Constructions of combinatorial configurations

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Table of Contents

Table of Contents

Incidence geometry

An **incidence geometry** of rank two is a triple (P, L, I) where

- P is a set of 'points',
- L is a set of 'blocks',
- I is an incidence relation between the elements in P and L.

When there are at most one block containing p_i and p_j for all pairs of points, then we may call the blocks (combinatorial) **lines**.

Incidence graph of an incidence geometry

We can use a graph to represent the incidences of points and blocks.

The **incidence graph** of the incidence structure (P, L, I) is the bipartite graph with vertex set $P \cup L$ and an edge between the vertices p and b if p is a point on b.

The girth of the incidence graph of an incidence geometry of points and **combinatorial lines** is at least 6.

Girth g is equivalent to avoiding n-gons for all $n < g/2$.

Combinatorial configurations

A **combinatorial** (v, b, r, k)-**configuration** is an incidence structure with v points and b lines/blocks such that

- every point appears on r lines,
- every line has k points,
- every pair of points is in at most one line, or equivalently,
- every pair of lines intersect in at most one point.

The four parameters (v, b, r, k) are redundant.
We only need the three parameters (d, r, k) with

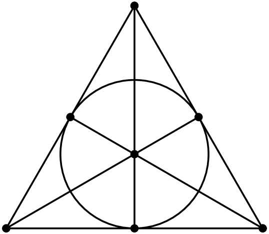
$$d := \frac{v \gcd(r, k)}{k} = \frac{b \gcd(r, k)}{r} = \frac{vr}{\operatorname{lcm}(r, k)} = \frac{bk}{\operatorname{lcm}(r, k)}.$$

Reduced parameters: (d, r, k)-configuration.

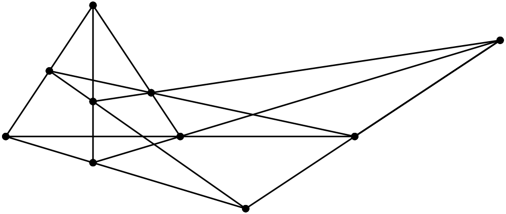
A combinatorial (v, b, r, k) configuration is also called an r-regular and k-uniform **partial linear space**.

Balanced configurations

We say that a combinatorial configuration is **balanced** if $r = k$. This implies that the number of points equals the number of lines and also, the associated integer, so $d = v = b$.



The Fano plane,
$(v, b, r, k) = (7, 7, 3, 3)$
$(d, r, k) = (7, 3, 3)$

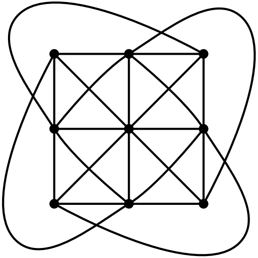


The Desargues' configuration
$(v, b, r, k) = (10, 10, 3, 3)$
$(d, r, k) = (10, 3, 3)$

Non-balanced configurations

When $r \neq k$, then $v \neq b$ and $d = \frac{v \gcd(r,k)}{k}$.

- The affine plane $AG(2, q)$ over the finite field $\mathbb{F}_q$ has parameters $(q^2, q^2 + q, q + 1, q)$ so $d = q$.
  Reduced parameters: $(q, q + 1, q)$.
- A Steiner triple systems $STS(v)$ of order v has parameters $(v, v(v-1)/6, (v-1)/2, 3)$.
  Reduced parameters: $(v \gcd(v-1, 3)/3, (v-1)/2, 3)$.



$AG(2, 3)$ / $STS(9)$

$$\begin{aligned} (v, b, r, k) &= (9, 12, 4, 3) \\ (d, r, k) &= (3, 4, 3) \end{aligned}$$

Necessary conditions for existence of configurations

The following necessary conditions for existence of configurations are well-known.

Lemma.
Suppose that there exists a (v, b, r, k)-configuration. Then

1. $v \geq r(k-1)+1$ and $b \geq k(r-1)+1$, and
2. $vr = bk$.

We say that parameters satisfying these conditions are **admissible**.

What about sufficient conditions?

