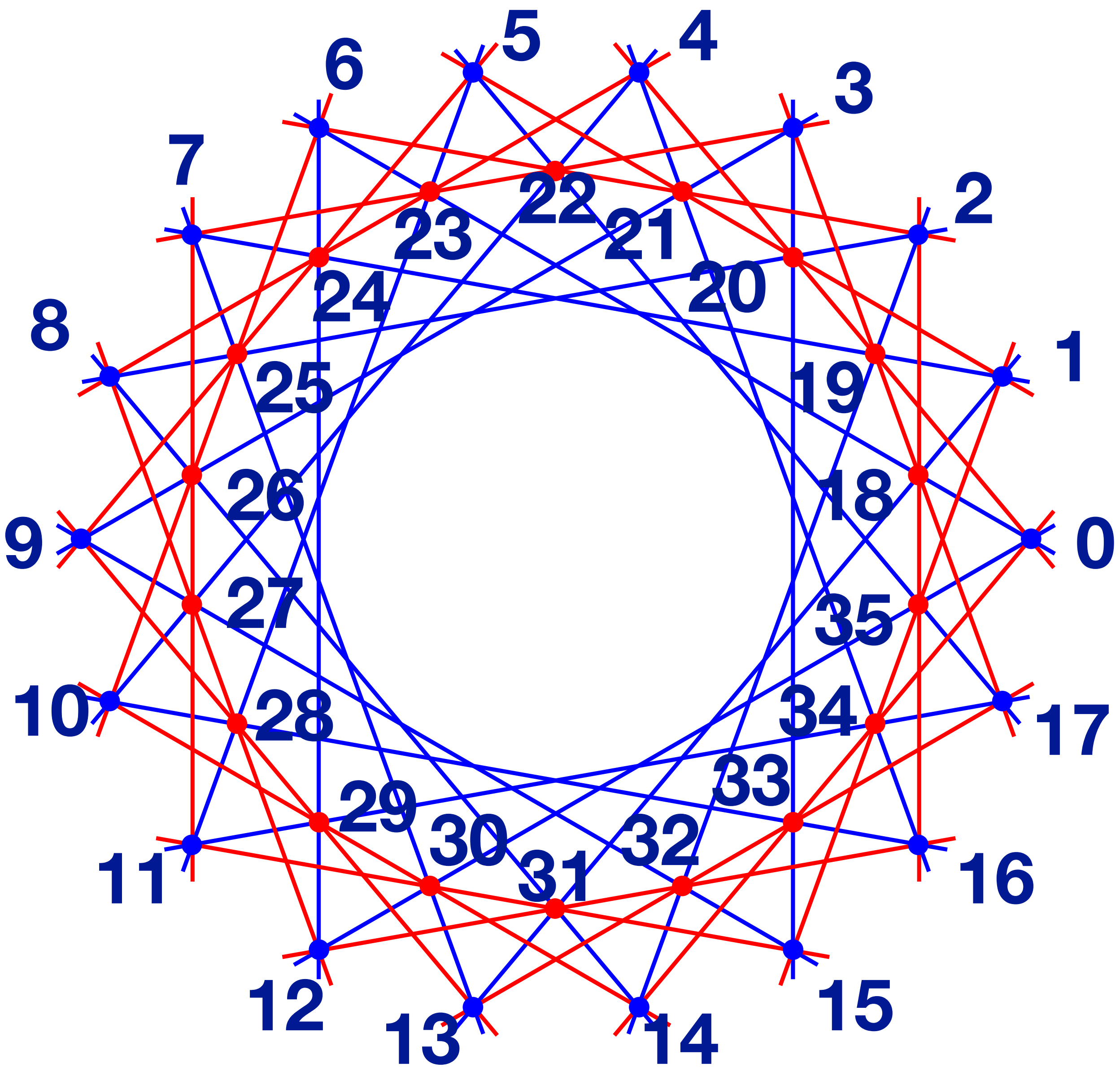
1	1	1	2	2	3	3	4	7
2	6	5	4	6	4	5	5	8
3	8	7	7	9	8	9	6	9

