Complete Regular Dessins and Skew Morphisms of Cyclic Groups

Kan Hu

University of Primorska

Algebraic Graph Theory International Webinar

Overview

- Graph embeddings
- Skew morphisms
- Complete regular dessins
- Skew morphisms of cyclic groups
- Observations and new problems
- References

Graph embeddings

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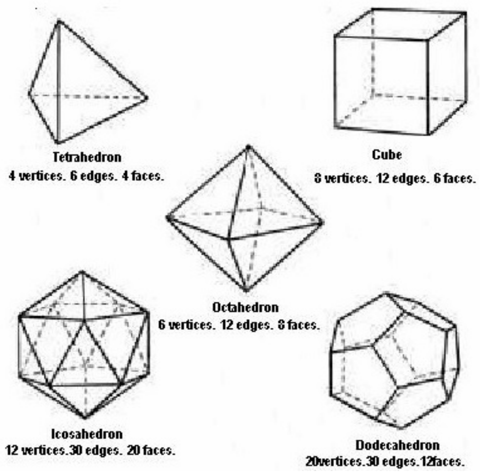
- $\mathrm{Aut}^+(M)$ acts semi-regularly on Ω, and in the case where $\mathrm{Aut}^+(M)$ is transitive, and hence regular on Ω, the map M is called orientably regular.

The Platonic maps

Example (Orienatbly regular maps on the sphere: The Platonic maps)



Regular Cayley maps

Definition

- Let A be a finite group, let $X \subset A$ with the property:

$$1 \notin X, \quad A = \langle X \rangle, \quad \text{and} \quad X^{-1} = X,$$

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- If G is also regular on the arcs, then M is called an (orientably) regular Cayley map.

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If M is a regular Cayley map, then

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- $G := \mathrm{Aut}(M)$ is regular on the arcs of M, so the stabilizer G_v of a vertex v is transitive on its neighborhood.
- G has a cyclic complementary factorization

$$G = AG_v, \quad \text{where } A \cap G_v = 1,\ G_v \text{ is cyclic.}$$

Platonic maps revisited

Example

M	$\mathrm{Aut}^+(M)$	Is Cayley?	Cayley subgroup
Tetrahedron	$\mathbf{A}_4$	Yes	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
Cube	$\mathbf{S}_4$	Yes	$\mathbf{D}_8$
Octahedron	$\mathbf{S}_4$	Yes	$\mathbf{D}_6$
Icosahedron	$\mathbf{A}_5$	Yes	$\mathbf{A}_4$
Dodecahedron	$\mathbf{A}_5$	No	N/A

Dessins d'enfants

Theorem (Belyĭ, 1979)

A compact Riemann surface $\mathbb{S}$, regarded as an algebraic curve, can be defined over the field $\bar{\mathbb{Q}}$ of algebraic numbers iff there exists a non-constant meromorphic function $\beta : \mathbb{S} \to \Sigma$, with at most three critical values, say $\{0, 1, \infty\}$, where Σ is the Riemann sphere.

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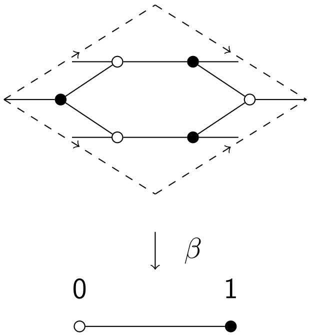
- the embedded graph is the preimage $\beta^{-1}[0, 1]$ of the closed interval $[0, 1]$ with black vertices $\beta^{-1}(0)$ and white vertices $\beta^{-1}(1)$,
- the faces are the components of $\mathbb{S} \backslash \beta^{-1}[0, 1]$.