Sufficent conditions

- When $r = 3$, the necessary conditions are sufficent [Gropp (1994)].
- When $r = 4$, it is conjectured that the necessary conditions are sufficient [Gropp (2001)].
- When $r = 5$, the necessary conditions are not sufficent. Sufficient conditions are not known for $k > r$.
- In general sufficient conditions are not known.

Balanced configurations

$r = k$	π	d :	$\exists(d, r, k)$	conf.					
3	7	7	$\rightarrow$						
4	13	13	$\rightarrow$						
5	21	21	~~22~~[1]	23	$\rightarrow$				
6	31	31	~~32~~[2]	~~33~~[3]	34	$\rightarrow$			
7	43	~~43~~[4]	~~44~~[1]	45	?46?	?47?	48	$\rightarrow$	
8	57	57	~~58~~[1]	?59?	?60?	?61?	?62?	63	$\rightarrow$
9	73	73	~~74~~[5]	?75?	?76?	?77?	78	?79?	80 $\rightarrow$

For unbalanced configurations in general, less is known!

[1][Bose and Connor (1952)]
[2][Schellenberg (1975)]
[3][Kaski and Östergård (2007)]
[4][Bose (1938)]
[5][Gropp (1992)])

Parameter sets of combinatorial configurations

For which parameter sets do (v, b, r, k)-configurations exist?
We saw that a (v, b, r, k)-configuration has reduced parameter set (d, r, k) with

$$d = \frac{v \gcd(r, k)}{k} = \frac{b \gcd(r, k)}{r},$$

and we say that (d, r, k) is **configurable** if there is a configuration with these parameters.

The set of (r, k)-configurable tuples

Define $S_{(r,k)} = \{d \in \mathbb{Z}_+ : (d, r, k) \text{ is configurable}\}$.

Theorem (Bras-Amorós and S., 2012 (2009))

For every pair of integers $r, k \geq 2$, $S_{(r,k)}$ is closed under addition and has finite complement in the set of non-negative integers.

We say that $S_{(r,k)}$ is a numerical semigroup (think: the stamp problem).

Numerical semigroups

Denote $\mathbb{Z}_+$ the set of non-negative integers. A numerical semigroup is a subset $S \subset \mathbb{Z}_+$, such that

- S is closed under addition,
- $0 \in S$ and
- the complement $(\mathbb{Z}_+) \setminus S$ is finite.

The stamp problem

The stamp problem: given stamps of only two values x and y, what is the largest stamp value that cannot be put on an envelope?

This value is the Frobenius number of the numerical semigroup $\langle x, y\rangle$ generated by x and y.

- The Frobenius number of $\langle 5, 8\rangle$ is $(5-1)(8-1)-1=27$.
- The Frobenius number of $\langle 5, 12\rangle$ is $(5-1)(12-1)-1=43$.
- The Frobenius number of $\langle 8, 12\rangle$ is ∞.

For more stamps there is no closed formula for the Frobenius number.

Numerical semigroups

The *multiplicity* of a numerical semigroup is its smallest non-zero element.

The *conductor* of a numerical semigroup is the smallest element such that all subsequent natural numbers belong to the numerical semigroup (the Frobenius number $+1$).

The *gaps* of a numerical semigroup are the elements in the complement of the numerical semigroup.

Example

$$\langle 3, 7 \rangle = \{0, 3, 6, 7, 9, 10, 12, 13, 14, 15, 16, \dots\}$$

is the numerical semigroup generated by 3 and 7. In this numerical semigroup

- the multiplicity is 3,
- the conductor is 12 and
- the gaps are $\{1, 2, 4, 5, 8, 11\}$.

Lemma

A set of positive integers generate a numerical semigroup if and only if they are coprime.

Assuming $0 \in S_{(r,k)}$, to prove finite complement it is therefore enough to prove:

- $S_{(r,k)}$ is closed under addition,
- at least two elements of $S_{(r,k)}$ are coprime.

The affine plane over the finite field with q elements is a combinatorial configuration with q^2 points and $q^2 + q$ lines with parameters $(d, r, k) = (q, q + 1, q)$.

