The least Euclidean distortion constant of a distance-regular graph

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- contraction$(\rho) := \max\limits_{x,y \in V} \dfrac{d(x,y)}{||\rho(x) - \rho(y)||_2}$.
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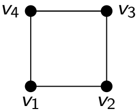

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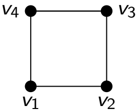

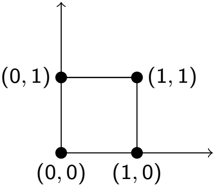

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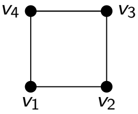

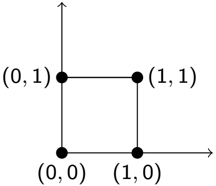

- $\exp(\rho) = \max\{\frac{1}{1}, \frac{\sqrt{2}}{2}\} = 1$
- $\text{cont}(\rho) = \max\{\frac{1}{1}, \frac{2}{\sqrt{2}}\} = \sqrt{2}$
- $\text{dist}(\rho) = \sqrt{2}$

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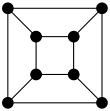
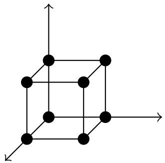


$\longmapsto$



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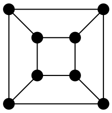

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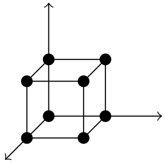

$\longrightarrow$

$c_2(Q_3) \leq \sqrt{3}$

Enflo 1969, Linial and Magen 2000, Vallentin 2008.

- ρ is an optimal embedding and $c_2(Q_d) = \sqrt{d}$.

Grand ancestor of finite metric embedding

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Linial, London, Rabinovich 1995, Matousek 1997

The bound above is tight and can be attained by the graph metric of expander graphs.

Equivalent formulation

Definition

$$c_2(G) = \min_{\rho}\{\mathrm{dist}(\rho)\} = \min_{\rho}\{\exp(\rho)\cdot \mathrm{cont}(\rho)\},$$

where $\exp(\rho) = \max_{x,y} \frac{||\rho(x)-\rho(y)||_2}{d(x,y)}$ and $\mathrm{cont}(\rho) = \max_{x,y} \frac{d(x,y)}{||\rho(x)-\rho(y)||_2}$.

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Note: A matrix Q is positive semidefinite $\iff Q = UU^T$ for a matrix U.

$c_2(G) = \min C,$ $d(x,y) \le ||\rho(x)-\rho(y)||_2 \le C\, d(x,y)$ $\forall x, y \in V$, ρ is embedding. $\rho : x \mapsto x^{\text{th}}$ row of U	$\iff$	$c_2(G) = \min C,$ $d(x,y)^2 \le q_{x,x} + q_{y,y} - 2q_{x,y} \le C^2\, d(x,y)^2$ $\forall x, y \in V$, Q is PSD. $Q = (q_{x,y}) = (\langle \rho(x), \rho(y)\rangle)$ is PSD

Semidefinite programming and Duality

Linial, London, Rabinovich 1995

$c_2(G) = C$, where,

$$\begin{aligned} \textbf{minimize} \quad & C^2, \\ \textbf{subject to} \quad & Q = (q_{x,y})_{x,y \in V} \text{ is positive semi definite,} \\ & d(x,y)^2 \le q_{x,x} + q_{y,y} - 2q_{x,y} \le C^2 \cdot d(x,y)^2, \forall x, y \in V. \end{aligned}$$

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Let $\mathcal{O}_n := \{Q | Q \text{ is } n \times n \text{ PSD and } Q\vec{1} = \vec{0}\}$.

$$c_2(G) \le C$$

$$\iff \forall Q \in \mathcal{O}_n : \sum_{q_{x,y}>0} d^2(x,y) q_{x,y} + C^2 \sum_{q_{x,y}<0} d^2(x,y) q_{x,y} \le 0.$$

