Vertex-transitive graphs: from derangements to rigid cells

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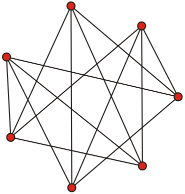
Lovász's problem, 1969

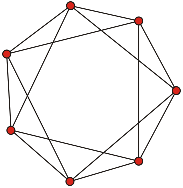
Lovász's problem

Does every connected vertex-transitive graph have a Hamilton path?

- A Hamilton path is a spanning path in a graph.
- A graph is vertex-transitive if its automorphism group acts transitively on vertices.



Is not VT



Is VT

Derangements

Derangement

A derangement is a fixed-point-free element of a transitive permutation group.

Jordan, 1872

A transitive permutation group on a finite set of cardinality at least 2 contains derangements.

Semiregular

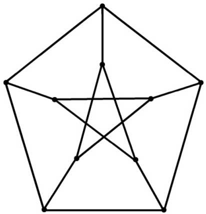
A derangement that has all cycles in its cycle decomposition of the same length is called semiregular.

(Semi)regular subgroups/automorphisms

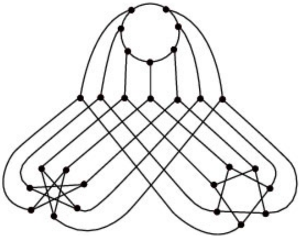
Non-Cayley vertex-transitive graphs = without regular subgroups.
Properties of subgroups close to regularity?

An automorphism is (m, n)-semiregular if it has m orbits of size n.

The Petersen and the Coxeter graph ($G = PSL(2,7)$ and $H = S_3$).



Has a (2,5)-semiregular automorphism



Has a (4,7)-semiregular automorphism

DM, '81; for transitive 2-closed groups, Klin, '96 (Polycirculant conjecture)

Does every vertex-transitive (di)graph have a semiregular automorphism?

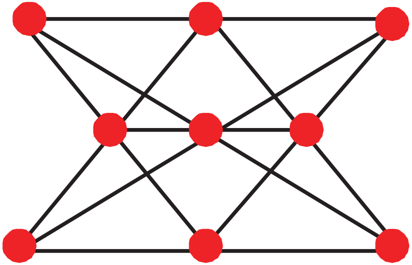
It is known that every finite transitive permutation group contains a fixed-point-free element of prime power order, but not necessarily a fixed-point-free element of prime order and, hence, no semiregular element.

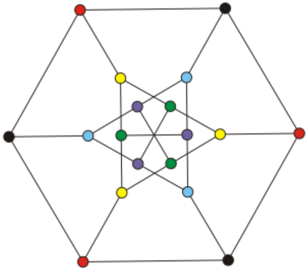
Why semiregularity matters

Why is it useful to know whether a vertex-transitive graph admits a semiregular automorphism?

It allows a quotienting with respect to semiregular automorphisms, and when the latter is wisely chosen, properties of the original graph can be analyzed via its quotient.

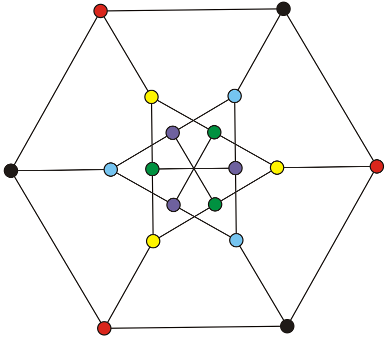
The Pappus configuration and the Pappus graph