Configuration Isomorphism

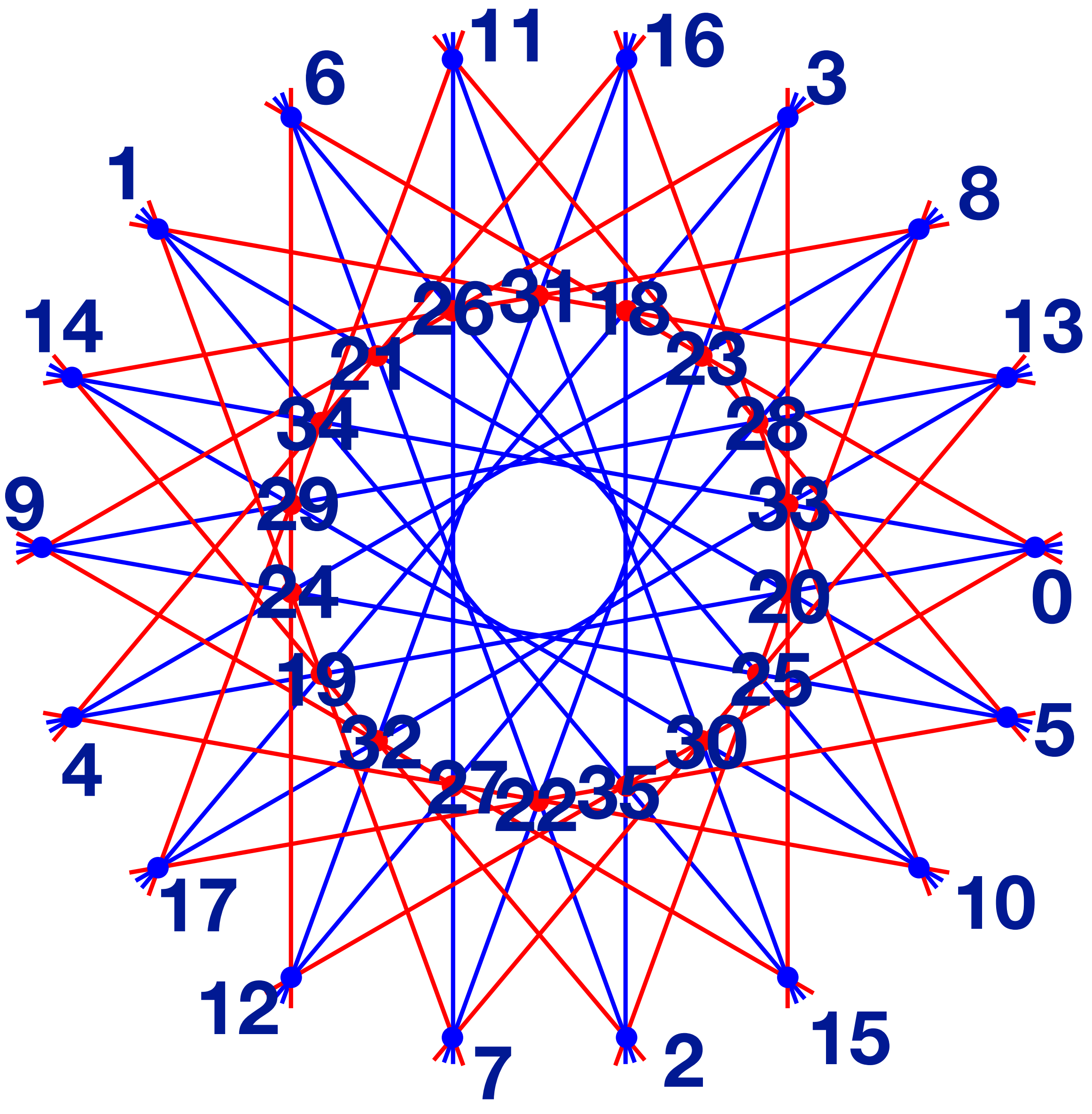


Not isomorphic!

Isomorphism Question



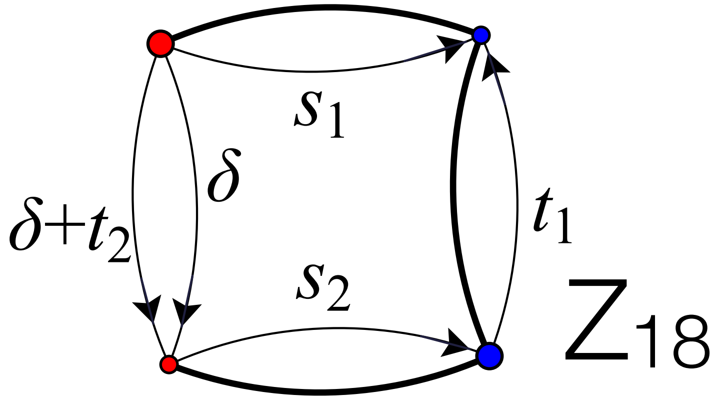
18#(6,5;1,4)

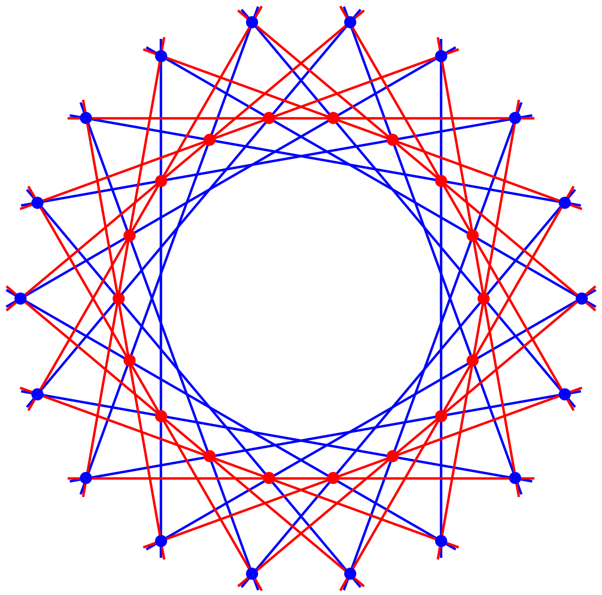


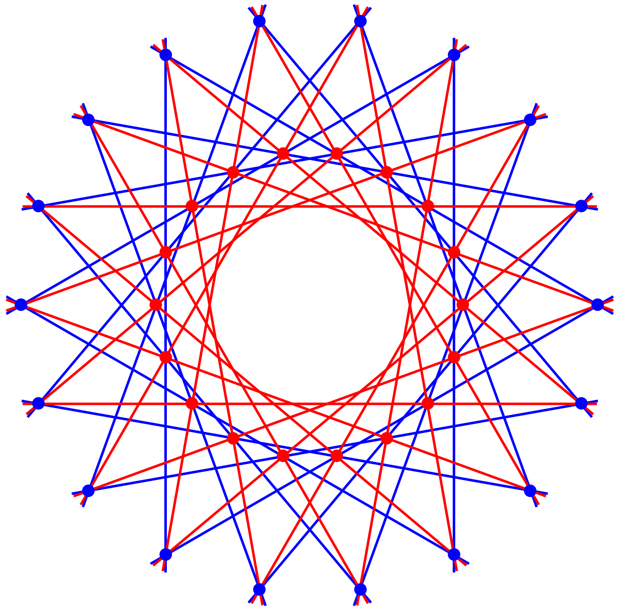
18#(8,7;1,6)