The least distortion $c_2(G)$

Linial, London, Rabinovich 1995

$$c_2(G) = \max_{Q \in \mathcal{O}_n} \delta(Q),$$

where

$$\delta(Q) = \sqrt{\frac{\sum_{q_{x,y}>0} d^2(x,y) q_{x,y}}{\sum_{q_{x,y}<0} d^2(x,y)(-q_{x,y})}}.$$

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$$\max_{Q \in \mathcal{O}_n} \{\delta(Q)\} = c_2(G) = \min_{\rho} \{\mathrm{dist}(\rho)\}$$

- Any $Q \in \mathcal{O}_n$ provides a lower bound on $c_2(G)$.
- Any embedding ρ provides an upper bound on $c_2(G)$.

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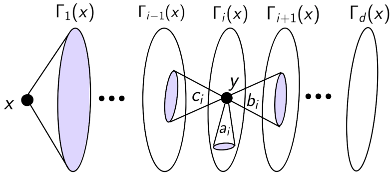
$$\stackrel{\text{def}}{\Longleftrightarrow} \exists\ a_i, b_i, c_i\ (i = 0, 1, \ldots, d) \text{ s.t. } \forall x, y \in V \text{ with } d(x, y) = i :$$



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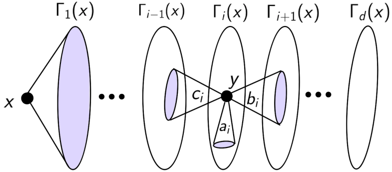

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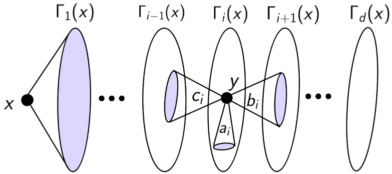
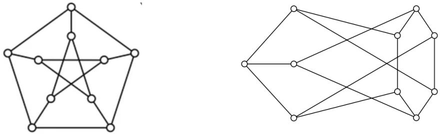


- G is regular of valency $k = b_0$ and $a_i + b_i + c_i = k$.
- $\mathtt{i}(G) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$: the **intersection array** of G

Examples of distance-regular graphs

- **The Petersen graph**



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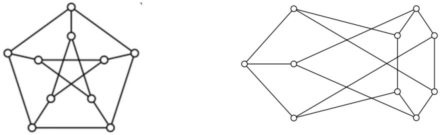

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- Distance-regular graphs of diameter 2 are called **strongly regular**.

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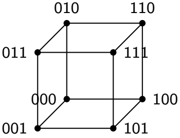
The Hamming graph $H(n, d)$

- $V = \mathbb{Z}_n^d$
- $x = (x_i) \sim y = (y_i) \overset{\textbf{def}}{\iff} |\{i : x_i \neq y_i\}| = 1$
- $b_i = (d - i)(n - 1),\ c_i = i\ (i = 0, 1, \ldots, d)$



$\mathtt{i}(H(2, 3)) = \{3, 2, 1; 1, 2, 3\}$

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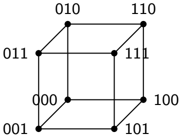
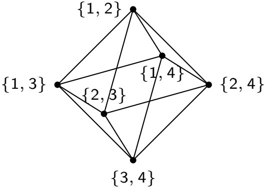


$\mathrm{i}(H(2, 3)) = \{3, 2, 1; 1, 2, 3\}$

The Johnson graph $J(n, d)$

- $V = \binom{[n]}{d}$
- $x \sim y \overset{\text{def}}{\iff} |x \cap y| = d - 1$
- $b_i = (d - i)(n - d - i),\ c_i = i^2$ $(i = 0, 1, \ldots, d)$



$\mathrm{i}(J(4, 2)) = \{4, 1; 1, 4\}$

Distance Matrices

A_i **is the distance** i **matrix of** $G \overset{\textbf{def}}{\iff} (A_i)_{xy} = \begin{cases} 1, & \text{if } d(x,y) = i \\ 0, & \text{otherwise.} \end{cases}$

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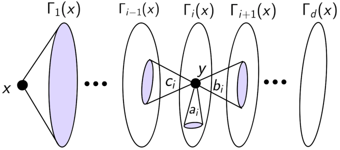
Combinatorial definition:



In terms of matrices:

$$
\begin{aligned}
A_1 A_i &= c_{i+1} A_{i+1} + a_i A_i + b_{i-1} A_{i-1} \\
\implies A_i &= v_i(A_1), \\
v_0(x) = 1,\ & v_1(x) = x \text{ and} \\
x \cdot v_i(x) &= c_{i+1} v_{i+1}(x) + a_i v_i(x) + b_{i-1} v_{i-1}(x).
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- They are the eigenvalues of the following matrix:

$$L = \begin{bmatrix} a_0 & b_0 & 0 & \ldots & 0 \\ c_1 & a_1 & b_1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & \ldots & \ldots & c_d & a_d \end{bmatrix}$$

Eigenmatrix of a distance-regular graph

Recall: $A_i = v_i(A_1)$

$$\mathbf{P} = \begin{array}{c} \begin{array}{cccccc} A_0 & A_1 & A_2 & \cdots & A_{d-1} & A_d \end{array} \\ \begin{bmatrix} 1 & k & v_2(k) & \cdots & v_{d-1}(k) & v_d(k) \\ 1 & \theta_1 & v_2(\theta_1) & \cdots & v_{d-1}(\theta_1) & v_d(\theta_1) \\ 1 & \theta_2 & v_2(\theta_2) & \cdots & v_{d-1}(\theta_2) & v_d(\theta_2) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \theta_{d-1} & v_2(\theta_{d-1}) & \cdots & v_{d-1}(\theta_{d-1}) & v_d(\theta_{d-1}) \\ 1 & \theta_d & v_2(\theta_d) & \cdots & v_{d-1}(\theta_d) & v_d(\theta_d) \end{bmatrix} \end{array}$$

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1 & \theta_2 & v_2(\theta_2) & \cdots & v_{d-1}(\theta_2) & v_d(\theta_2) \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
1 & \theta_{d-1} & v_2(\theta_{d-1}) & \cdots & v_{d-1}(\theta_{d-1}) & v_d(\theta_{d-1}) \\
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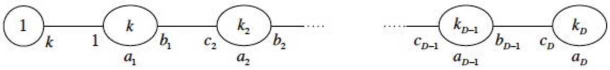
$$



Where $k_i := v_i(k)$ is the degree of the distance i graph G_i.

The least distortion of drgs

Vallentin 2008

If G is a drg with diameter d and eigenvalues $k > \theta_1 > \ldots > \theta_d$, then

$$c_2(G)^2 \geq \frac{d^2 v_d(k)}{k} \min_{1 \leq j \leq d} \left\{ \frac{k - \theta_j}{v_d(k) - v_d(\theta_j)} \right\}.$$

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The least distortion of drgs

Vallentin 2008

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This lower bound is tight for all

1. Hamming graphs, $c_2(H(n,d)) = \sqrt{d}$, $(x_1, \cdots, x_d) \to (e_{x_1}, \cdots, e_{x_d}) \in \mathbb{R}^{nd}$.
2. Johnson graphs, $c_2(J(n,d)) = \sqrt{d}$, $A \to \sum_{x \in A} e_x \in \mathbb{R}^n$.
3. Strongly regular graphs, $c_2(G) = 2\sqrt{1 + \frac{1}{s}}$, where s is the negative eigenvalue of G.

$c_2(G) = \min_{\rho}\{\mathrm{dist}(\rho)\}$

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Conjecture (Vallentin 2008)

The lower bound above is tight for all drgs.

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1. The conjecture is not true in general.
2. The conjecture is true for several families of drgs.