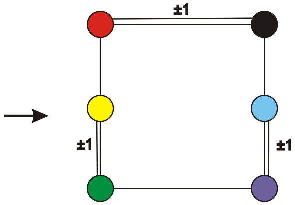




Connection to hamiltonicity of VTG

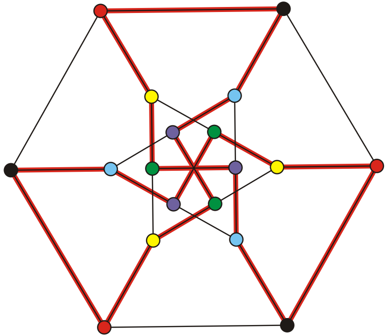
The Pappus graph

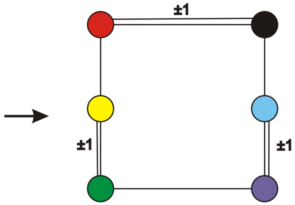




Connection to hamiltonicity of VTG

The Pappus graph





Why semiregularity matters

With this approach many questions in algebraic graph theory have been successfully answered:

- Partial results on hamiltonicity of vertex-transitive graphs.
- Structural results for vertex-transitive graphs with construction of infinite families of graphs with prescribed properties.
- Classification results (e.g. vertex-transitive graphs of particular orders).
- Construction of new strongly regular (di)graphs with previously unknown parameters.

Known results - graphs of a particular valency

- All cubic VTG have SA (Marušič, Scapellato, '93).
- Every arc-transitive graph (AGT) of prime valency has SA (Xu, '07).
- All quartic VTG have SA (Dobson, Malnič, DM, Nowitz, '07).
- All VTG of valency $p+1$ admitting a transitive $\{2,p\}$-group for p odd have SA (Dobson, Malnič, Marušič, Nowitz, '07).
- All ATG with valency pq, p, q primes, such that Aut(X) has a nonabelian minimal normal subgroup N with at least 3 vertex orbits, have SA (Xu, '08).
- All ATG with valency 8 (Verret, '15).
- All ATG with valency $2p$ (Verret, Giudici, '20).

Known results - graphs of a particular order

- All transitive permutation groups of degree p^k or mp, for some prime p and $m < p$, have SE of order p (Marušič, '81).
- All VTD of order $2p^2$ have SA of order p (Marušič, Scapellato, '93).
- There are no elusive 2-closed groups of square-free degree (Dobson, Malnič, Marušič, Nowitz, '07).
- All VTG of order $3p^2$ have SA of order p (Marušič, '17).
- There are no elusive 2-closed groups of degree p^2q and p^2qr, where p, q and r are (not necessarily distinct) three primes (Arezoomand, Ghasemi, '20).

Known results - other

- All vertex-primitive graphs have SA (Giudici, '03).
- All vertex-quasiprimitive graphs have SA (Giudici, '03).
- All vertex-transitive bipartite graphs where only system of imprimitivity is the bipartition, have SA (Giudici, Xu, '07).
- Every 2-arc-transitive graph has SA (Xu, '07).
- Every VT, edge-primitive graph has SA (Giudici, Li, '09).
- All distance-transitive graphs have SA (KK, Šparl, '09).
- All generalized Cayley graphs (Hujdurović, KK, Marušič, '13).

See also a recent survey paper:
M. Arezoomand, A. Abdollahi and P. Spiga, On problems concerning fixe–point–free permutations and on the polycirculant conjecture - a survey, Trans. Combin. Vol. 8 No. 1 (2019), pp. 15–40.

Summary - Current situation

X = VT graph

G transitive subgroup of Aut(X)

Giudici

Every normal subgroup of G transitive

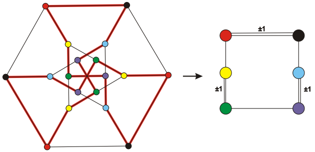
?

G has an intransitive normal subgroup

Typical situation:
G solvable.

Possible future research directions

- Path 1: Deciding which vertex-transitive graphs admit semiregular automorpisms whose quotient "multigraphs" are in fact simple graphs - the so called simplicial automorphisms.



- Path 2: Studying vertex-transitive graphs with "almost semiregular" automorphisms, that is, automorphisms whose cycle decomposition, apart from a small number of fixed vertices, only involves cycles of the same length.

