On certain edge-transitive bicirculants

István Kovács

University of Primorska, Slovenia

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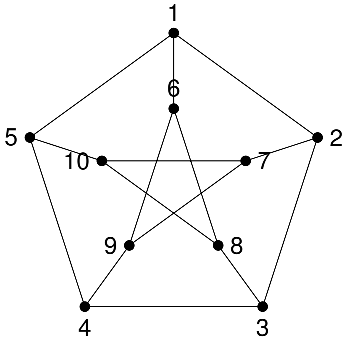
We consider a class of graphs, whose definition captures two properties of the Petersen graph.



The permutation (1 2 3 4 5)(6 7 8 9 10) is an automorphism (a rotational symmetry), and the subgraph induced by any of its orbits is a cycle.

Definition (R. Jajcay, P. Šparl, Š. Miklavič, G. Vasiljević, 2019)

Let $\mathcal{C}$ denote the class of regular graphs Γ of valency at least 3 such that

- Γ admits an automorphism with two orbits of the same size (in other word, Γ is a **bicirculant**),
- at least one of the subgraphs induced by these orbits is a cycle.

- R. Jajcay, P. Šparl, Š. Miklavič, G. Vasiljević, On certain edge-transitive bicirculants, Electron. J. Combin. 26 (2) (2019) #P2.6.

In this paper, the authors studied the **edge-transitive** graphs in class $\mathcal{C}$ of valency at least 6. The graphs of valency at most 5 have been studied earlier by different people under different names.

The trivalent graphs in $\mathcal{C}$ are the **generalised Petersen graphs** (M. Watkins, 1969).

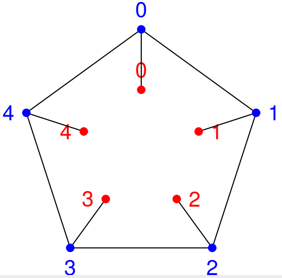
A generalised Petersen graph is determined by two parameters **n**, **r**, and denoted by $\mathbf{GP}(\mathbf{n}, \mathbf{r})$.

n is the size of the orbits of the distinguished automorphism.

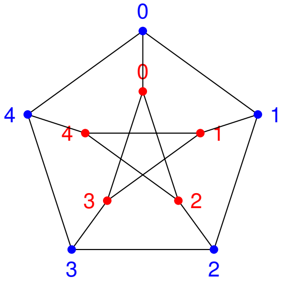
The vertices in each of the two orbits are labelled by $\mathbb{Z}_n$ so that we distinguish the orbits by colours (blue and red vertices).

The distinguished automorphism acts as the translation $x \mapsto x + 1$ ($x \in \mathbb{Z}_n$).

There are edges $\{0, 1\}$ and $\{0, 0\}$.



r determines the red neighbours of the red 0, i.e., we have the edges $\{0, r\}$ and $\{0, -r\}$.



The Petersen graph is $GP(5,2) = GP(5,3)$.

Theorem (R. Frucht, J.E. Graver, M.E. Watkins, 1972)

The class $\mathcal{C}$ contains seven edge-transitive trivalent graphs:

$GP(4, 1) =$ 3-cube,

$GP(5, 2) =$Petersen graph,

$GP(8, 3) =$Möbius–Kantor graph,

$GP(10, 2) =$ dodecahedron graph,

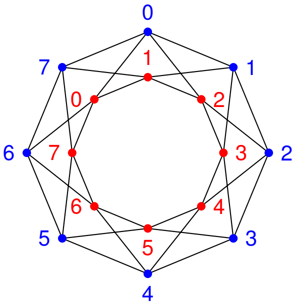
$GP(10, 3) =$ Desargues graph,

$GP(12, 5) =$ Nauru graph,

$GP(24, 5)$.

The tetravalent graphs in $\mathcal{C}$ are the **Rose Window graphs** (S. Wilson, 2008).

A Rose Window graph is determined by three parameters **n**, **a**, **r**, and denoted by $\mathbf{R_n}(\mathbf{a}, \mathbf{r})$.