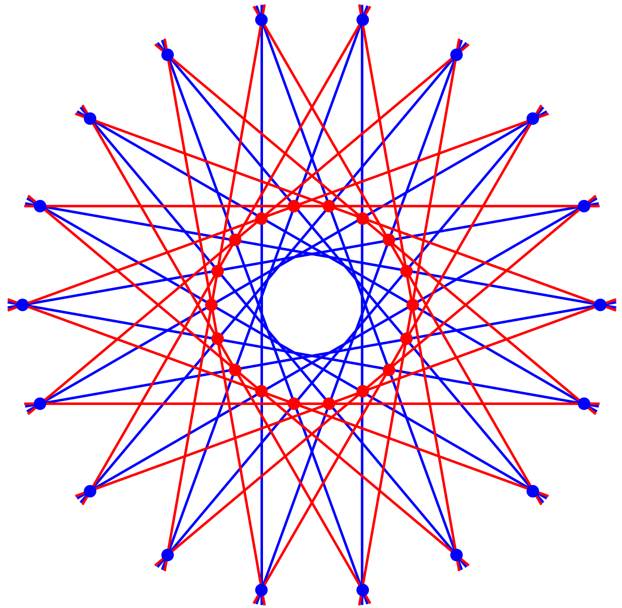
Isomorphic 4-configurations

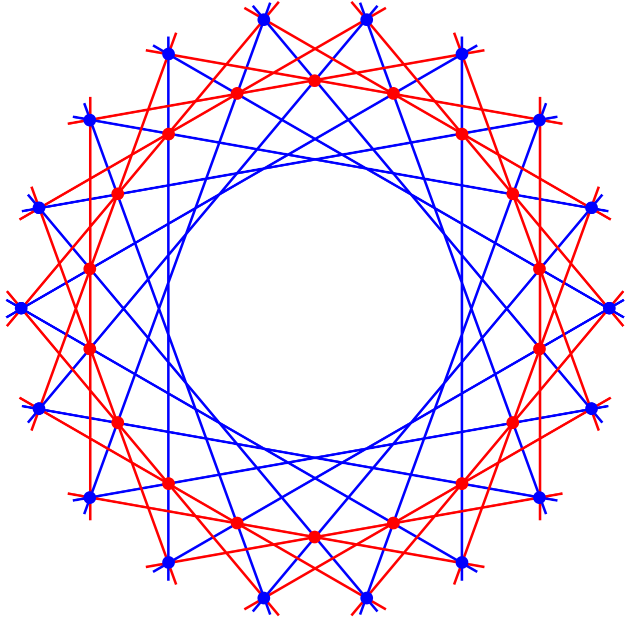
Isomorphism Question

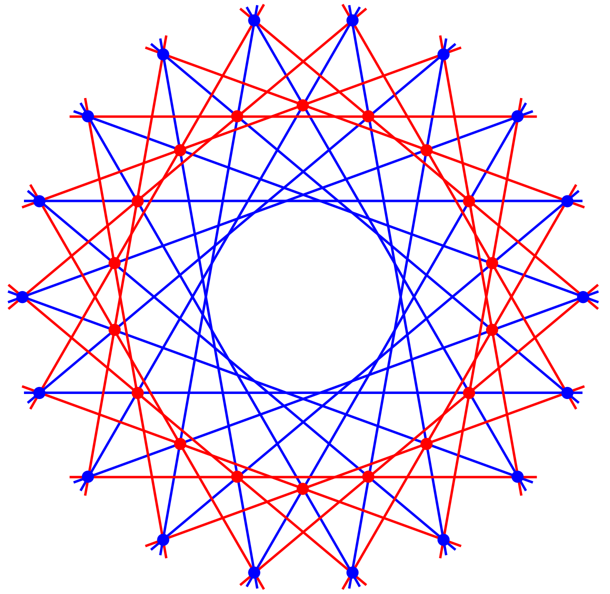


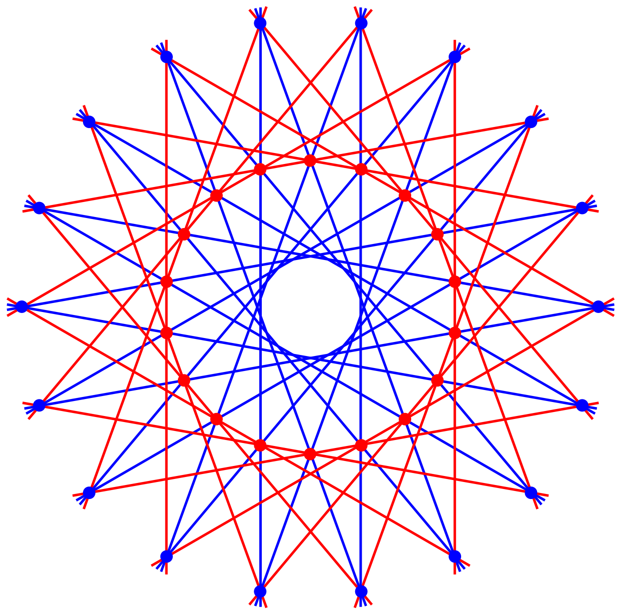

18#(6,4;1,5)


18#(6,2;5,7)


18#(8,6;1,7)


18#(6,5;1,4)


18#(7,6;2,5)


18#(8,7;1,6)