Possible future research directions

P1: Deciding which vertex-transitive graphs admit semiregular automorpisms whose quotient "multigraphs" are in fact simple graphs - the so called simplicial automorphisms.

Again the motivation comes again from the Lovász problem on Hamilton paths/cycles in vertex-transitive graphs.

Since cubic vertex-transitive graphs admit a semiregular automorphism a quotienting is possible. It sometimes helps if the corresponding quotient remains a simple graph.

Such is the case with the only infinite family of cubic arc-transitive graphs with a primitive automophism group.

17 types of cubic arc-transitive graphs (Conder, Nedela, 2009)

s	Type	Bipartite?	s	Type	Bipartite?	s	Type	Bipartite?
1	$\{1\}$	Sometimes	3	$\{2^1, 3\}$	Never	5	$\{1, 4^1, 4^2, 5\}$	Always
2	$\{1, 2^1\}$	Sometimes	3	$\{2^2, 3\}$	Never	5	$\{4^1, 4^2, 5\}$	Always
2	$\{2^1\}$	Sometimes	3	$\{3\}$	Sometimes	5	$\{4^1, 5\}$	Never
2	$\{2^2\}$	Sometimes	4	$\{1, 4^1\}$	Always	5	$\{4^2, 5\}$	Never
3	$\{1, 2^1, 2^2, 3\}$	Always	4	$\{4^1\}$	Sometimes	5	$\{5\}$	Sometimes
3	$\{2^1, 2^2, 3\}$	Always	4	$\{4^2\}$	Sometimes			

Examples: $K_4 = \{1, 2^1\}$, $K_{3,3} = \{1, 2^1, 2^2, 3\}$, $Q_3 = \{1, 2^1\}$, $F010A = \{2^1, 3\}$, $F014A = \{1, 4^1\}$, $F016A = \{1, 2^1\}$, $F018A = \{1, 2^1, 2^2, 3\}$, $F020A = \{1, 2^1\}$, $F020B = \{2^1, 2^2, 3\}$.

Possible future research directions

Corollary of Lorimer's result, 1984

Let X be a cubic arc-transitive graph whose automorphism group contains a non-trivial normal subgroup N with at least three orbits on $V(X)$. Then X admits a simplicial automorphism.

By Conder & Nedela characterization of cubic arc-transitive graphs (17 families) we have:

- every graph of type $\{1, 4^1\}$ is a regular cover of the Heawood graph, which does not admit simplicial automorphisms.
- every graph of type $\{1, 4^1, 4^2, 5\}$ is a regular cover of the Biggs-Conway graph of order 2352, which admits simplicial automorphisms.

Proposition

With the exception of the Heawood graph every cubic arc-transitive graph of type $\{1, 4^1\}$ admits a simplicial automorphism.
Every cubic arc-transitive graph of type $\{1, 4^1, 4^2, 5\}$ admits a simplicial automorphism.

Possible future research directions

Corollary of Wong's result, 1967

The list of cubic vertex-transitive graph with a primitive automorphism group consists of the complete graph K_4, the Petersen graph $F10A$, the Coxeter graph $F28A$, $F234B$, and the Sextet graphs $S(p)$, where $p \equiv \pm 1 \pmod{16}$ is a prime – graphs arising from the action of $\mathrm{PSL}(2, p)$, with $p \equiv \pm 1 \pmod{16}$ a prime, on cosets of S_4.

Theorem (KK, Marušič, 2021)

A cubic vertex-transitive graph with a primitive automorphism group admits a simplicial automorphism of prime order unless it is isomorphic to one of the following four exceptional graphs:

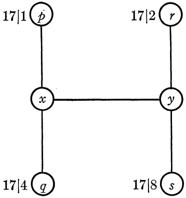
- the complete graph K_4 ($\mathrm{F4A}$ in Foster notation);
- the Petersen graph $\mathrm{F10A}$;
- the Coxeter graph $\mathrm{F28A}$;
- the Sextet graph $S(17)$ ($\mathrm{F102A}$ in Foster notation).