The wreath graph $W(8, 2)$ is $R_8(2, 1)$.

Theorem (I. K., K. Kutnar, D. Marušič, 2011)

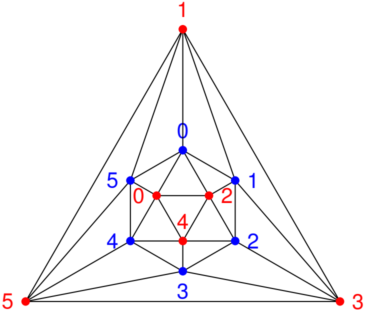
The class $\mathcal{C}$ contains four infinite families of tetravalent edge-transitive graphs:

a. $R_n(2,1)$, $n \geq 3$ (wreath graphs).
b. $R_{2m}(m-2, m-1)$, $m \geq 2$.
c. $R_{12m}(3m+2, 3m-1)$ and $R_{12m}(3m-2, 3m+1)$, $m \geq 2$.
d. $R_{2m}(2b, r)$, where $b^2 = \pm 1 \pmod{m}$, $2 \leq 2b \leq m$, and $r \in \{1, m-1\}$ is odd.

The edge-transitivity of the above graphs was proved by S. Wilson, and he also conjectured that these are all.

The pentavalent graphs in $\mathcal{C}$ are the **Tabačjn graphs** (A. Arroyo, I. Hubard, K. Kutnar, E. O'Reilly, P. Šparl, 2015).

A Tabačjn graph is determined by four parameters **n**, **a**, **b**, **r**, and denoted by $\mathbf{T}(\mathbf{n};\mathbf{a},\mathbf{b};\mathbf{r})$.



The icosahedron graph is $T(6;1,2;2)$.

Theorem (A. Arroyo, I. Hubard, K. Kutnar, E. O'Reilly, P. Šparl, 2015)

The class $\mathcal{C}$ contains three pentavalent edge-transitive graphs:

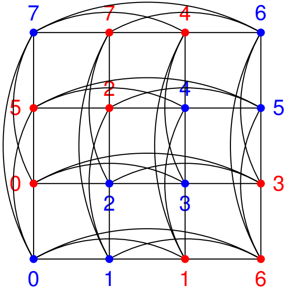
$T(3;1,2;1) = K_6$,

$T(6;2,4;1) = K_{6,6} - 6K_2$,

$T(6;1,2;2) =$ icosahedron graph.

The hexavalent graphs in $\mathcal{C}$ are the **Nest graphs** (R. Jajcay., Š. Miklavič, P. Šparl, G. Vasiljević, 2019).

A Nest graph is determined by five parameters **n**, **a**, **b**, **c**, **r**, and denoted by $\mathcal{N}(\mathbf{n}; \mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{r})$.



The lattice graph $L_2(4)$ is $\mathcal{N}(8; 1, 5, 6; 5)$.

Theorem (R. Jajcay., Š. Miklavič, P. Šparl, G. Vasiljević, 2019)

The class $\mathcal{C}$ contains three families of hexavalent edge-transitive graphs of girth 3:

a. $\mathcal{N}(4;1,2,3;1)$, $\mathcal{N}(5;1,2,3;2)$, $\mathcal{N}(8;1,2,5;3)$,

   $\mathcal{N}(8;1,3,4;3)$, $\mathcal{N}(10;1,3,4;3)$, $\mathcal{N}(12;1,3,10;5)$.

b. $\mathcal{N}(n;1,2m+1,2m+2;1)$, $m \geq 1$ and n is an even divisor of $2(m^2+m+1)$ with $n \geq 4m+2$.

c. $\mathcal{N}(2m;1,b,b+m+1;m-1)$, $b = 4b_0 - 1$, $b_0 > 1$ and m is a divisor of b^2+3 with $m \equiv 2 \pmod 4$ and $b < 2m$.

