Schurity problem for finite groups: overview and new results

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S-rings

- G is a finite group, e is the identity of G.
- $\mathbb{Z}G$ is the integer group ring.

A subring $\mathcal{A} \subseteq \mathbb{Z}G$ is called an S-ring (Schur ring) over G if there exists a partition $\mathcal{S} = \mathcal{S}(\mathcal{A})$ such that:

- $\{e\} \in \mathcal{S}$,
- $X \in \mathcal{S} \;\Rightarrow\; X^{-1} \in \mathcal{S}$,
- $\mathcal{A} = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : \; X \in \mathcal{S}\}$, where $\underline{X} = \sum_{x \in X} x$.

- The elements of $\mathcal{S}$ are called the basic sets of $\mathcal{A}$.
- The trivial S-ring $\mathcal{T}(G) = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : \; X \in \{\{e\}, G \setminus \{e\}\}\}$ if $G \neq \{e\}$.
- $\mathbb{Z}G = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : \; X \in \{\{g\} : \; g \in G\}\}$.
- The center $Z(\mathbb{Z}G)$ is an S-ring, basic sets are conjugacy classes of G.

Schurian S-rings and Schur groups

- $G_{right} = \{x \mapsto xg,\ x \in G :\ g \in G\} \leq \mathrm{Sym}(G)$.
- $\mathrm{Orb}(K, G)$ is the set of all orbits of $K \leq \mathrm{Sym}(G)$ on G.

Theorem (Schur, 1933)

Let $K \leq \mathrm{Sym}(G)$ and $K \geq G_{right}$. Then
$V(K, G) = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} :\ X \in \mathrm{Orb}(K_e, G)\}$ is an S-ring over G.

An S-ring $\mathcal{A}$ over G is called schurian if $\mathcal{A} = V(K, G)$ for some $K \leq \mathrm{Sym}(G)$ such that $K \geq G_{right}$.

- There exists a nonschurian S-ring over $E_{p^2} = C_p \times C_p$, where $p \geq 5$ is prime (Wielandt, 1964).

A finite group G is called a Schur group if every S-ring over G is schurian (Pöschel, 1974).

- A section of a Schur group is Schur.

Problem (Pöschel, 1974)

Determine all Schur groups.

Leung-Man theory

- $H \leq G$ is an $\mathcal{A}$-subgroup if $\underline{H} \in \mathcal{A}$.
- If $L \trianglelefteq U \leq G$ and $\underline{L}, \underline{U} \in \mathcal{A}$ then $S = U/L$ is an $\mathcal{A}$-section.
- $\mathcal{A}_S = \mathrm{Span}_{\mathbb{Z}}\{\underline{X}^{\pi}: \ X \in \mathcal{S}(\mathcal{A}), \ X \subseteq U\}$, where $\pi : U \to U/L$ is the canonical epimorphism, is an S-ring over S.

- U and L are proper $\mathcal{A}$-subgroups of G such that $G = U \times L$.
- $\mathcal{A} = \mathcal{A}_U \otimes \mathcal{A}_L$ is the tensor product of $\mathcal{A}_U$ and $\mathcal{A}_L$ if $\mathcal{S}(\mathcal{A}) = \{X_1 \times X_2 : \ X_1 \in \mathcal{S}(\mathcal{A}_U), \ X_2 \in \mathcal{S}(\mathcal{A}_L)\}$.
- The tensor product of schurian S-rings is schurian.

- $S = U/L$ is an $\mathcal{A}$-section such that $\{e\} < L \trianglelefteq G$ and $U < G$.
- $\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}$ is the generalized wreath product of $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ if every $X \in \mathcal{S}(\mathcal{A}) \setminus \mathcal{S}(\mathcal{A}_U)$ is a union of some L-cosets.
- A necessary and sufficient condition of schurity for a generalized wreath product (Evdokimov-Ponomarenko, 2012).

- $\mathcal{A}$ is cyclotomic if $\mathcal{S}(\mathcal{A}) = \mathrm{Orb}(K, G)$ for some $K \leq \mathrm{Aut}(G)$.
- $\mathcal{A} = V(G_{right}K, G)$.

Leung-Man theory

Theorem (Leung-Man, 1996)

Let $\mathcal{A}$ be an S-ring over a cyclic group. Then one of the following statements holds:

- $\mathcal{A}$ is trivial;
- $\mathcal{A}$ is a tensor product of two S-rings;
- $\mathcal{A}$ is a generalized wreath product of two S-rings;
- $\mathcal{A}$ is cyclotomic.

A finite group G is called an LM-group if for every S-ring over G one of the statements of the Leung-Man theorem holds.

- Every cyclic group is LM-group.
- There are infinitely many both abelian and nonabelian non-LM groups.

Problem

Determine all LM-groups.

Cyclic Schur groups

Theorem (Pöschel, 1974)

Let p be an odd prime. Cyclic p-groups are Schur and if $p \geq 5$, then a Schur p-group is cyclic.

- The above theorem also holds for $p = 2$ (Golfand-Najmark-Pöschel, 1985).

Theorem (Klin-Pöschel, 1981)

A cyclic group of order pq, where p and q are distinct primes, is Schur.

Theorem (Evdokimov-Kovács-Ponomarenko, 2013)

Let $n \geq 1$ be an integer. The cyclic group of order n is Schur if and only if n belongs to one of the following families of integers:

$$p^k, pq^k, 2pq^k, pqr, 2pqr,$$

where p, q, r are primes and $k \geq 0$ is an integer.