A Belyi function

Example

$\beta : \mathbb{S} \to \Sigma, (x, y) \mapsto x^3$ is a Belyi function, where $\mathbb{S}$ is the Fermat curve defined by the polynomial $x^3 + y^3 = 1$. Topologically $\mathbb{S}$ is a torus, which may be obtained by identifying the opposite sides of the rhombus, as shown below:



$K_{3,3}$ embedded in $\mathbb{S}$

K_2 embedded in Σ

Grothendieck's observation

Galois actions: The absolute Galois group $\mathbf{G} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the Galois group of the separable closure $\bar{\mathbb{Q}}$ over $\mathbb{Q}$, has a natural action on the algebraic curves $\mathbb{S}$ and Belyĭ functions β. This induces an action of $\mathbf{G}$ on (isomorphism classes of) the bipartite maps on $\mathbb{S}$.

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Theorem (González-Diez, Jaikin-Zapirain, 2015)

$\mathbf{G}$ *acts faithfully on regular dessins.*

G. González-Diez, A. Jaikin-Zapirain, The absolute Galois group acts faithfully on regular dessins and on Beauville surfaces, Proc. London Math. Soc., 2015, 111(4): 775–796.

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- A regular dessin is reflexible if it is isomorphic to its mirror image. Otherwise, it is chiral.

Skew morphisms

Definition

A skew morphism of a finite group A is a permutation $\varphi \in \mathrm{Sym}(A)$ such that $\varphi(1) = 1$ and

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for some integer function $\pi : A \to \mathbb{Z}$, called the power function of φ.

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- A skew morphism of A is proper if it is not an automorphism.

Cayley skew morphisms

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Example

A proper Cayley skew morphism of the cyclic additive group $(\mathbf{Z}_9, +)$:

$$\varphi = (0)(1, 2, 7, 5, 4, 8)(3, 6);$$
$$\pi = [1]\ [3, 5, 3, 5, 3, 5]\ [1, 1].$$

Regular Cayley maps and skew morphisms

Theorem (Jajcay and Širáň, 2002)

A Cayley map $CM(A, X, P)$ is regular iff P extends to a Cayley skew morphism φ of A.

R. Jajcay and J. Širáň, Skew morphisms of regular Cayley maps, **Discrete Math.**, 224 (2002) 167–179.

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- For all $a, b \in A$,

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Remark

The subgroup $\langle\varphi\rangle$ is core-free in G.

I. Kovács and R. Nedela, Decomposition of skew morphisms of cyclic groups, **Ars Math. Contemp.**, 4 (2011) 329–249.

Cyclic complementary factorizations

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- Thus, $\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b)$, and so φ is a skew morphism of A, and π is the associated power function.

M. Conder, R. Jajcay and T. Tucker, Cyclic complements and skew morphisms of groups, **J. Algebra**, 453 (2016) 68–100.

Bicyclic groups

Theorem (Itô, 1956; Douglas, 1961)

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Theorem (Janko, 2008)

A non-metacyclic 2-group G has a factorization $G = \langle a \rangle \langle b \rangle$ iff $d(G) = 2$ and G contains a unique non-metacyclic maximal subgroup.

More general factorizations

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Theorem (Wielandt, 1958; Kegal, 1961)

If a finite group G admits a factorization $G = AB$ of two nilpotent subgroups A and B, then G is solvable.

Complete regular dessins

Problem (A)

Classify regular dessins with underling graphs $K_{m,n}$ (=Classify orientably edge-transitive embeddings of $K_{m,n}$).

Correspondence

Theorem (Jones, Nedela and Škoviera, 2007)

The color- and orientation-preserving automorphism group $H := \mathrm{Aut}_0^+(M)$ *of a regular dessin* M *with underlying graph* $K_{m,n}$ *has a factorization*

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- $M = (H; a, b)$ is symmetric $\iff \exists \tau \in \mathrm{Aut}(H) : a \mapsto b, b \mapsto a$.
- $M = (H; a, b)$ is circular $\iff \langle a \rangle \cap \langle ab \rangle = \langle b \rangle \cap \langle ab \rangle = 1$.

G.A. Jones, R. Nedela and M. Škoviera, Complete bipartite graphs with a unique regular embedding, **J. Combin. Theory Ser. B**, 98 (2) (2008) 241–248

The standard embedding

Example

Let $H = \langle a, b \mid a^m = b^n = [a, b] = 1 \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_n$.