Problem (R. Jajcay., Š. Miklavič, P. Šparl, G. Vasiljević, 2019)

Classify the edge-transitive Nest graphs.

An exhaustive computer search for edge-transitive graphs in $\mathcal{C}$ of valency d and order n has been carried out under the conditions:

- $d = 6$, $n \leq 220$,
- $7 \leq d \leq 10$, $n \leq 100$.

This has resulted in 66 hexavalent graphs and none of valency larger than 6.

Questions (R. Jajcay., Š. Miklavič, P. Šparl, G. Vasiljević, 2019)

1. For which integer d, $d > 6$ does the class $\mathcal{C}$ contain at least one edge-transitive graph of valency d?
2. For which integer d, $d > 6$ does the class $\mathcal{C}$ contain infinitely many edge-transitive graphs of valency d?

Among the 66 edge-transitive hexavalent graphs, there is only one graph of twice odd order, the complement of the Petersen graph.

This motivated us to consider the edge-transitive graphs in $\mathcal{C}$ whose order is twice an odd number (no restriction on the valency).

Theorem (I. K., J. Ruff, 2022)

The class $\mathcal{C}$ contains the following edge-transitive graphs of twice odd order:

K_6, the Petersen graph and its complement, and the wreath graphs $W(2n+1,2)$.

The first step in our proof is to consider the graphs with a primitive group. Notice that the edge-transitive graphs in $\mathcal{C}$ are also vertex-transitive.

Theorem (P. Müller, 2013)

Let G be a primitive permutation group of degree $2n$ containing an element with two orbits of the same size. Then one of the following holds.

1. (**Affine action**) $\mathbb{Z}_2^m \lhd G \leq AGL(m,2)$ is an affine permutation group, where $n = 2^{m-1}$, and $n = 2, 4$ or 8.
2. (**Almost simple action**) G is an almost simple group and one of the following holds.
   a. $n \geq 3$, $\mathrm{soc}(G) = A_{2n}$, and $A_{2n} \leq G \leq S_{2n}$ in its natural action.
   b. $n = 5$, $\mathrm{soc}(G) = A_5$, and $A_5 \leq G \leq S_5$ in its action on the set of 2-subsets of $\{1,2,3,4,5\}$.
   c. $n = (q^d - 1)/2(q-1)$, $\mathrm{soc}(G) = PSL_d(q)$, and $PSL_d(q) \leq G \leq P\Gamma L_d(q)$ for some odd prime power q and even number $d \geq 2$ such that $(d,q) \neq (2,3)$.
   d. $n = 6$ and $\mathrm{soc}(G) = G = M_{12}$.
   e. $n = 11$, $\mathrm{soc}(G) = M_{22}$, and $M_{22} \leq G \leq \mathrm{Aut}(M_{22})$.
   f. $n = 12$ and $\mathrm{soc}(G) = G = M_{24}$.

Corollary

Let Γ be a G-edge-transitive graph in $\mathcal{C}$ such that G is primitive on $\mathrm{V}(\Gamma)$. Then Γ is isomorphic to one of the following graphs:

$$K_6,\ L_2(4),\ \text{Pet (Petersen graph)},\ \overline{\text{Pet}}.$$

Therefore, if Γ is a G-edge-transitive graph in $\mathcal{C}$ and its order is larger than 16, then G admits a non-trivial **block of imprimitivity** (a **block** for short).

In the next slides we study blocks and the corresponding quotient graphs.

We recall first some general concepts.

Definition

Let Γ be a graph and $\pi = \{V_1, \ldots, V_k\}$ be a partition of $\mathsf{V}(\Gamma)$.
The **quotient graph** Γ/π has vertex set π, and edges

$$\{V_i, V_j\} \text{ if there are } u \in V_i \text{ and } v \in V_j \text{ with } \{u, v\} \in \mathsf{E}(\Gamma).$$

We say that Γ is an **r-cover** of Γ/π if

$$\forall \{u, v\} \in \mathsf{E}(\Gamma) : \; u \in V_i, \; v \in V_j, \; V_i \neq V_j, \text{ and } |\Gamma(u) \cap V_j| = r.$$

A block system $\mathcal{B}$ for a group G is **normal** if $\mathcal{B}$ is formed by the orbits of some normal subgroup N of G. In this Γ/N is also written for $\Gamma/\mathcal{B}$.