Possible future research directions

Theorem (KK, DM, '21)

Let $p \equiv \pm 1 (mod\, 16)$ be a prime greater than 17, let $X = S(p)$ be the unique (up to isomorphism) cubic orbital graph arising from the action of $G = PSL(2,p)$ on the cosets of $H = S_4$, and let $\pi \in G$ be of order p. Then π is a simplicial automorphism.

The smallest case $p = 17$ gives rise to the well-known *H*-graph on 102 vertices with the automorphism group isomorphic to $PSL(2,17)$.



Open problem

Which vertex-transitive graphs have simplicial automorphisms?

Possible future research directions

P2: Studying vertex-transitive graphs with "almost semiregular" automorphisms, that is, automorphisms whose cycle decomposition, apart from a small number of fixed vertices, only involves cycles of the same length.

The motivation comes from another open problem involving existence of the so-called odd automorphisms in vertex-transitive graphs, that is, automorphisms which act as odd permutations on vertices (Hujdurović, Kutnar, DM, '16). In the complete solution of the problem for cubic arc-transitive graphs (Kutnar, DM, '19) one of the main tools was an analysis of all possible configurations induced by the set of fixed vertices of an 'almost semiregular' automorphisms'.

Possible future research directions

A non-trivial automorphism g of a graph X is called quasi-semiregular if it fixes one vertex and the only power g^i fixing another vertex is the identity mapping.

Theorem (Feng, Hujdurović, Kovács, Kutnar, DM, '19)

Let X be a connected arc-transitive graph of valency $d \in \{3, 4\}$, and suppose that X admits a quasi-semiregular automorphism.

- If $d = 3$ then X is isomorphic to K_4 or the Petersen graph or the Coxeter graph.
- If $d = 4$ and X is 2-arc-transitive, then X is isomorphic to K_5.
- If $d = 5$ and X is G-arc-transitive, where G is solvable and contains a quasi-semiregular automorphism, then X is isomorphic to $Cay(A, S)$, where A is an abelian group of odd order and S is an orbit of a subgroup of $Aut(A)$.

Given a graph X and an automorphism α of X let $\mathrm{Fix}(\alpha)$ denote the set of all vertices of X fixed by α.

Rigid cell I

Given a graph X and an automorphism α of X let $\mathrm{Fix}(\alpha)$ denote the set of all vertices of X fixed by α.

With the assumption that $\mathrm{Fix}(\alpha) \neq \emptyset$ we call the subgraph $X[\mathrm{Fix}(\alpha)]$ induced on $\mathrm{Fix}(\alpha)$ the rigid subgraph of α or, in short, the α-rigid subgraph.

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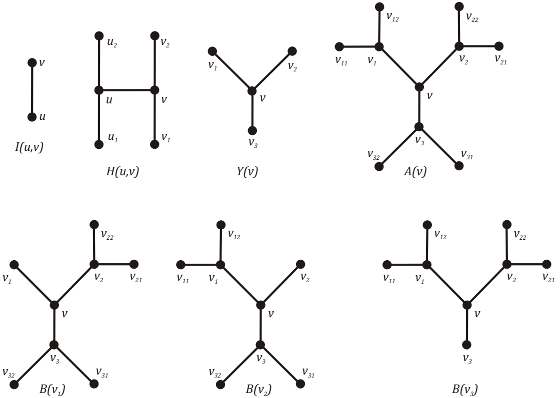
Every component of $X[\mathrm{Fix}(\alpha)]$ is referred to as an α-rigid cell.

Rigid cells in cubic symmetric graphs



We will use the terms *I-tree*, *H-tree*, *Y-tree*, *A-tree* and *B-tree* for the graphs given in the figure. More precisely, we will denote these graphs by $I(u, v)$, $H(u, v)$, $Y(v)$, $A(v)$ and $B(v)$, respectively.