The lines come in $q + 1$ parallel classes of q lines each.

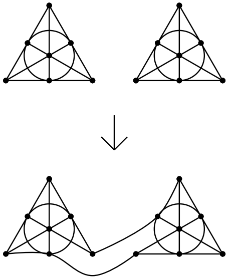
Construct a configuration:

- take the points P on k lines from one parallel class,
- take the lines L of r (other) parallel classes and restrict them P.

This configuration has parameters (kq, rq, r, k) or reduced parameters $(q \gcd(r, k), r, k)$, where $q \geq \max(r, k)$ is a prime power.

Table of Contents

Addition



In terms of points and lines:

$$(\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1) \oplus (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2) = (\mathcal{P}_1 \cup \mathcal{P}_2, \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{I})$$

In terms of reduced parameters (d, r, k):

$$d, d' \in S_{(r,k)} \Rightarrow d + d' \in S_{(r,k)}$$

.

Theorem (S. and Bras-Amorós, 2013)

Let $S_{(r,k)} = \{d \in \mathbb{Z}_+ : (d,r,k) \text{ is configurable}\}$.

If $d_1, d_2 \in S_{(r,k)}$ *then* $d_1 + d_2 - n \in S_{(r,k)}$ *for all* $n \in \{1, \ldots, \gcd(r,k)\}$.

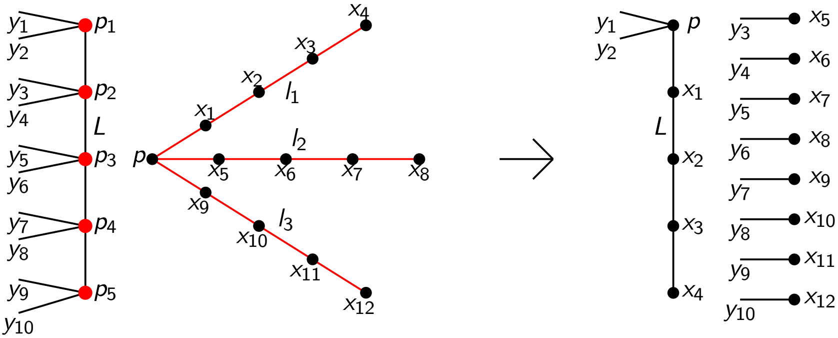
Construction.

- Take two (r,k)-configurations A and B with v_A and v_B points and b_A and b_B lines, respectively.
- Remove $a := nk/\gcd(r,k)$ points and $b := nr/\gcd(r,k)$ lines and match missing incidences.
- Obtain an (r,k)-configuration with $v = v_A + v_B - a$ points and $b = b_A + b_B - b$ lines. It has parameter $d = d_A + d_B - n$.

Example

An example of this construction for $(r, k) = (3, 5)$.

In this case $\gcd(r, k) = 1$, so the only possible choice of n is $n = 1$.



The red points and lines in the two combinatorial configurations on the left are removed and the resulting configuration is shown on the right.

Bound 1

Theorem (S. and Bras-Amorós, 2013)

The conductor c of a numerical semigroup $S_{(r,k)}$ associated to the (r,k)-configurations is bounded by

$$c \leq (x+1)m - x\,\mathrm{gcd}(r,k)$$

where m is the multiplicity of $S_{(r,k)}$ and $x = \left\lfloor \frac{m-2}{\mathrm{gcd}(r,k)} \right\rfloor$.

Bound 2

The construction from affine planes gave an (r,k)-configuration with $d = q\gcd(r,k)$ for any prime power $q \geq \max(r,k)$.

Lemma

If $\gcd(r,k) = 1$, then any prime power $q \geq \max(r,k)$ belongs to $S(r,k)$.

We combine this with the additive structure of the numerical semigroup and get:

Theorem (S. and Bras-Amorós, 2013)

The conductor c of a numerical semigroup $S_{(r,k)}$ associated to the (r,k)-configurations is bounded by

$$c \leq 2 \prod_{p \text{ prime},\ p<\max(r,k)} \left(\lfloor \log_p(\max(r,k)-1)\rfloor + 1\right).$$

Flowers bound

Theorem (Flowers, 2015)

The conductor c of a numerical semigroup $S_{(r,k)}$ associated to the (r,k)-configurations is bounded by $c = c(r,k) < k^2 \max(r+1, r/2+k)$ for all $r \geq k$.