We need the following properties.

Let Γ be a connected G-vertex- and G-edge-transitive graph. Let $\mathcal{B}$ be a normal block system for G and K be the kernel of the action of G on $\mathcal{B}$. Then

1. Γ is an r-cover of $\Gamma/\mathcal{B}$, where

   $$r = |\Gamma(v) \cap B|,$$

   v is any vertex and B is any block in $\mathcal{B}$ intersecting $\Gamma(v)$.

2. If $r = 1$ in case (1), then Γ and $\Gamma/\mathcal{B}$ have the same valency, the kernel K is regular on every block in $\mathcal{B}$, and $\Gamma/\mathcal{B}$ is G/K-vertex- and G/K-edge-transitive.

We study next blocks for groups acting on graphs in class $\mathcal{C}$ and for this purpose set the following notation:

- Γ is a G-edge-transitive graph in $\mathcal{C}$.
- $2n$ is the order of Γ.
- d is the valency of Γ, $d \geq 6$.
- C is a cyclic semiregular subgroup of G, it has two orbits, and at least one of the subgraphs induced by these orbits is a cycle.
- B is a block for G, $1 < |B| < 2n$.
- $\mathcal{B}$ is the block system induced by B.

We say that B is **cyclic** if it is contained in one of the two orbits of C.

Lemma (cyclic block)

Suppose that B is cyclic such that $1 < |B| < |C|/2$.

1. $\mathcal{B}$ is formed by the orbits of $C_{\{B\}}$ (setwise stabiliser of B in C).
2. Γ is a 1-cover of $\Gamma/\mathcal{B}$.
3. $\Gamma/\mathcal{B} \in \mathcal{C}$.

Lemma (non-cyclic block I)

Suppose that B is non-cyclic.

1. B is a union of two $C_{\{B\}}$-orbits.
2. C acts transitively on $\mathcal{B}$.
3. If $|B| > 2$ and B is minimal, then $\mathcal{B}$ is normal.

Lemma (non-cyclic block II)

If $n > 5$ is odd and B is non-cyclic, then $|B| > 2$.

Sketch of proof: Deny it and let $B = \{u, v\}$. Define the involution t as

$$t := (u\,v)(u^c\,v^c)\cdots(u^{c^{n-1}}\,v^{c^{n-1}}),$$

where c is a generator of C. Then $gt = tg$ for every $g \in G$.

Fix an edge $\{u, w\}$ and define the graph Γ' as

$$\mathsf{V}(\Gamma') := \mathsf{V}(\Gamma) \text{ and } \mathsf{E}(\Gamma') := \{\{u, w\}^x : x \in \langle G, t\rangle\}.$$

Γ' is a **circulant graph** because $\langle c, t\rangle \cong \mathbb{Z}_{2n}$ and it is regular on $\mathsf{V}(\Gamma')$.

Γ' is arc-transitive.

We use the classification of arc-transitive circulant graphs to get a contradiction. Q.e.d.

Lemma (existence of a non-cyclic block)

If $n > 5$ is odd, then G admits a non-trivial cyclic block.

Sketch of proof: We construct a cyclic block.

There is a minimal non-trivial block B. We may assume that B is non-cyclic and $|B| > 2$. Fix a vertex $u \in B$.

We apply Müller's theorem to $G_{\{B\}}$ and show that every $B' \in \mathcal{B}$ contains a unique vertex u' such that $K_u = K_{u'}$, where K is the kernel of the action of G on $\mathcal{B}$.

We show that the vertices u' form a block of size $|\mathcal{B}|$. As $|\mathcal{B}|$ is odd, the latter block must be cyclic. Q.e.d.