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Let $H = \langle a, b \mid a^m = b^n = [a, b] = 1 \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_n$. Then

- $M = (H; a, b)$ and $M^* := (H; b, a)$ form a reciprocal pair of regular dessins with underlying graph $K_{m,n}$.

The standard embedding

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- $M = (H; a, b)$ and $M^* := (H; b, a)$ form a reciprocal pair of regular dessins with underlying graph $K_{m,n}$.
- If $m = n$, then $M \cong M^*$, so it is symmetric, corresponding to an orientably regular embedding of $K_{n,n}$.

Uniqueness

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Every group of order n is cyclic iff $\gcd(n, \phi(n)) = 1$.

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$K_{n,n}$ *underlies a unique orientably regular embedding iff* $\gcd(n, \phi(n)) = 1$.

G.A. Jones, R. Nedela and M. Škoviera, Complete bipartite graphs with a unique regular embedding, **J. Combin. Theory Ser. B**, 98 (2) (2008) 241–248

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Corollary

$K_{m,n}$ underlies a unique orientably edge-transitive embedding iff $\gcd(n, \phi(m)) = \gcd(m, \phi(n)) = 1$.

W. Fan and C.H. Li, The complete bipartite graphs with a unique edge-transitive embedding, **J. Graph Theory**, 87(2018) 581–586.

Progress

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- A complete classification of orientably regular embeddings of $K_{n,n}$ [Jones, 2010].

More progress

- Unique orientably edge-transitive embedding of $K_{m,n}$ [Fan and Li, 2018].

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- Regular dessins with underlying graphs K_{p^e,p^f}, p odd prime [Hu, Nedela and Wang, 2019].
- Regular dessins with underlying graphs $K_{2^e,2^f}$: Metacyclic case [Cai and Hu, 2023+].

Orientably edge-transitive embeddings of K_{p^e,p^f}

Theorem (Hu, Nedela and Wang, 2019)

Let p be an odd prime, and $1 \le f \le e$, then the number of orientably edge-transitive embeddings M of K_{p^e,p^f} is

$$\mu(p^e, p^f) = \begin{cases} \frac{1}{2}p^{e-1}(1+p^{e-1}) & 1 \le f = e, \\ p^{2(f-1)} & 1 \le f < e. \end{cases}$$

(Low Exponent Advantage)

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Moreover, $\mathrm{Aut}_0^+(M)$ is isomorphic to one of the following groups:

- $\mathbf{M}_1(e,f,r) = \langle a, b \mid a^{p^e} = b^{p^f} = 1, a^b = a^{1+p^r} \rangle$, $1 \leq f \leq e$ *and* $1 \leq r \leq e \leq f + r$.
- $\mathbf{M}_2(e,f,r) = \langle a, b \mid a^{p^f} = b^{p^e} = 1, a^b = a^{1+p^r} \rangle$, $1 \leq r < f < e$.
- $\mathbf{M}_3(e,f,h,r) = \langle a, b \mid a^{p^h} = 1, b^{p^{e+f-h}} = a^{p^f}, a^b = a^{1+p^r} \rangle$, *where* $h - f \leq r < f < h < e$.

Orientably edge-transitive embeddings of $K_{2^e,2^f}$

Theorem (Cai and Hu, 2023+)

Let $1 \le f \le e$. Up to isomorphism, the number $\mu'(2^e,2^f)$ of orientably edge-transitive embeddings M of $K_{2^e,2^f}$ with metacyclic automorphism groups is given by the following formula:

$$\mu'(2^e,2^f) = \begin{cases} 1 & \text{if } 1 = f = e, \\ 4 & \text{if } 1 = f < e, \\ 14 & \text{if } 2 = f < e, \\ 2^{e-3}(1+2^{e-1}+2^{e-2}) & \text{if } 2 \le f = e, \\ 3 \cdot 2^{2f-3} & \text{if } 2 < f = e-1, \\ 3 \cdot 2^{2f-2} & \text{if } 2 < f < e-1. \end{cases}$$

X. Cai and K. Hu, Regular dessins with underlying graphs $K_{2^e,2^f}$: The metacyclic case, preprint, 2023.

Cont.