We will call this bound $F(r,k)$.

How good are these bounds?

Projective planes: When they exist, finite projective planes of order $r-1$ are the smallest (r,r)-configurations. Therefore $P(r) = r^2 - r + 1$ is a lower bound for the smallest (r,r)-configuration.

Golomb rulers: A Golomb ruler of order r is an ordered set of r integers $a_1, a_2, \ldots, a_r$ such that $0 \leq a_1 < a_2 < \cdots < a_r$ and all the differences $\{a_i - a_j : 1 \leq j < i \leq r\}$ are distinct. The length $L_G(r)$ of the ruler $G(r)$ is equal to $a_r - a_1$.

Every Golomb ruler defines a cyclic configuration (see Gropp 1990). Therefore $G(r) = 2L_G(r) + 1$ is an upper bound for the conductor of $S_{(r,r)}$, where L_G is the shortest known Golomb ruler of order r.

r	$P(r)$	$G(r)$	Bound 1	$F(r,r)$
3	7	7	21	40
4	13	13	52	96
5	21	23	105	187
6	31	35	258	324
7	43	48	301	514
8	57	63	456	768
9	73	80	857	1093

Bounds for the conductor $c(r,k)$ with $\gcd(r,k)=1$.

r	k	Bound 2	Bound 1	$F(r,k)$
3	4	8	10	45
3	5	12	17	54
3	7	24	37	72
3	8	48	50	81
3	10	96	101	99
3	11	96	101	108
4	5	12	17	104
4	7	24	37	128
4	9	64	65	160
4	11	96	101	192
5	6	24	37	200
5	7	24	37	212
5	8	48	50	225
5	9	64	65	250
5	11	96	101	300
5	12	192	145	325

Theorem (S. and Bras-Amorós, 2013)

For any $n = rk/\gcd(r,k)$ *numbers* $d_1, \dots, d_n \in S_{(r,k)}$ *the number* $d_1 + \dots + d_n + 1$ *also belongs to* $S_{(r,k)}$.

Proof.

- Take n combinatorial configurations $C_1, \dots, C_n$ with reduced parameter sets $(d_1, r, k), \dots, (d_n, r, k)$.
- On each configuration C_i, remove one point-line incidence (p_i, l_i).
- Instead let the n lines l_i all meet in sets of r in $k/\gcd(r,k)$ new points $p'_1, \dots, p'_{k/\gcd(r,k)}$, and join the n points in sets of k over $r/\gcd(r,k)$ new lines $l'_1, \dots, l'_{r/\gcd(r,k)}$.
- The configurations C_i have $v_i = d_i k/\gcd(r,k)$ points and $b_i = d_i r/\gcd(r,k)$ lines, the new configuration has parameters $(v_1 + \dots + v_n + k/\gcd(r,k), b_1 + \dots + b_n + r/\gcd(r,k), r, k)$, i.e. reduced parameters $(d_1 + \dots + d_n + 1, r, k)$.

□

New constructions give better bounds on the number $c(r, k)$ such that $d \in S_{(r,k)}$ for all $d \geq c(r, k)$.

But determining the set $S_{(r,k)}$ exactly is a difficult problem.

Table of Contents

Extending balanced configurations [Martinetti, 1886]

Given a $(v, 3, 3)$-configuration, add a point and a line to construct a $(v + 1, 3, 3)$-configuration.

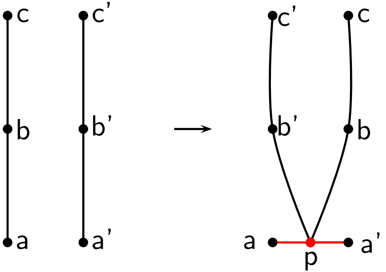
How?