The order of automorphisms fixing a vertex in a cubic arc-transitive graphs

The structure of vertex stabilizers in cubic arc-transitive graphs implies that only automorphisms of order 2, 3, 4 and 6 can fix a vertex.

s	$\mathrm{Aut}(X)_v$	$\mathrm{Aut}(X)_e$
1	$\mathbb{Z}_3$	id
2	S_3	$\mathbb{Z}_2^2$ or $\mathbb{Z}_4$
3	$S_3 \times \mathbb{Z}_2$	D_8
4	S_4	D_{16} or QD_{16}
5	$S_4 \times \mathbb{Z}_2$	$(D_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$

Proposition (KK, DM, 2017)

Let X be a cubic arc-transitive graph and let $\alpha \in \mathrm{Aut}(X)$ be an automorphism of X fixing a vertex.

(i) If α is of order 3 or 6 then the only α-rigid cells are isolated vertices.

(ii) If α is of order 4 then the only possible α-rigid cells are I-trees.

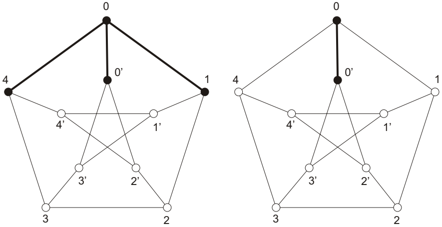
Rigid cells of involutions

We will say that an involution in the automorphism group of a cubic arc-transitive graph is

- an I-involution if all of its rigid cells are isomorphic to the I-tree,
- a Y-involution if all of its rigid cells are isomorphic to the Y-tree,
- an H-involution if all of its rigid cells are isomorphic to the H-tree,
- an A-involution if all of its rigid cells are isomorphic to the A-tree,
- an M-involution if it admits non-isomorphic rigid cells (and thus its rigid cells are of *mixed* structure), and
- an S-involution if it is semiregular.

The Petersen graph - an example of type $\{2^1, 3\}$

Any involution in X is either conjugate to the involution $(0)(0')(1)(4)(1'\ 2)(3\ 4')(2'\ 3')$ with one rigid cell isomorphic to the Y-tree, or to the involution $(0)(0')(1\ 4)(1'\ 3')(1'\ 4')(2\ 3)$ with one rigid cell isomorphic to the I-tree.



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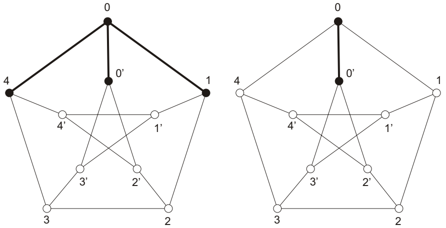
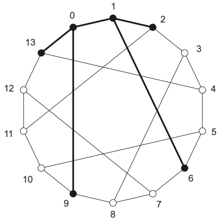


Proposition (KK, Marušič, JCTB, 2019)

Every cubic 3-arc-regular graph admitting a 2-arc-regular subgroup has two conjugacy classes of non-semiregular involutions, one consisting of I-involutions, and one consisting of Y-involutions.

The Heawood graph - an example of type $\{1, 4^1\}$

There exist two conjugacy classes of involutions in $\mathrm{Aut}(X)$, one consisting of S-involutions and one consisting of H-involutions. In particular, any non-semiregular involution in X is conjugate to the involution (0)(1)(2)(3 11)(4 12)(5 7)(6)(8 10)(9)(13) with one rigid cell which is isomorphic to the H-tree.



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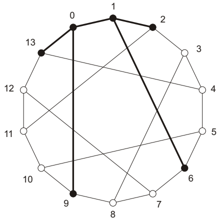


Proposition (KK, Marušič, JCTB, 2019)

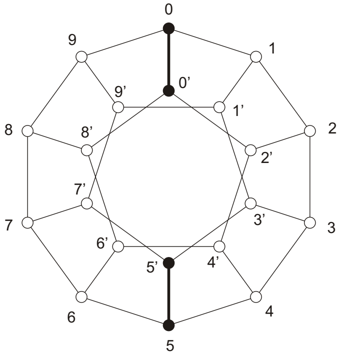
In every cubic 4-arc-regular graph non-semiregular involutions are H-involutions.