Moreover, if the group $\mathrm{Aut}_0^+(M)$ is abelian, or contains a cyclic maximal subgroup, then it is isomorphic to one of the following groups:

- $\mathbf{A}(e,f) = \langle a, b \mid a^{2^e} = b^{2^f} = [a,b] = 1 \rangle$, where $e \geq f \geq 0$.
- $\mathbf{D}_{2^{e+1}} = \langle a, b \mid a^{2^e} = b^2 = 1, a^b = a^{-1} \rangle$, where $e \geq 2$.
- $\mathbf{SD}_{2^{e+1}} = \langle a, b \mid a^{2^e} = b^2 = 1, a^b = a^{-1+2^{e-1}} \rangle$, where $e \geq 3$.
- $\mathbf{M}_{2^{e+1}} = \langle a, b \mid a^{2^e} = b^2 = 1, a^b = a^{1+2^{e-1}} \rangle$, where $e \geq 3$.

Cont.

On the other hand, if the group $\mathrm{Aut}_0^+(M)$ is non-abelian and contains no cyclic maximal subgroup, then it is isomorphic to one of the following groups:

- $\mathbf{E}^+(e,f,\ell) = \langle a,b \mid a^{2^e} = b^{2^f} = 1, a^b = a^{1+2^\ell}\rangle$, where either $2 \le f \le \ell < e \le f+\ell$ or $2 \le \ell < f \le e \le f+\ell$.
- $\mathbf{F}^+(e,f,\ell) = \langle a,b \mid a^{2^f} = b^{2^e} = 1, a^b = a^{1+2^\ell}\rangle$, where $2 \le \ell < f < e$.
- $\mathbf{G}^+(e,f,h,\ell) = \langle a,b \mid a^{2^h} = 1, b^{2^{e+f-h}} = a^{2^f}, a^b = a^{1+2^\ell}\rangle$, where $\max\{2, h-f\} \le \ell < f < h < e$.
- $\mathbf{E}^-(e,f,\ell) = \langle a,b \mid a^{2^e} = b^{2^f} = 1, a^b = a^{-1+2^\ell}\rangle$, where either $2 \le f \le \ell \le e \le f+\ell$ or $2 \le \ell < f \le e \le f+\ell$.
- $\mathbf{F}^-(e,f,\ell) = \langle a,b \mid a^{2^f} = b^{2^e} = 1, a^b = a^{-1+2^\ell}\rangle$, where $2 \le \ell \le f < e$.
- $\mathbf{G}^-(e,f,\ell) = \langle a,b \mid a^{2^{f+1}} = 1, b^{2^{e-1}} = a^{2^f}, a^b = a^{-1+2^\ell}\rangle$, where $2 \le \ell \le f+1 < e$.

Reciprocal skew morphisms

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$$ab^x = b^{\varphi(x)}a^{\pi(x)} \quad \text{and} \quad ba^y = a^{\varphi'(y)}a^{\pi'(y)}$$

determine a pair of skew morphisms $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\varphi' : \mathbb{Z}_m \to \mathbb{Z}_m$, the associated power functions $\pi : \mathbb{Z}_n \to \mathbb{Z}_{|\varphi|}$ and $\pi' : \mathbb{Z}_m \to \mathbb{Z}_{|\varphi'|}$ such that

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- the orders $|\varphi|\big|m$ and $|\varphi'|\big|n$;
- $\pi(x) = -\varphi'^{-x}(-1)$ and $\pi'(y) = -\varphi^{-y}(-1)$.

Such a pair (φ, φ') of skew-morphisms are called reciprocal.

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Remark

If $G = \langle a\rangle\langle b\rangle$ has an automorphism $\tau : a \mapsto b, b \mapsto a$, then $m = n$ and $\varphi = \varphi'$, so

$$|\varphi| \text{ divides } n \quad \text{and} \quad \pi(x) = -\varphi^{-x}(-1), \quad \text{for all } x \in \mathbb{Z}_n.$$

In this case, φ is a symmetric skew morphism of $\mathbb{Z}_n$.