The least distortion of drgs

If G is a drg with eigenvalues $k > \theta_1 > \ldots > \theta_d$, then for any $1 \leq r \leq d$,

$$c_2(G)^2 \geq \frac{r^2 v_r(k)}{k} \min_{1 \leq j \leq d} \left\{ \frac{k - \theta_j}{v_r(k) - v_r(\theta_j)} \right\}.$$

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$\max_{Q \in \mathcal{O}_n} \{\delta(Q)\} = c_2(G)$

Counterexamples to Vallentin's Conjecture

- $c_2(G)^2 \geq \frac{d^2 v_d(k)}{k} \min_{1 \leq j \leq d} \left\{ \frac{k-\theta_j}{v_d(k)-v_d(\theta_j)} \right\}$. [Vallentin 2008]
- $c_2(G)^2 \geq \frac{(d-1)^2 v_{d-1}(k)}{k} \min_{1 \leq j \leq d} \left\{ \frac{k-\theta_j}{v_{d-1}(k)-v_{d-1}(\theta_j)} \right\}$.

Hadamard graphs

A Hadamard graph is a distance-regular graph with intersection array $\{2\mu, 2\mu-1, \mu, 1; 1, \mu, 2\mu-1, 2\mu\}$, where $\mu \geq 2$ an even number.

Computing the bounds above

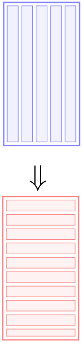
1. $\frac{8(\sqrt{2\mu}-1)}{\sqrt{2\mu}}$. [Vallentin 2008]
2. $\frac{9(\sqrt{2\mu}-1)}{\sqrt{2\mu}+1}$.

When $\mu \geq 34$, $\frac{9(\sqrt{2\mu}-1)}{\sqrt{2\mu}+1} > \frac{8(\sqrt{2\mu}-1)}{\sqrt{2\mu}}$.

The cosine sequence/the standard sequence

Graph representation

For an eigenvalue θ, let U_θ be a $n \times m$ matrix whose columns form an orthonormal basis for the eigenspace of θ. For $1 \leq x \leq d$, let $u_\theta(x)$ denote the x-th row of U_θ.



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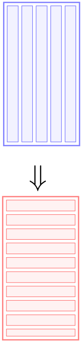

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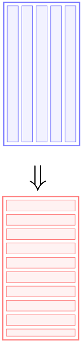

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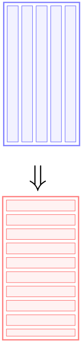

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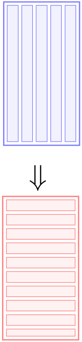

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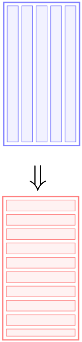

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- $w_r(\theta) = v_r(\theta)/v_r(k)$.

Two interpretations of the cosine sequences

- $w_r(\theta) = \dfrac{v_r(\theta)}{v_r(k)}$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \color{blue}{A_0} & \color{blue}{A_1} & \color{blue}{A_2} & \color{blue}{\cdots} & \color{blue}{A_{d-1}} & \color{blue}{A_d} \end{array} \\
\mathbf{W} = & \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & \theta_1/k & w_2(\theta_1) & \cdots & w_{d-1}(\theta_1) & w_d(\theta_1) \\
1 & \theta_2/k & w_2(\theta_2) & \cdots & w_{d-1}(\theta_2) & w_d(\theta_2) \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
1 & \theta_{d-1}/k & w_2(\theta_{d-1}) & \cdots & w_{d-1}(\theta_{d-1}) & w_d(\theta_{d-1}) \\
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\end{bmatrix}
\end{array}
$$

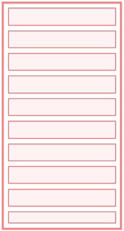
- $\theta \cdot w_r(\theta) = c_r w_{r-1}(\theta) + a_r w_r(\theta) + b_r w_{r+1}(\theta)$

$$
\begin{bmatrix}
a_0 & b_0 & 0 & \dots & 0 \\
c_1 & a_1 & b_1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & c_{d-1} & a_{d-1} & b_{d-1} \\
0 & \dots & \dots & c_d & a_d
\end{bmatrix}
\begin{bmatrix} 1 \\ w_1(\theta_i) \\ \vdots \\ w_{d-1}(\theta_i) \\ w_d(\theta_i) \end{bmatrix}
= \theta_i \begin{bmatrix} 1 \\ w_1(\theta_i) \\ \vdots \\ w_{d-1}(\theta_i) \\ w_d(\theta_i) \end{bmatrix}
$$

General Embedding for any drg

Embedding/Eigenpolytope

Let $\rho : V(G) \to \mathbb{R}^m$ be defined as $\rho(x) := \frac{u_{\theta_1}(x)}{\sqrt{2(u_{\theta_1}(x), u_{\theta_1}(x))(1 - w_1(\theta_1))}}$.