We need one more lemma.

Lemma

Suppose that $\Gamma \in \mathcal{C}$ is G-edge-transitive and has valency 6. Let $C \leq G$ be a semiregular cyclic subgroup with two orbits. Then G contains an element g of order n such that

1. $HgH = Hg^{-1}H$ and $|H| = 6|H \cap H^g| = \frac{1}{2}|H\langle g\rangle \cap HgH|$, or
2. $HgH \neq Hg^{-1}H$ and $|H| = 3|H \cap H^g| = |H\langle g\rangle \cap HgH|$.

We are ready to give a sketch of the proof of our theorem.

Theorem (I. K., J. Ruff, 2022)

The class $\mathcal{C}$ contains the following edge-transitive graphs of twice odd order:

K_6, the Petersen graph and its complement, and the wreath graphs $W(2n+1, 2)$.

Sketch of proof: Deny it and let Γ be a counter example of smallest order.

There is a non-trivial cyclic block B for $G := \mathrm{Aut}(\Gamma)$ (Lemma).

$C_{\{B\}}$ is the kernel of G acting on $\mathcal{B}$ (Lemma), in particular, $C_{\{B\}} \lhd G$.

Then $P \lhd G$, where $P \leq C_{\{B\}}$ and $|P| = p$ for a prime p.

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Γ/P is G/P-edge-transitive and it is in $\mathcal{C}$ (Lemma).

As Γ is a minimal counter example,

$$n = 5p,\ \Gamma/P \cong \overline{\mathrm{Pet}},\ G/P \cong A_5 \text{ or } S_5.$$

If $p > 5$, then $G = P \times A_5$ or $P \times S_5$ or $P \rtimes S_5$, and in the last case S_5 acts on P as

$$z^\lambda = \begin{cases} z & \text{if } \lambda \text{ is even,} \\ z^{-1} & \text{if } \lambda \text{ is odd,} \end{cases} \quad (z \in P, \lambda \in S_5).$$

We show that H is a vertex-stabiliser, where

$$H := \begin{cases} \langle (1,2,3),(1,2) \rangle & \text{if } G/P = A_5, \\ \langle (1,2,3),(1,2),(4,5) \rangle & \text{if } G/P = S_5. \end{cases}$$

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We show that H is a vertex-stabiliser, where

$$H := \begin{cases} \langle (1,2,3), (1,2) \rangle & \text{if } G/P = A_5, \\ \langle (1,2,3), (1,2), (4,5) \rangle & \text{if } G/P = S_5. \end{cases}$$

The final contradiction arises after applying our last Lemma to G and H.

There is $g \in G$ of order $5p$ such that

1. $HgH = Hg^{-1}H$ and $|H| = 6|H \cap H^g| = \frac{1}{2}|H\langle g\rangle \cap HgH|$.
2. $HgH \neq Hg^{-1}H$ and $|H| = 3|H \cap H^g| = |H\langle g\rangle \cap HgH|$.

Now $g = z\sigma$ such that $P = \langle z\rangle$ and σ is a 5-cycle in S_5.

If $G = P \times A_5$ or $P \times S_5$, then case (2) occurs, and

$$|H| = 3|H \cap H^\sigma| = |H\langle\sigma\rangle \cap H\sigma H|.$$

If $G = P \rtimes S_5$, then case (1) occurs, and

$$|H| = 3|H \cap H_1^\sigma| = |H\langle\sigma\rangle \cap H_1\sigma H| + |H\langle\sigma\rangle \cap (4,5)H_1\sigma H| = 2|H|,$$

where $H_1 = H \cap A_5$.

Notice that in either case the identities hold in S_5 and independent of p. We check that none of them are possible. Q.e.d.

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where $H_1 = H \cap A_5$.

Notice that in either case the identities hold in S_5 and independent of p. We check that none of them are possible. Q.e.d.

The final contradiction arises after applying our last Lemma to G and H.