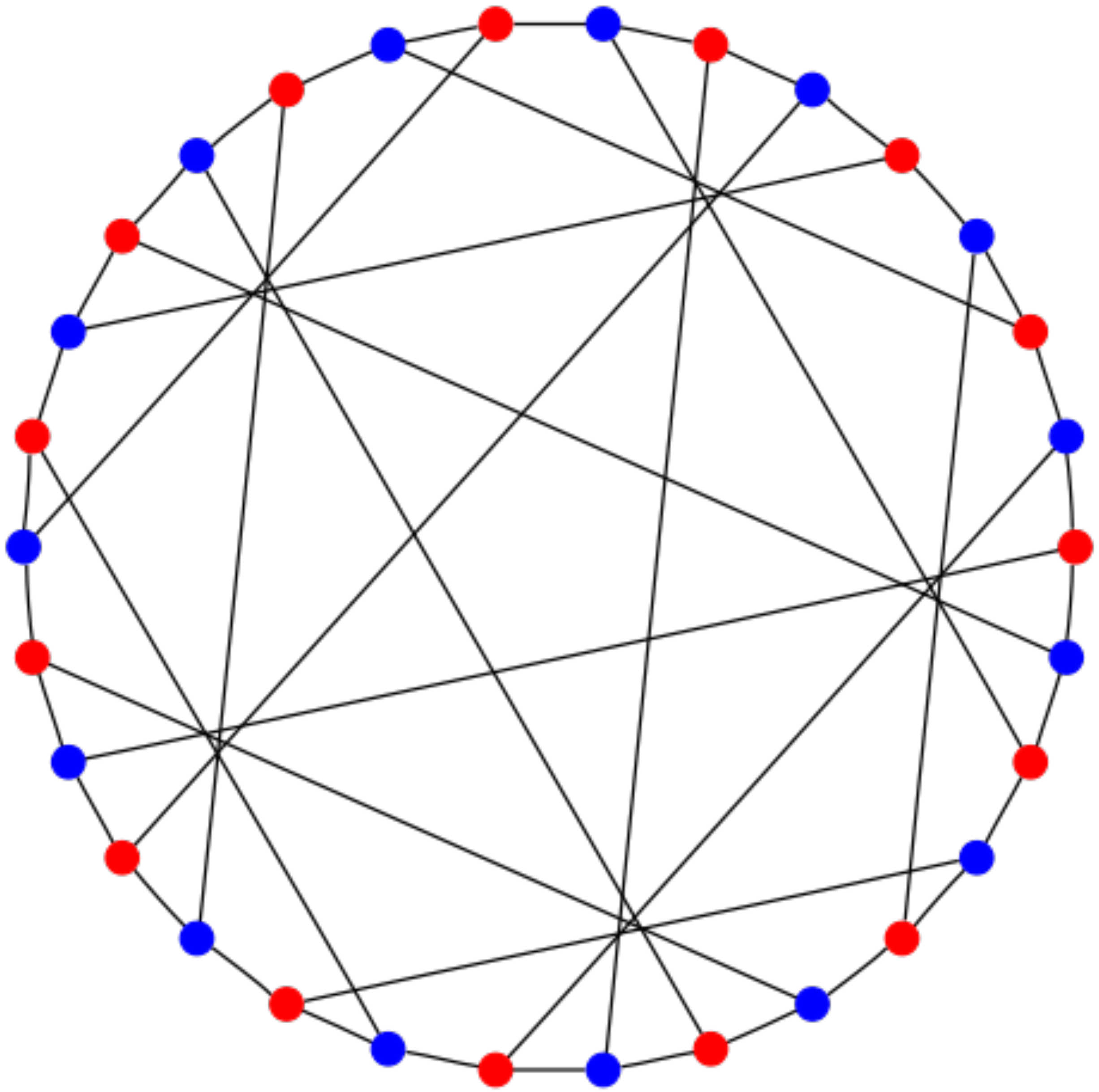
Isomorphism Question

- Which infinite families of configurations have isomorphic members?
- (Levi) graph isomorphism problem is hard
  - Determining isomorphism via Reduced Levi Graph?
- **Isomorphism relationships within/across known infinite families?**
- **Combinatorial properties of classes of geometric configurations?**

Application:

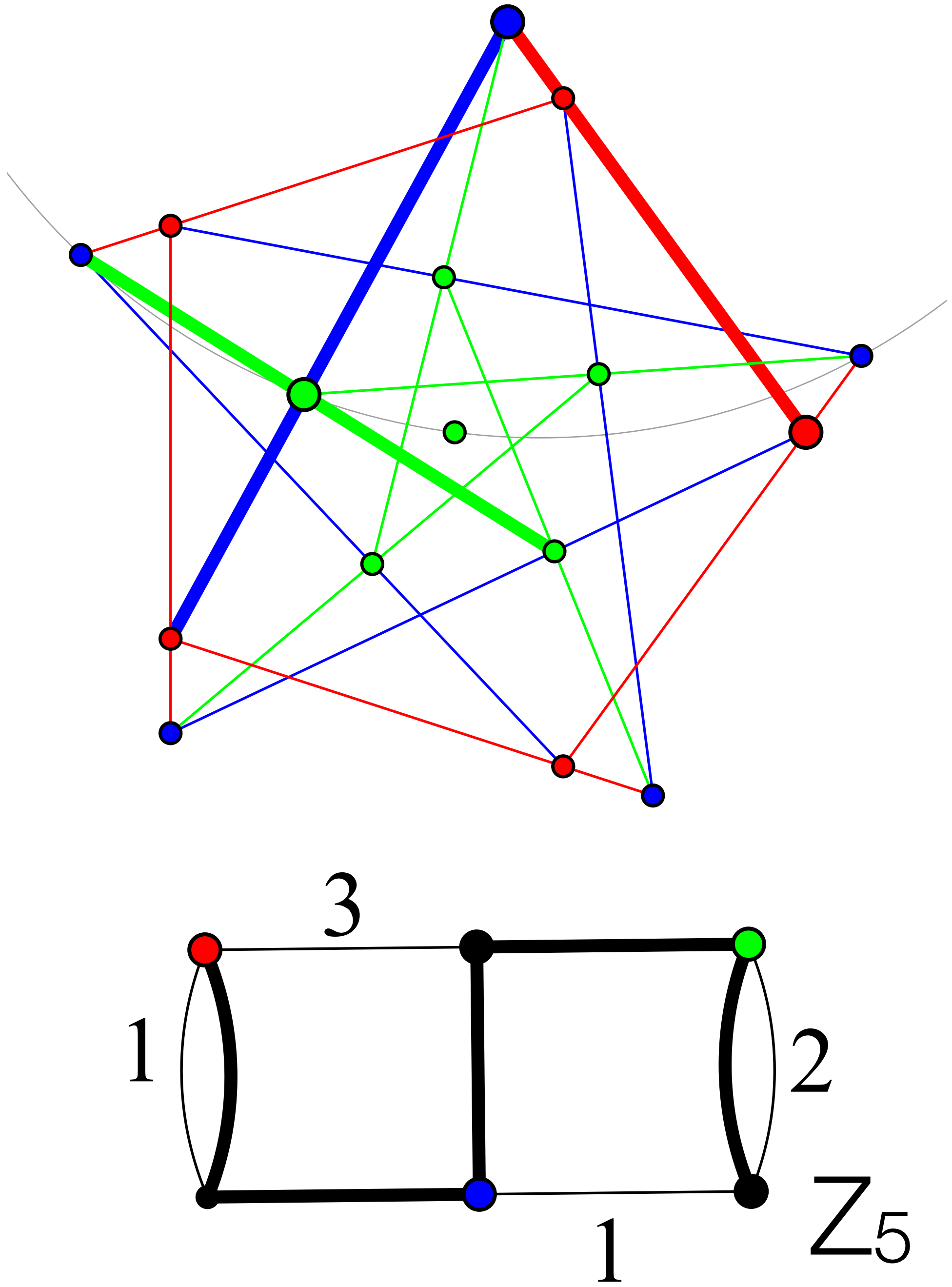
How can we use configurations to answer other questions?