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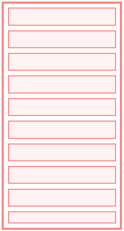

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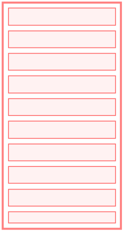

- $d(x, y) = r \Rightarrow ||\rho(x) - \rho(y)||_2 = \sqrt{\frac{1 - w_r(\theta_1)}{1 - w_1(\theta_1)}}$.
- $\exp(\rho) = \max_{1 \le r \le d} \frac{\sqrt{\frac{1 - w_r(\theta_1)}{1 - w_1(\theta_1)}}}{r} = \sqrt{\frac{1 - w_1(\theta_1)}{1 - w_1(\theta_1)}} = 1.$

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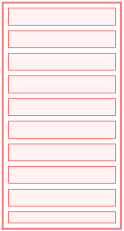

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- $\mathrm{cont}(\rho) = \max_{1 \leq r \leq d} \frac{r}{\sqrt{\frac{1-w_r(\theta_1)}{1-w_1(\theta_1)}}}$.

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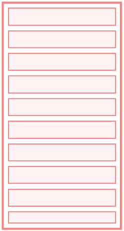

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- $\operatorname{dist}^2(\rho) = \max\limits_{1 \le r \le d} \left\{ r^2 \left(\frac{1-w_1(\theta_1)}{1-w_r(\theta_1)} \right) \right\}$.

Bounds on $c_2(G)$ for drgs

Recall

- $\max_{Q \in \mathcal{O}_n}\{\delta(Q)\} = c_2(G) = \min_\rho\{\mathrm{dist}(\rho)\}$
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$$\max_{1 \leq r \leq d}\left\{ r^2 \min_{1 \leq j \leq d}\left\{ \frac{1 - w_1(\theta_j)}{1 - w_r(\theta_j)} \right\}\right\} \leq c_2(G)^2 \leq \max_{1 \leq r \leq d}\left\{ r^2 \frac{1 - w_1(\theta_1)}{1 - w_r(\theta_1)} \right\}$$

Revised Conjecture

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$$\max_{1\le r\le d}\left\{r^2\min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_r(\theta_j)}\right\}\right\}\le c_2(G)^2\le\max_{1\le r\le d}\left\{r^2\frac{1-w_1(\theta_1)}{1-w_r(\theta_1)}\right\}$$

Conjecture (Vallentin 2008)

$$c_2(G)^2=d^2\min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_d(\theta_j)}\right\}.$$

Revised Conjecture

Cioabă, G., Ihringer and Kurihara 2022+

$$\max_{1\le r\le d}\left\{r^2\min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_r(\theta_j)}\right\}\right\}\le c_2(G)^2\le \max_{1\le r\le d}\left\{r^2\frac{1-w_1(\theta_1)}{1-w_r(\theta_1)}\right\}$$

Conjecture (Vallentin 2008)

$$c_2(G)^2=d^2\min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_d(\theta_j)}\right\}.$$

Conjecture (Cioabă, G., Ihringer and Kurihara 2022+)

$$c_2(G)^2=\max_{r=d-1,d}\left\{r^2\min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_r(\theta_j)}\right\}\right\}$$

Revised Conjecture

Cioabă, G., Ihringer and Kurihara 2022+

$$\max_{1\le r\le d}\left\{r^2\min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_r(\theta_j)}\right\}\right\}\le c_2(G)^2\le \max_{1\le r\le d}\left\{r^2\frac{1-w_1(\theta_1)}{1-w_r(\theta_1)}\right\}$$