There is $g \in G$ of order $5p$ such that

1. $HgH = Hg^{-1}H$ and $|H| = 6|H \cap H^g| = \frac{1}{2}|H\langle g \rangle \cap HgH|$.
2. $HgH \neq Hg^{-1}H$ and $|H| = 3|H \cap H^g| = |H\langle g \rangle \cap HgH|$.

Now $g = z\sigma$ such that $P = \langle z \rangle$ and σ is a 5-cycle in S_5.

If $G = P \times A_5$ or $P \times S_5$, then case (2) occurs, and

$$|H| = 3|H \cap H^\sigma| = |H\langle \sigma \rangle \cap H\sigma H|.$$

If $G = P \rtimes S_5$, then case (1) occurs, and

$$|H| = 3|H \cap H_1^\sigma| = |H\langle \sigma \rangle \cap H_1\sigma H| + |H\langle \sigma \rangle \cap (4,5)H_1\sigma H| = 2|H|,$$

where $H_1 = H \cap A_5$.

Notice that in either case the identities hold in S_5 and independent of p. We check that none of them are possible. Q.e.d.

Theorem (I. K., 2022+)

The class $\mathcal{C}$ contains three families of hexavalent edge-transitive graphs:

a. $\mathcal{N}(5; 1, 2, 3; 2), \mathcal{N}(8; 1, 3, 4; 3), \mathcal{N}(8; 1, 2, 5; 3),$

   $\mathcal{N}(10; 1, 3, 4; 3), \mathcal{N}(10; 2, 4, 6; 3)$ and $\mathcal{N}(12; 2, 4, 8; 5)$.

b. $\mathcal{N}(2m; 2s, s - s^2 + m, s + s^2 + m; 1)$, m is odd and $s^3 \equiv 1 \pmod{m}$

c. $\mathcal{N}(2m; 2s, s - s^2 + m/2, s + s^2 + m/2; m - 1)$, $m \equiv 2 \pmod{4}$ and $s^3 \equiv 1 \pmod{m}$

In the proof we follow the same strategy, namely we consider cyclic blocks.

Let Γ be an edge-transitive Nest graph of order $2n$. It turns out the largest cyclic block has size either 1 or 2 or $n/2$.

- If the size is 1, then Γ is one of:

$$\mathcal{N}(5;1,2,3;2),\ \mathcal{N}(8;1,3,4;3),\ \mathcal{N}(8;1,2,5;3) \text{ and } \mathcal{N}(12;2,4,8;5).$$

- If the size is 2, then Γ is $\mathcal{N}(10;1,3,4;3)$ or $\mathcal{N}(10;2,4,6;3)$.
- If the size is $n/2 = m$, then

$$\Gamma = \mathcal{N}(2m; 2s, s - s^2 + m, s + s^2 + m; 1),$$

where m is odd and $s^3 \equiv 1 \pmod{m}$; or

$$\Gamma = \mathcal{N}(2m; 2s, s - s^2 + m/2, s + s^2 + m/2; m - 1),$$

where $m \equiv 2 \pmod{4}$ and $s^3 \equiv 1 \pmod{m}$.

To sum up, all edge-transitive graphs in the class $\mathcal{C}$ of valency at most 6 are known, and we have shown that no such graph exists if the valency is larger than 6 and the order is twice an odd number.

A possible future work is to extend the method from twice odd to arbitrary order.

I would like to finish my talk with two questions.

Question 1

Is there a polynomial algorithm for recognising whether a given graph belongs to the class $\mathcal{C}$?

Question 2

Is there a polynomial algorithm for recognising whether a given graph is a bicirculant?

To sum up, all edge-transitive graphs in the class $\mathcal{C}$ of valency at most 6 are known, and we have shown that no such graph exists if the valency is larger than 6 and the order is twice an odd number.

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Question 1

Is there a polynomial algorithm for recognising whether a given graph belongs to the class $\mathcal{C}$?

Question 2

Is there a polynomial algorithm for recognising whether a given graph is a bicirculant?

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Thank you for your attention!