Configurations and Cages



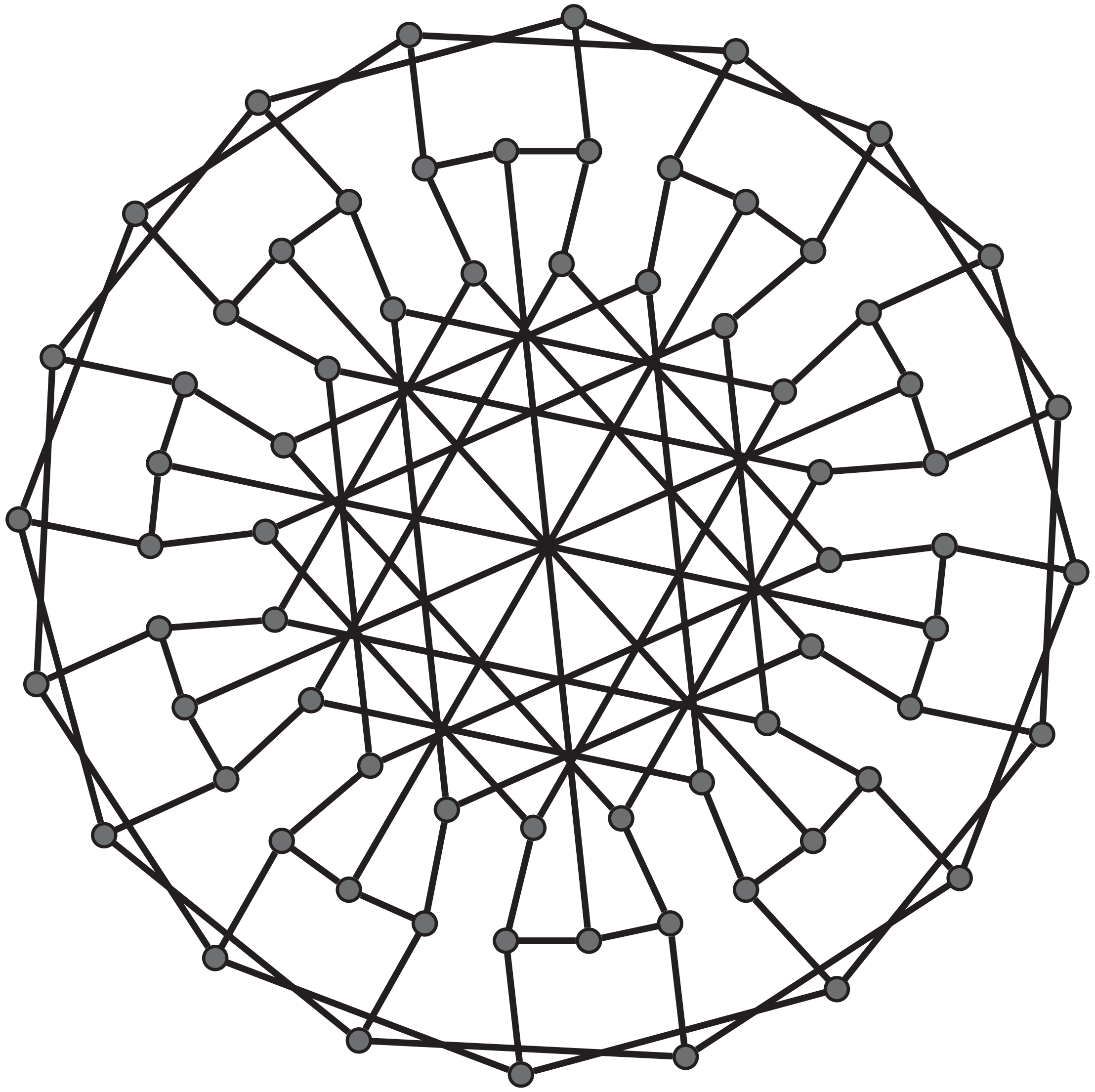
Tutte-Coxeter cage

(8,3)-cage

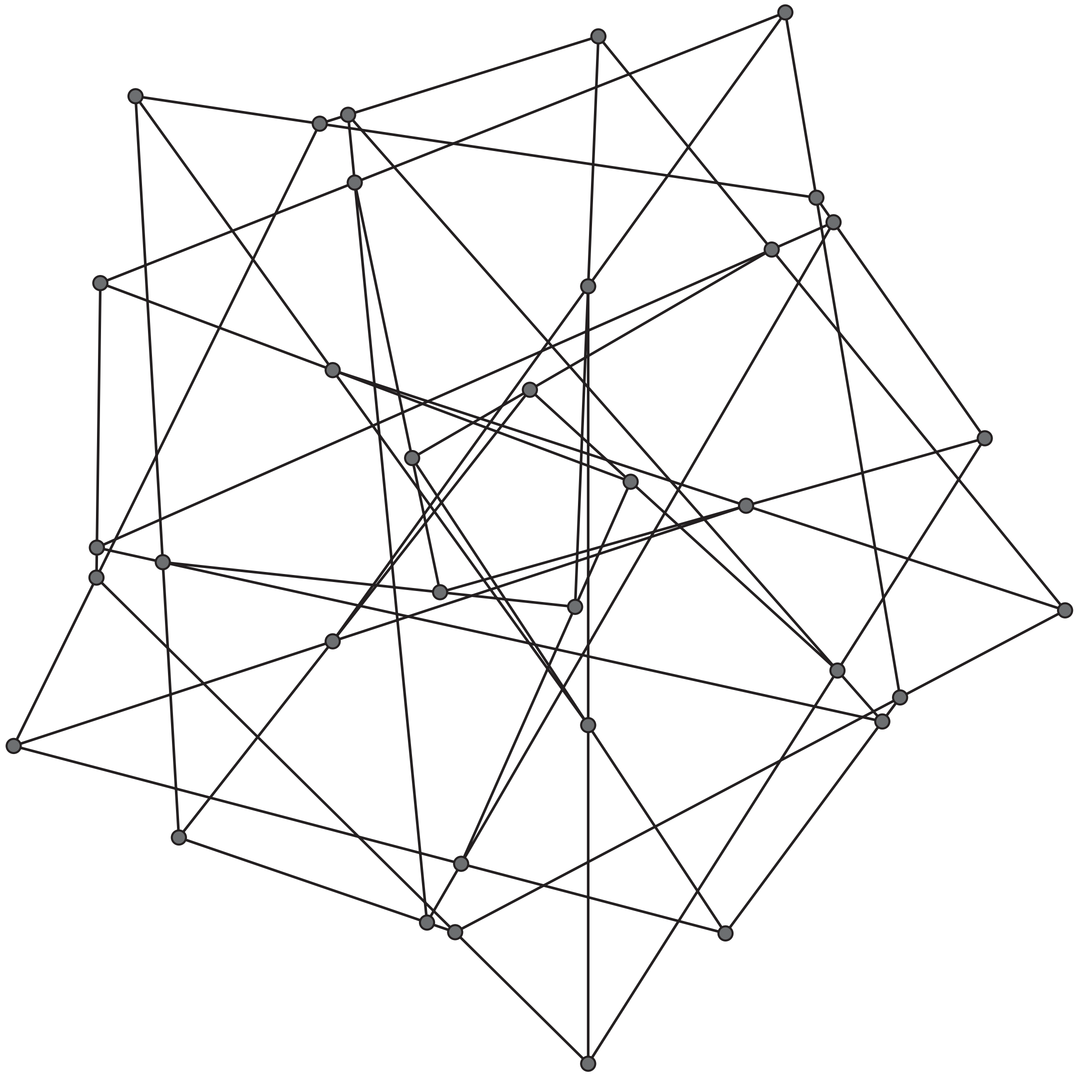


(15_3) Cremona-Richmond

Configurations and Cages



Balaban 10-cage



(35_3) configuration

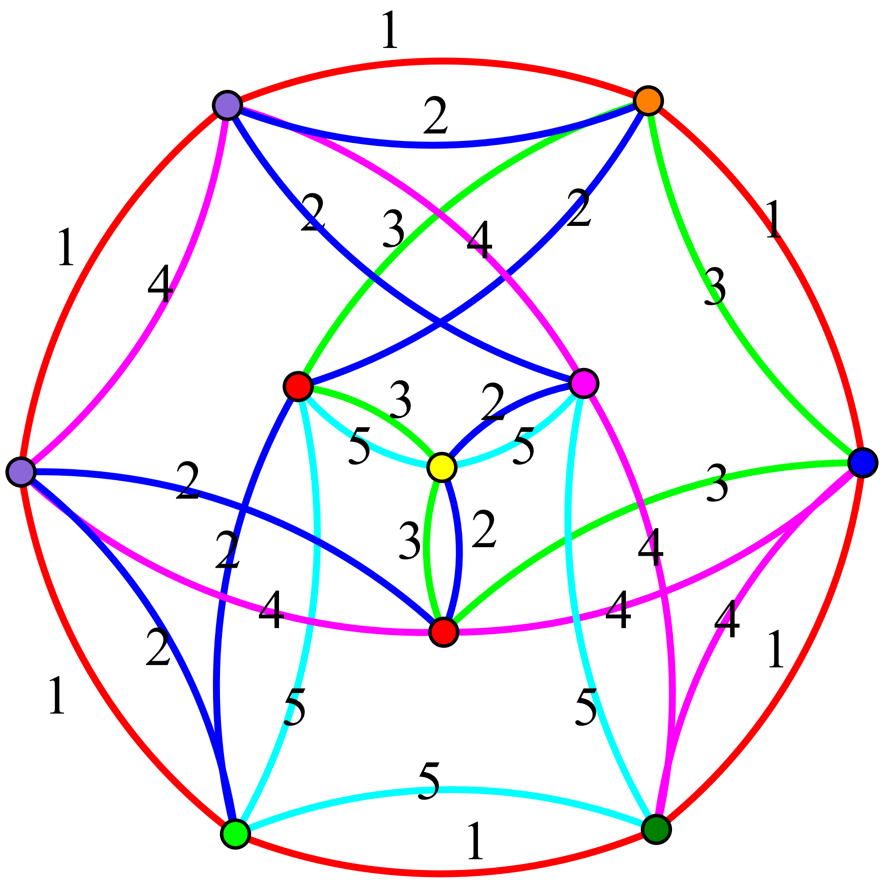
(Triangle and quadrangle-free)