Conjecture (Vallentin 2008)

$$c_2(G)^2=d^2\min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_d(\theta_j)}\right\}.$$

Conjecture (Cioabă, G., Ihringer and Kurihara 2022+)

$$c_2(G)^2=\max_{r=d-1,d}\left\{r^2\min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_r(\theta_j)}\right\}\right\}=\max_{r=d-1,d}\left\{r^2\frac{1-w_1(\theta_1)}{1-w_r(\theta_1)}\right\}.$$

Revised Conjecture

Conjecture (Cioabă, G., Ihringer and Kurihara 2022+)

$$c_2(G)^2 = \max_{r=d-1,d}\left\{r^2 \min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_r(\theta_j)}\right\}\right\} = \max_{r=d-1,d}\left\{r^2\frac{1-w_1(\theta_1)}{1-w_r(\theta_1)}\right\}.$$

Holds true for:

- Grassmann graph $G_q(n,d)$.
- Odd graphs O_{d+1}.
- Bilinear forms graph $B_q(n,d)$.
- Hadamard graphs.
- All drgs of diameter 3.
- many more...

Conjectures

Main Conjecture (Cioabă, G., Ihringer and Kurihara 2022+)

$$c_2(G)^2 = \max_{r=d-1,d}\left\{r^2 \min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_r(\theta_j)}\right\}\right\} = \max_{r=d-1,d}\left\{r^2\frac{1-w_1(\theta_1)}{1-w_r(\theta_1)}\right\}.$$

Conjectures

Main Conjecture (Cioabă, G., Ihringer and Kurihara 2022+)

$$c_2(G)^2 = \max_{r=d-1,d} \left\{ r^2 \min_{1 \le j \le d} \left\{ \frac{1 - w_1(\theta_j)}{1 - w_r(\theta_j)} \right\} \right\} = \max_{r=d-1,d} \left\{ r^2 \frac{1 - w_1(\theta_1)}{1 - w_r(\theta_1)} \right\}.$$

Conjecture 1

$$\min_{1 \le j \le d} \left\{ \frac{1 - w_1(\theta_j)}{1 - w_r(\theta_j)} \right\} = \frac{1 - w_1(\theta_1)}{1 - w_r(\theta_1)}.$$

Conjectures

Main Conjecture (Cioabă, G., Ihringer and Kurihara 2022+)

$$c_2(G)^2 = \max_{r=d-1,d}\left\{r^2 \min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_r(\theta_j)}\right\}\right\} = \max_{r=d-1,d}\left\{r^2\frac{1-w_1(\theta_1)}{1-w_r(\theta_1)}\right\}.$$

Conjecture 1

$$\min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_r(\theta_j)}\right\} = \frac{1-w_1(\theta_1)}{1-w_r(\theta_1)}.$$

Conjecture 2

$$\min_{1\le r\le d}\left\{\frac{1-w_r(\theta_1)}{r^2}\right\} = \min_{r=d-1,d}\left\{\frac{1-w_r(\theta_1)}{r^2}\right\}.$$

Conjecture 1 + Conjecture 2 imply Main Conjecture.

Conjecture 1 & Conjecture 2 true for all IA on Brouwer's list.

Conjecture 1

Conjecture 1

$$\min_{1\le j\le d}\left\{\frac{1-w_1(\theta_j)}{1-w_r(\theta_j)}\right\}=\frac{1-w_1(\theta_1)}{1-w_r(\theta_1)}.$$

Conjecture 1 implies that ρ is optimal and $c_2(G)=\mathrm{dist}(\rho)=\mathrm{cont}(\rho)$.