Assume that there are two parallel lines $\{a, b, c\}$ and $\{a', b', c'\}$, with a and a' noncollinear.

Add a point p and replace the two parallel lines with the lines $\{p, b, c\}$, $\{p, b', c'\}$, $\{p, a, a'\}$.

The result is a $(v + 1, 3, 3)$-configuration.

The Martinetti extension



Reduction of configurations I [Martinetti, 1886]

A configuration is called irreducible if it cannot be constructed from a smaller configuration using the extension construction.

Theorem. [Martinetti- Boben]
The irreducible configurations à la Martinetti are:

- Cyclic configurations with base line $\{0, 1, 3\}$ (starting with the Fano plane).
- Three infinite families $T_1(n)$, $T_2(n)$, $T_3(n)$, on $10n$ points. The smallest configuration in $T_1(n)$ is the Desargues' configuration.
- The Pappus' configuration.

Reduction of configurations II [Carstens et al., 2001]

Given a $(v, 3, 3)$-configuration, remove a point and a line to construct a $(v-1, 3, 3)$-configuration.

How?

A complicated family of several Martinetti-like reductions defined on the incidence graph.

Their goal was to show that the only irreducible configuration was the Fano plane.

Unfortunately, in 2005, Ravnik used a computer to show that they failed to reduce at least the Desargues' configuration.

Reduction of configurations III [Boben, 2005]

In the incidence graph of a $(v, 3, 3)$-configuration, remove a point-vertex p and a line-vertex ℓ and connect their neighbors so that the result is an incidence graph of a $(v - 1, 3, 3)$-configuration.

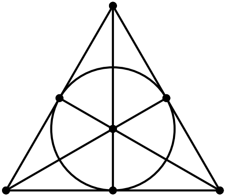
The incidence graph of a $(v, 3, 3)$-configuration is a bipartite cubic graph of girth at least 6: the construction removes two vertices and preserves other properties.

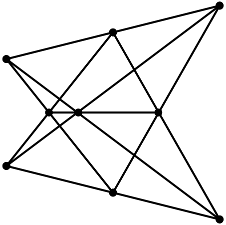
Martinetti's reduction is a special case of this reduction.

Irreducible configurations

Theorem. [Boben (2005)] Boben's irreducible configurations are:

- The Fano plane.
- The Pappus' configuration.





Extending balanced configurations with $r = k \geq 3$

Theorem. (S. 2015)

Assume there is

- a (balanced) (d, k, k)-configuration (P, L, I) with k points $Q \subseteq P$ and k lines $M \subseteq L$, and
- a bijection $f : Q \to M$ defined as follows:
  - the image of a point $q \in Q$ is a line $f(q) \in M$ through that point,
  - two points $q, q' \in Q$ can be collinear only on the line $f(q)$ or $f(q')$,
  - two lines $m, m' \in M$ can meet only in the point $f^{-1}(m)$ or $f^{-1}(m')$.

Then there is a $(d + 1, k, k)$-configuration constructed from C through an extension procedure.

Extending balanced configurations with $r = k \geq 3$

Theorem. (S. 2015)
Assume there is

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  - two lines $m, m' \in M$ can meet only in the point $f^{-1}(m)$ or $f^{-1}(m')$.

Then there is a $(d + 1, k, k)$-configuration constructed from C through an extension procedure.

- Disconnect all incidences $(q, f(q)) \in I$.
- Add a new line ℓ and the incidences (q, ℓ) for all points $q \in Q$.
- Add a new point p and the incidences (p, m) for all lines $m \in M$.

The result is a configuration with parameters $(d + 1, k, k)$.

Example: Extending $(v, 3, 3,)$-configurations

Lemma. Every $(v, 3, 3)$-configuration admits an extension.

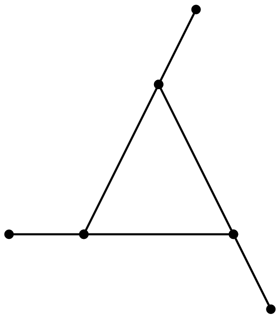
Proof.