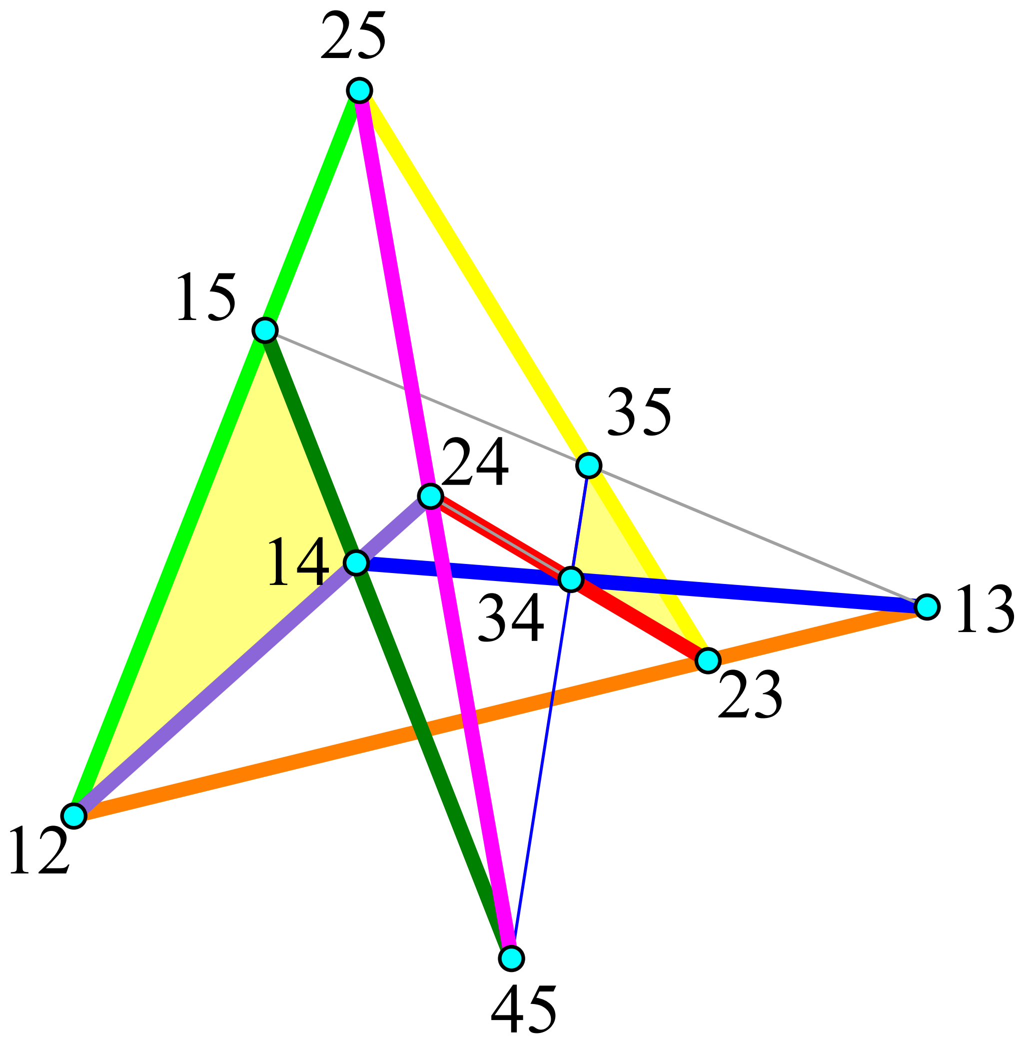
(Both images from Pisanski et al., 2004)

Conjecture: All (*3*,*g*)-cages where *g* is an even integer are bipartite graphs *(Pisanski et al, 2004)*

Can we use (symmetric, polycyclic) configurations to find small bipartite graphs with large girth?