$$\begin{array}{c}
\begin{matrix} A_0 & \quad A_1 & \quad \cdots & \quad A_r & \quad \cdots & \quad A_d \end{matrix}\\
\begin{bmatrix}
1 & 1 & \cdots & 1 & \cdots & 1\\
1 & w_1(\theta_1) & \cdots & w_r(\theta_1) & \cdots & w_d(\theta_1)\\
1 & w_1(\theta_2) & \cdots & w_r(\theta_2) & \cdots & w_d(\theta_2)\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
1 & w(\theta_{d-1}) & \cdots & w_r(\theta_{d-1}) & \cdots & w_d(\theta_{d-1})\\
1 & w_d(\theta_d) & \cdots & w_r(\theta_d) & \cdots & v_d(\theta_d)
\end{bmatrix}
\end{array}$$

$$\begin{bmatrix}
a_0 & b_0 & 0 & \cdots & 0\\
c_1 & a_1 & b_1 & 0 & 0\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
0 & \cdots & c_{d-1} & a_{d-1} & b_{d-1}\\
0 & \cdots & \cdots & c_d & a_d
\end{bmatrix}
\begin{bmatrix}
1\\ w_1(\theta_j)\\ \vdots\\ w_{d-1}(\theta_j)\\ w_d(\theta_j)
\end{bmatrix}
=\theta_j
\begin{bmatrix}
1\\ w_1(\theta_j)\\ \vdots\\ w_{d-1}(\theta_j)\\ w_d(\theta_j)
\end{bmatrix}$$

Conjecture 2

Conjecture 2

$$\min_{1\le r\le d}\left\{\frac{1-w_r(\theta_1)}{r^2}\right\} = \min_{r=d-1,d}\left\{\frac{1-w_r(\theta_1)}{r^2}\right\}.$$

Conjecture 2 implies that $c_2(G) \le \mathrm{cont}(\rho) = \max_{r=d-1,d}\left\{\frac{r\sqrt{1-w_1(\theta_1)}}{\sqrt{1-w_r(\theta_1)}}\right\}$.

$$\begin{array}{c}
\begin{array}{cccccc} A_0 & A_1 & \cdots & A_r & \cdots & A_d \end{array} \\
\begin{bmatrix}
1 & 1 & \cdots & 1 & \cdots & 1 \\
1 & w_1(\theta_1) & \cdots & w_r(\theta_1) & \cdots & w_d(\theta_1) \\
1 & w_1(\theta_2) & \cdots & w_r(\theta_2) & \cdots & w_d(\theta_2) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & w(\theta_{d-1}) & \cdots & w_r(\theta_{d-1}) & \cdots & w_d(\theta_{d-1}) \\
1 & w_d(\theta_d) & \cdots & w_r(\theta_d) & \cdots & v_d(\theta_d)
\end{bmatrix}
\end{array}$$

$$\begin{bmatrix}
a_0 & b_0 & 0 & \dots & 0 \\
c_1 & a_1 & b_1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & c_{d-1} & a_{d-1} & b_{d-1} \\
0 & \dots & \dots & c_d & a_d
\end{bmatrix}
\begin{bmatrix} 1 \\ w_1(\theta_1) \\ \vdots \\ w_{d-1}(\theta_1) \\ w_d(\theta_1) \end{bmatrix}
= \theta_1
\begin{bmatrix} 1 \\ w_1(\theta_1) \\ \vdots \\ w_{d-1}(\theta_1) \\ w_d(\theta_1) \end{bmatrix}$$

Conjecture 2

Conjecture 2

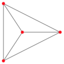
$$\min_{1\le r\le d}\left\{\frac{1-w_r(\theta_1)}{r^2}\right\}=\min_{r=d-1,d}\left\{\frac{1-w_r(\theta_1)}{r^2}\right\}.$$

Cioabă, G., Ihringer and Kurihara 2022+

$$\min_{1\le r\le d}\left\{\frac{1-w_r(\theta_1)}{r^2}\right\}=\min_{\lceil\frac{d+1}{2}\rceil\le r\le d}\left\{\frac{1-w_r(\theta_1)}{r^2}\right\}.$$

- Conjecture 2 true for drgs with diameter 3 or 4.