The dodecahedron - an example of type $\{1, 2^1\}$

There exist three conjugacy classes of involutions in $\mathrm{Aut}(X)$, two consisting of S-involutions and one consisting of I-involutions. Any non-semiregular involution in X is conjugate to the involution

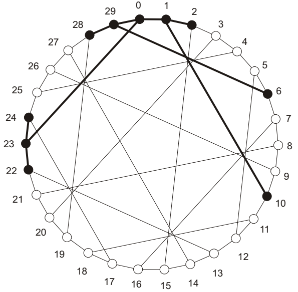
$$(0)(1\ 9)(2\ 8)(3\ 7)(4\ 6)(5)(0')(1'\ 9')(2'\ 8')(3'\ 7')(4'\ 6')(5')$$

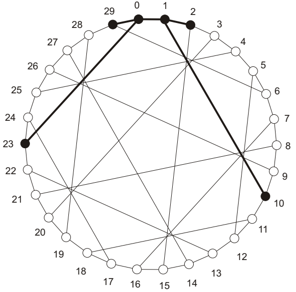
which has two rigid cells, both isomorphic to the I-tree.



Tutte's 8-cage - an example of type $\{4^1, 4^2, 5\}$

There exist three conjugacy classes of involutions in $\mathrm{Aut}(X)$, one consisting of S-involutions, one consisting of H-involutions, and one consisting of A-involutions. Any non-semiregular involution in X is either conjugate to the A-involution (0)(1)(2)(6)(10)(22)(23)(24)(28)(29)(3 15)(5 7)(9 11)(13 21)(17 25)(19 27)(4 16)(8 12)(14 20)(18 26) with one rigid cell which is isomorphic to the A-tree, or to the H-involution (0)(1)(2)(10)(23)(29)(3 15)(6 28)(9 11)(22 24)(4 14)(5 27)(7 19)(8 18)(12 26)(13 25)(16 20)(17 21) with one rigid cell which is isomorphic to the H-tree.





Results and examples

Proposition (KK, Marušič, JCTB, 2019)

Every cubic 5-arc-regular graph admitting a 4-arc-regular subgroup has two conjugacy classes of non-semiregular involutions, one consisting of H-involutions and one consisting of A-involutions.

Back to 17 types

s	Type	Bipartite?	s	Type	Bipartite?	s	Type	Bipartite?
1	$\{1\}$	Sometimes	3	$\{2^1, 3\}$	Never	5	$\{1, 4^1, 4^2, 5\}$	Always
2	$\{1, 2^1\}$	Sometimes	3	$\{2^2, 3\}$	Never	5	$\{4^1, 4^2, 5\}$	Always
2	$\{2^1\}$	Sometimes	3	$\{3\}$	Sometimes	5	$\{4^1, 5\}$	Never
2	$\{2^2\}$	Sometimes	4	$\{1, 4^1\}$	Always	5	$\{4^2, 5\}$	Never
3	$\{1, 2^1, 2^2, 3\}$	Always	4	$\{4^1\}$	Sometimes	5	$\{5\}$	Sometimes
3	$\{2^1, 2^2, 3\}$	Always	4	$\{4^2\}$	Sometimes			

Type $\{3\}$ and type $\{5\}$ graphs

Proposition (KK, Marušič, JCTB, 2019)

Let X be a cubic 3-arc-regular graph and let $\alpha \in \mathcal{I}(X)$ be a non-semiregular involution of X. Then the possible α-rigid cells are I-trees and Y-trees, with both types of cells possibly occurring simultaneously only when the graph X is of type $\{3\}$.