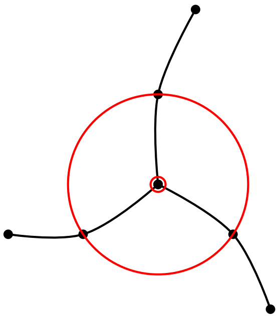
- If the configuration contains a triangle, take Q and M the three points and the three lines in the triangle.
- If the configuration contains no triangle, there are still three points a, b, c such that (a, b) and (b, c) are collinear on the two lines A, B, and a third line C through a not meeting A nor B. □

This implies the following well-known result.

Corollary. There is a $(v, 3, 3)$ configuration whenever the parameters are admissible.

Extending $(v, v, 3, 3)$-configurations





Deficiency of a configuration

The **distance** between two points is the number of lines in a “shortest path” between them.

In a (d, r, k)-configuration all points have the same number $r(k-1)$ of points at distance 1.

The **deficiency** of a configuration is the number of points at distance at least 2 from a given point.

Extending $(v, 4, 4)$-configurations

Lemma. A $(v, 4, 4)$-configuration admits an extension if and only if it has deficiency at least 1.

Proof.

- If deficiency is 0 then it is the finite projective plane of order 3, which is not extendable.
- If deficiency is ≥ 1 then there are always points a, b, c, d such that the pairs (a, b), (b, c), (c, d) are collinear on the lines A, B, C, and the pairs (a, c), (b, d) are at distance at least two. The forth line D can be taken as the line through a and d if there is such a choice of points and lines. Otherwise D can be taken through d such that it does not meet A, B, C. □

There are $(v, 4, 4)$-configuration with deficiency 0 and 1, so we get the following well-known result.

Corollary. There is a $(v, 4, 4)$-configuration whenever the parameters are admissible.

Extending configurations with $r, k \geq 3$

Theorem. Let $t = rk/\gcd(r,k)$. Assume there is

- a (d,r,k)-configuration (P,L,I) with t points $Q \subseteq P$ and t lines $M \subseteq L$, and
- a bijection $f : Q \to M$ defined as follows:
  - the image of a point $q \in Q$ is a line $f(q) \in M$ through that point,
  - $Q = \bigcup_{i=1}^{r/\gcd(r,k)} Q_i$ such that $|Q_i| = k$, $Q_i \cap Q_j = \emptyset$, and two points $q_i, q_i' \in Q_i$ can be collinear only on the line $f(q_i)$ or $f(q_i')$,
  - $M = \bigcup_{i=1}^{k/\gcd(r,k)} M_i$ such that $|M_i| = r$, $M_i \cap M_j = \emptyset$, and two lines $m_i, m_i' \in M_i$ can meet only in the point $f^{-1}(m_i)$ or $f^{-1}(m_i')$.

Then there is a $(d+1,r,k)$-configuration constructed from C through an extension procedure.

Extending configurations with $r, k \geq 3$

Proof.

- Disconnect all incidences $(q, f(q)) \in I$.
- For each Q_i add a new line ℓ_i and the incidences (q_i, ℓ_i) for all points $q_i \in Q_i$.
- For each M_i add a new point p_i and the incidences (p_i, m_i) for all lines $m_i \in M_i$.

The result is a configuration with parameters $(d+1, r, k)$.

$$\begin{array}{lcl} (d, r, k) & \rightarrow & (d+1, r, k) \\ (v, b, r, k) & \rightarrow & (v + k/\gcd(r,k), b + r/\gcd(r,k), r, k) \end{array}$$

□

Reduction of **balanced** configurations with $r = k \geq 3$

A **reduction** of a balanced configuration (P, L, I) is a triple (p, ℓ, f) where

- p is a point,
- ℓ is a line,
- f' is a bijection $f' : Q' \to M'$,
  where
  - $Q' = \{q : q \in \ell \text{ and } q \neq p\}$, and
  - $M' = \{m : p \in m \text{ and } m \neq \ell\}$,

  such that q is not collinear with $s \in f'(q)$ except possibly through ℓ or with p.

Now delete p and l (and their incidences) and add incidences $(q, f'(q))$ for $q \in Q'$.