5-cycle double-cover conjecture



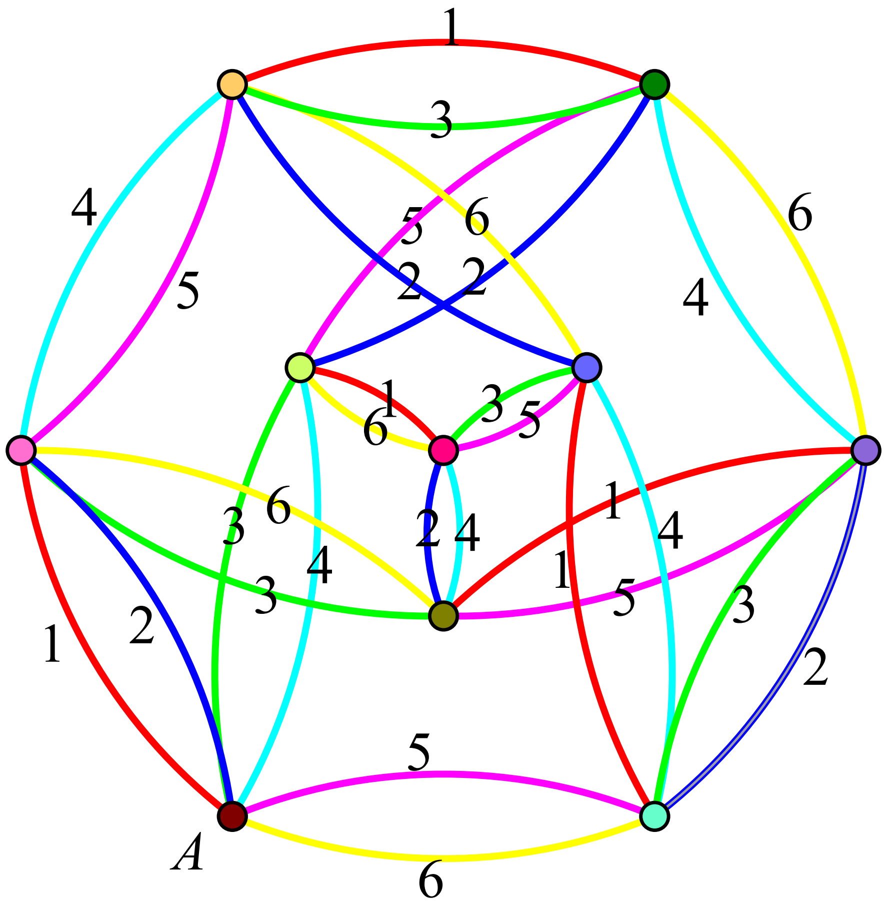


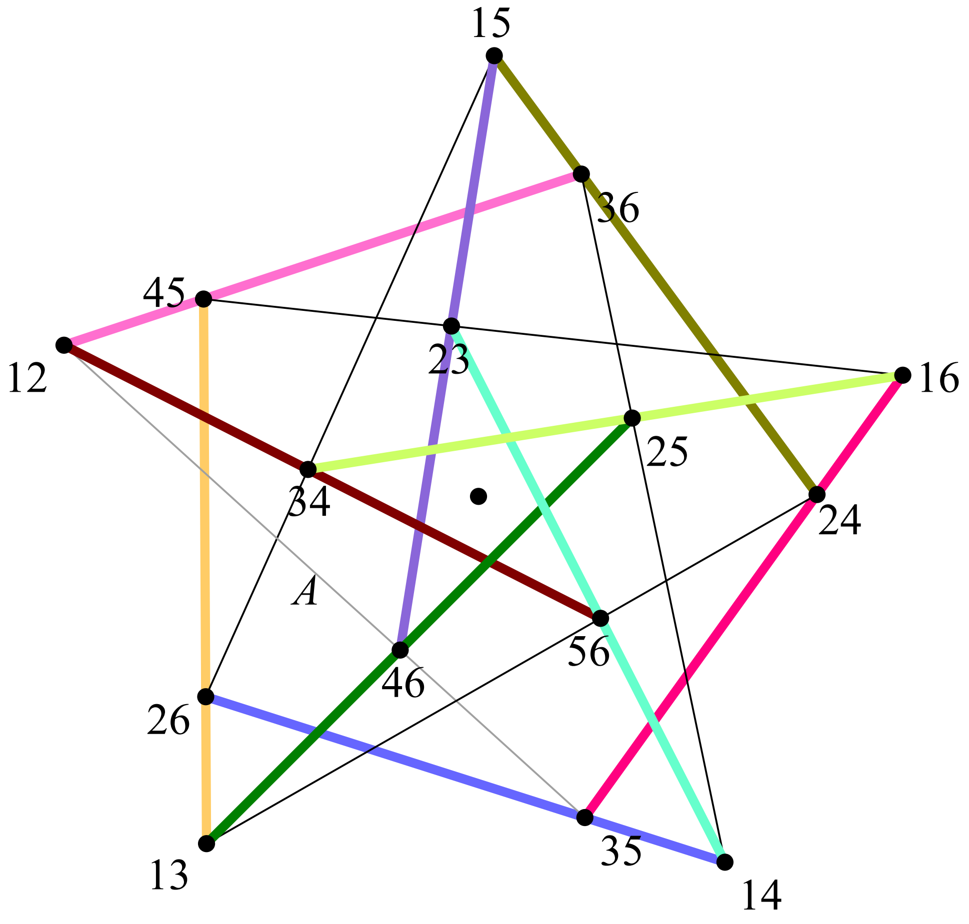
Cover every bridgeless cubic graph with 5 even subgraphs so that each edge is in exactly two of the cycles

↔

Color with Desargues Configuration (Kral et al, 2009)

Fulkerson conjecture





Every bridgeless cubic graph has 6 perfect matchings so that each edge is in two matchings

⟷

Edge-color with the Cremona-Richmond Configuration (Kral, et al 2009)

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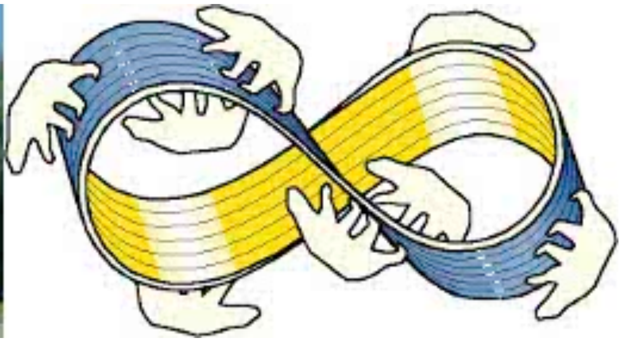

Thank you!