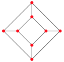
Least distortion of graphs of five platonic solids



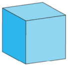
$\mathrm{i}(G_1) = \{3; 1\}$

Tetrahedron

$$c_2(G_1) = 1$$

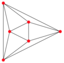


$\mathrm{i}(G_2) = \{3, 2, 1; 1, 2, 3\}$

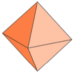


Cube

$$c_2(G_2) = \sqrt{3}$$

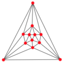


$\mathrm{i}(G_3) = \{4, 1; 1, 4\}$

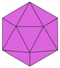


Octahedron

$$c_2(G_3) = \sqrt{2}$$

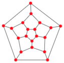


$\mathrm{i}(G_4) = \{5, 2, 1; 1, 2, 5\}$

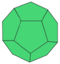


Icosahedron

$$c_2(G_4) = 3\sqrt{\frac{\sqrt{5}-1}{2\sqrt{5}}}$$



$\mathrm{i}(G_5) = \{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$



Dodecahedron

$$c_2(G_5) = 5\sqrt{\frac{3-\sqrt{5}}{6}}$$

Least distortion of several families of drgs

name	$c_2(G)$
Hamming graph	$\sqrt{d}$
Johnson graph	$\sqrt{d}$
Halved cube	$\sqrt{d}$
Doob graph	$\sqrt{d}$
Grassmann graph	$d\sqrt{q^{d-1}/\begin{bmatrix} d \\ 1 \end{bmatrix}_q}$
Twisted Grassmann graph	$d\sqrt{q^{d-1}/\begin{bmatrix} d \\ 1 \end{bmatrix}_q}$
Bilinear forms graph	$d\sqrt{q^{d-1}/\begin{bmatrix} d \\ 1 \end{bmatrix}_q}$
Dual Polar graph	$d\sqrt{q^{d-1}/\begin{bmatrix} d \\ 1 \end{bmatrix}_q}$
Alternating forms graph	$d\sqrt{b^{d-1}/\begin{bmatrix} d \\ 1 \end{bmatrix}_b}$
Quadratic forms graph	$d\sqrt{b^{d-1}/\begin{bmatrix} d \\ 1 \end{bmatrix}_b}$
Half dual polar graph	$d\sqrt{b^{d-1}/\begin{bmatrix} d \\ 1 \end{bmatrix}_b}$
Gosset graph $E_7(1)$	$\sqrt{3}$
Exceptional Lie graph	$3\sqrt{b^2/\begin{bmatrix} 3 \\ 1 \end{bmatrix}_b}$
Affine $E_6(q)$ graph	$\sqrt{b^2/\begin{bmatrix} 3 \\ 1 \end{bmatrix}_b}$

name	$c_2(G)$
Witt graph M_{24}	$\frac{2}{5}\sqrt{45}$
Witt graph M_{23}	$2\sqrt{\frac{21}{13}}$
Triality graph	$\sqrt{\frac{q^5}{(q^3+1)(q^2+1)}}$
Unitary dual polar graph	$d\sqrt{q^{d-1}/\begin{bmatrix} d \\ 1 \end{bmatrix}_q}$
Hermitian forms graph	$d\sqrt{\frac{(b^d+b^{d-1}+b+1)b^{d-1}}{(b^d+b+1)\begin{bmatrix} d \\ 1 \end{bmatrix}_b}}$
Hadamard graph	$3\sqrt{\frac{\sqrt{2\mu}-1}{\sqrt{2\mu}+1}}$
Taylor graph	$3\sqrt{\frac{k-z_1}{2k}}$
Generalized hexagons	$3\sqrt{\frac{t\sqrt{st}}{(\sqrt{st}+1)(t+1)}}$
Generalized octagons	$4\sqrt{\frac{st^2(st-\sqrt{2st}+1)}{(t+1)(s^2t^2+1)}}$
Odd graph O_{d+1}	$\begin{cases} 2\ell\sqrt{\frac{4\ell-2}{4\ell^2+\ell-1}}, & \text{if } d=2\ell, \\ (2\ell+1)\sqrt{\frac{4\ell+2}{4\ell^2+7\ell+3}}, & \text{if } d=2\ell+1. \end{cases}$