A balanced configuration is **irreducible** if it does not admit a reduction.

Lemma. The reduction is the inverse operation of the extension.

Reduction of configurations with $r, k \geq 3$

A **reduction** of a configuration (P, L, I) is a triple (R, N, f') where

- R is a set of points,
- N is a set of lines,
- f' is a pairing between the elements of two multisets $f : Q' \to M'$, where
  - $Q' = \{q : q \in P$ and $\exists \ell \in N$ such that $q \in \ell$ and $q \notin R\}$,
  - $M' = \{m : m \in L$ and $\exists p \in R$ such that $p \in m$ and $m \notin N\}$,

  such that q is not collinear with $s \in f(q)$ except possibly through one of the lines in N or with one of the points in R.

Now delete R and N and their incidences and add incidences $(q, f'(q))$ for $q \in Q'$.

A configuration is **irreducible** if it does not admit a reduction.

Lemma. The reduction is the inverse operation of the extension.

Reduction of $(d, 3, 3)$-configurations

In the case $(d, 3, 3)$ this definition has the same implications as Boben's reduction.

Lemma There are only two irreducible $(d, 3, 3)$-configurations:

- The Fano plane,
- The Pappus' configuration.

Proof.

- The Fano plane is the smallest $(d, 3, 3)$-configuration.
- Only the Pappus' configuration has the property that given a point p, every line not through p meets two of the three lines through p. This property obstructs reduction.

□

Large configurations are reducible

Lemma. A (v, b, r, k)-configuration is reducible if
$b \geq 1 + r + r(k-1)(r-1) + r(k-1)^2(r-1)^2$.

However this bound is not sharp.

Table of Contents

Reduced Levi graphs of configurations with symmetries

Leah Berman talked in this seminar series some weeks ago about the reduced Levi graph of a configuration.

Remember: the **incidence graph** or **Levi graph** of the incidence structure (P, L, I) is the bipartite graph with vertex set $P \cup L$ and an edge between the vertices p and b if p is a point on b.

Assuming that the geometric (or combinatorial!) configuration has non-trivial automorphism group:

The voltage graph representing the quotient graph is the reduced Levi graph of the configuration.

Any automorphism of a the configuration is also an automorphism of the Levi graph.

Such automorphisms are type-preserving automorphisms: they send points to points and lines to lines.

Reduced Levi graphs of self-dual configurations

The Levi graph can also have type-permuting automorphisms: they send points to lines and lines to points.

Type-permuting automorphisms of the Levi graph of a configuration of points and lines are called dualities; if they have order two they are called polarities.

The voltage graph representing the quotient graph of the Levi graph under a polarity is also called the reduced Levi graph of the configuration (Artzy 1956).

Reduced Levi graphs as a tool for constructing configurations

As we saw in Berman's talk, reduced Levi graphs prescribing a type-preserving automorphism is THE TOOL for constructing configurations with symmetries.

Similarly, Artzy showed that reduced Levi graphs prescribing a polarity is THE TOOL for constructing self-dual configurations.

Both these constructions can be used to construct geometric configurations.

We introduced a generalisation of reduced Levi graphs for incidence geometries and used it to construct incidence geometries with trialities (Leemans and S. 2019).

Constructing interesting graphs from configurations

As Coxeter noted in his article “Self-dual configurations and regular graphs”, self-dual configurations have often interesting (regular) Levi graphs and Menger graphs.

For example, the Levi graphs of self-dual generalised polygons are examples of cage graphs.

The reduced Levi graphs of self-dual generalised polygons (polarity graphs) have also been used to construct cages, as we could see in Jozef Širáň’s talk a couple of weeks ago.

Some final words

There are, of course, other constructions of combinatorial configurations that never came up in this talk.

Focus has been on certain general constructions of combinatorial configurations proving that **eventually combinatorial configurations exist**.

Every construction of a geometric configuration gives a combinatorial configuration.

Sometimes a geometric construction can be found from the combinatorial construction.

It was recently proved that geometric balanced configurations eventually exist (Berman, Gevay, Pisanski 2021), but what about unbalanced configurations?

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