Proposition (KK, Marušič, JCTB, 2019)

Let X be a cubic 5-arc-regular graph and let $\alpha \in \mathcal{I}(X)$ be a non-semiregular involution of X. Then the only possible α-rigid cells are H-trees and A-trees, with both types of cells possibly occurring simultaneously only when the graph X is of type $\{5\}$.

Type $\{3\}$ and type $\{5\}$ graphs

Theorem (Conder, Hujdurović, KK, Marušič, JACO, 2020)

Let X be a cubic symmetric graph of type $\{s\}$, where $s \in \{3, 5\}$, and of order $2n$, where n is odd. Then $\mathcal{I}(X)$, the set of all involutions in $\mathrm{Aut}(X)$ which fix some vertex of X, is a conjugacy class. In particular, every involution $\alpha \in \mathcal{I}(X)$ is an M-involution, and

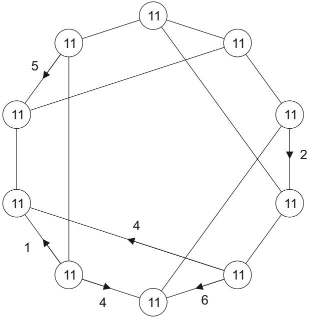
- if $s = 3$ then α has rigid cells isomorphic to the I-tree as well as rigid cells isomorphic to the Y-tree, and
- if $s = 5$ then α has rigid cells isomorphic to the H-tree as well as rigid cells isomorphic to the A-tree.

F110 - type $\{3\}$ graph of order 2 (mod 4)

There exist two conjugacy classes of involutions in $\mathrm{Aut}(X)$, one consisting of S-involutions and one consisting of M-involutions having rigid cells isomorphic to the I-tree as well as rigid cells isomorphic to the Y-tree. Every M-involution is conjugate to

$$u_0^0, u_0^4, u_1^0, u_1^7, u_3^4, u_3^9, u_3^{10}, u_4^1, u_4^2, u_4^7, u_7^4, u_7^{10}, u_8^1, u_8^7.$$

with two rigid cells isomorphic to the Y-tree and three rigid cells isomorphic to the I-tree.



Example of a type $\{3\}$ graph of order $0 \pmod 4$

Using Magma one can see that there exists a cubic symmetric graph of type $\{3\}$ and of order $39,916,800$ with the automorphism group isomorphic to the symmetric group S_{12}. For this graph, there are two conjugacy classes of involutions with fixed points. In one of these two classes each involution fixes 15360 vertices, which partition into 3840 rigid cells isomorphic to the Y-tree (and no rigid cell isomorphic to the I-tree), while in the other class, each involution fixes 2304 vertices, which partition into 1152 rigid cells isomorphic to the I-tree (and no rigid cell isomorphic to the Y-tree).

Using Magma one can see that there exists a cubic symmetric graph of type $\{5\}$ and of order $50,685,458,503,680,000$ with the automorphism group isomorphic to the symmetric group S_{20}. For this graph, there are two conjugacy classes of involutions with fixed points, one consisting of involutions with all rigid cells isomorphic to the A-tree and one consisting of involutions with all rigid cells isomorphic to the H-tree.

The transfer - Essential ingredient in the proof of the main theorem

For a group G and its subgroup $H \leq G$ the right transversal $\mathcal{T}$ for H in G is a set of right cosets representatives for H in G. Then G acts on $\mathcal{T}$, and for $t \in \mathcal{T}$ and $g \in G$ we define $t \cdot g$ to be the unique element of $\mathcal{T}$ that lies in the right coset Htg.

Suppose that H is a subgroup of G of finite index and suppose that M is a normal subgroup of H with H/M abelian. Then the *transfer from G to H/M* is the map $v\colon G \to H/M$ given by $v(g) = M\pi(g)$ where $\pi\colon G \to H$ is a map defined by

$$\pi(g) = \prod_{t \in \mathcal{T}} tg(t \cdot g)^{-1}$$

where $\mathcal{T}$ is a right transversal for H in G. The map v is a group homorophism.

THANK YOU!

HVALA!