Second correspondence

Theorem (Feng, Hu, Nedela, Škoviera, Wang, 2019)

The isomorphism classes of orientably edge-transitive embeddings of $K_{m,n}$ *are in 1-1 correspondence with the reciprocal pairs* (φ, φ') *of skew-morphisms* $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ *and* $\varphi' : \mathbb{Z}_m \to \mathbb{Z}_m$.

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Corollary

The isomorphism classes of orientably regular embeddings of $K_{n,n}$ are in 1-1 correspondence with the symmetric skew morphisms of $\mathbb{Z}_n$.

Y. Feng, K. Hu, R. Nedela, M. Škoviera and N.-E Wang, Complete regular dessins and skew morphisms of cyclic groups, **Ars Math. Contemp.** 16 (2019) 527–547.

Skew morphisms of cyclic groups

Problem (B)

Classify skew morphisms for the cyclic groups $\mathbb{Z}_n$.

A decomposition theorem

Theorem (Kovács and Nedela, 2011)

If $n = n_1 n_2$, *where*

$$\gcd(n_1, n_2) = \gcd(n_1, \phi(n_2)) = \gcd(\phi(n_1), n_2) = 1,$$

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then every skew morphism φ *of the cyclic group* $\mathbb{Z}_n$ *has a decomposition* $\varphi = \varphi_1 \times \varphi_2$*, where* φ_i *is a skew morphism of* $\mathbb{Z}_{n_i}$ $(i = 1, 2)$*.*

I. Kovács and R. Nedela, Decomposition of skew morphisms of cyclic groups, **Ars Math. Contemp.**, 4 (2011) 329–249.

Smooth skew morphisms

Definition

A skew morphism φ of A is smooth if $\pi(\varphi(x)) = \pi(x)$ for all $x \in A$. It is clear that the concept of smooth skew morphism generalizes naturally group automorphisms.

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Example

A smooth (non-Cayley) skew-morphism of the cyclic group $\mathbf{Z}_{20}$:

$$\varphi = (0)(4)(8)(12)(16)(1,5,9,13,17)(2,14,6,18,10)(3,11,19,7,15);$$
$$\pi = [1]\ [1]\ [1]\ [1]\ [1]\ [2,\ 2,\ 2,\ 2,\ 2]\ [4,\ 4,\ 4,\ 4,\ 4]\ [3,\ 3,\ 3,\ 3,\ 3].$$

A survey

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- Cyclic groups underlying only smooth skew morphisms [Hu, Kwon and Kovács, 2023+].

Automorphisms and smooth skew morphisms

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Every skew morphism of the cyclic group $\mathbb{Z}_n$ *is an automorphism iff* $n = 4$ *or* $\gcd(n, \phi(n)) = 1$.

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Theorem (Hu, Kovács and Kwon, 2023+)

Every skew morphism of the cyclic group $\mathbb{Z}_n$ is smooth iff $n = 2^e n_1$, where $0 \leq e \leq 4$ and n_1 is a square-free odd number.

Cont.

Theorem (Conder, Jajcay and Tucker, 2016)

Every skew morphism of a noncyclic abelian group A is an automorphism iff it is an elementary abelian 2-group.

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Every skew morphism of a noncyclic abelian group A is an automorphism iff it is an elementary abelian 2-group.

Corollary (Hu, Kovács and Kwon, 2023+)

Let A be a non-cyclic abelian group of order $|A| = 2^f n_1$, where $f \geq 0$ and n_1 is odd. If every skew morphism of A is smooth, then n_1 is square-free, and the (unique) Sylow 2-subgroup of A contains no direct factors $\mathbb{Z}_{2^e}$ for any $e \geq 5$,

Further problems

Problem

(1) *Classify regular dessins M with underlying graphs $K_{2^e,2^f}$, where $\mathrm{Aut}_0^+(M)$ is non-metacyclic.*

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Conjecture

Every skew morphism of a noncyclic abelian group A is smooth iff $A \cong \prod_{i=1}^{r} \mathbb{Z}_{2^{e_i}}$, where $r \geq 2$ and $0 \leq e_i \leq 4$ for all $1 \leq i \leq r$.

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Thank You Very Much!