Abelian Schur groups

Theorem (Evdokimov-Kovács-Ponomarenko, 2016)

An elementary abelian noncyclic group of order n is Schur if and only if $n \in \{4, 8, 9, 16, 27, 32\}$.

- Every elementary abelian Schur group is LM-group.

Theorem (Evdokimov-Kovács-Ponomarenko, 2016)

An abelian Schur group which is neither cyclic nor elementary abelian belongs to one of the following families of groups:

- $C_2 \times C_{2^k}$, $C_{2p} \times C_{2^k}$, $E_4 \times C_{p^k}$, $E_4 \times C_{pq}$, $E_{16} \times C_p$,
- $C_3 \times C_{3^k}$, $C_6 \times C_{3^k}$, $E_9 \times C_q$, $E_9 \times C_{2q}$,

where p and q are distinct primes, $p \neq 2$, and $k \geq 1$ is an integer.

- The following groups are Schur and LM-groups:
  - $E_4 \times C_p$ (Evdokimov-Kovács-Ponomarenko, 2016).
  - $C_2 \times C_{2^k}$ (Muzychuk-Ponomarenko, 2015).
  - $C_3 \times C_{3^k}$ (R., 2017).
  - $E_9 \times C_p$ (Ponomarenko-R., 2018).

Abelian Schur groups

Theorem 1

Let p be an odd prime. Then $C_{2p} \times C_{2^k}$ is Schur if and only if $k \leq 2$.

- The group $C_{2p} \times C_4$ is LM-group.
- To prove the "only if" part Theorem 1, it is sufficient to construct a nonschurian S-ring over $G = C_{2p} \times C_8$.
- For $p = 3$, a nonschurian S-ring over G was computed (Ziv-Av).
- For an arbitrary odd prime p, a nonschurian S-wreath product, where $S = (C_{2p} \times C_4)/C_p$, over G was constructed.

Theorem 2

Let p be an odd prime. Then $E_{16} \times C_p$ is Schur if and only if $p \not\equiv 1 \mod 3$.

- Theorem 2 for $p = 3$ follows from computer calculations (Pech, Reichard, Ziv-Av).
- $E_{16} \times C_p$ is LM-group.
- For a prime p such that $p \equiv 1 \mod 3$, a nonschurian S-wreath product $\mathcal{A}$, where $S = (E_4 \times C_p)/C_p$ and $\text{rk}(\mathcal{A}_S) = 2$, over $E_{16} \times C_p$ was constructed.

Abelian Schur groups

Theorem 3

The following groups are Schur and LM-groups:

- $E_4 \times C_{p^k}$, $E_4 \times C_{pq}$, $C_6 \times C_{3^k}$, $E_9 \times C_{2q}$,

where p and q are distinct primes, $p \neq 2$, and $k \geq 1$ is an integer.

Corollary 1

Let G be an abelian group which is neither cyclic nor elementary abelian. Then G is a Schur group if and only if G belongs to one of the following families of groups:

- $C_2 \times C_{2^k}$, $C_{2p} \times C_4$, $E_4 \times C_{p^k}$, $E_4 \times C_{pq}$, $E_{16} \times C_r$,
- $C_3 \times C_{3^k}$, $C_6 \times C_{3^k}$, $E_9 \times C_q$, $E_9 \times C_{2q}$,

where p and q are distinct primes, $p \neq 2$, r is a prime such that $r \not\equiv 1 \mod 3$, and $k \geq 1$ is an integer.

Corollary 2

Every abelian Schur group is LM-group.

Nonabelian Schur groups

- Every group of order at most 15 is Schur. In particular, there are nonabelian Schur groups (computer calculations, Fiedler, 1998).

Theorem (Ponomarenko-Vasil'ev, 2014)

Every Schur group G is metabelian and the set of prime divisors of $|G|$ is of size at most 7.

Theorem (Muzychuk, Ponomarenko, R., Vasil'ev, 2014-2015)

A nonabelian p-group is not Schur unless $p = 2$ and it is isomorphic to one of the groups $Q_8,\ D_8 * C_4,\ D_{2^k}$, where $k \geq 3$.

- The groups Q_8, $D_8 * C_4$, D_{2^k}, where $3 \leq k \leq 5$, are Schur.

Theorem (R., 2022)

A nonabelian nilpotent Schur group whose order has at least two distinct prime divisors is isomorphic to $Q_8 \times C_p$, where $p \geq 11$ is a prime such that $p \not\equiv 1 \mod 4$ and $p \not\equiv 1 \mod 6$.

Existence of an infinite family of nonabelian Schur groups

Question

Does an infinite family of nonabelian Schur groups exist?

- The largest known nonabelian Schur group has order 63.

Theorem

Let p be a prime. If p is a Fermat prime or $p = 4q + 1$, where q is a prime, then D_{2p} is Schur.

- The largest known Fermat prime is 65537
- There are infinitely many primes $p = 4q + 1$ modulo some famous (and widely believed) number-theoretical conjectures (Dickson, generalized Hardy-Littlewood).
- The keynote ingredient of the proof is nonexistence of a difference set in C_p.
- If $p \equiv 3 \mod 4$ and $p > 11$, then D_{2p} is not Schur (Ponomarenko-Vasil'ev, 2014).
- If $p = 4t^2 + 1$, where $t \geq 3$ is an odd integer, then D_{2p} is not Schur.
- D_{2p} is LM-group if and only if p is a Fermat prime.

Central S-rings

S-ring $\mathcal{A}$ is central if $\mathcal{A} \leq Z(\mathbb{Z}G)$ or, equivalently, each basic set of $\mathcal{A}$ is a union of some conjugacy classes of G.

- If G is abelian, then $Z(\mathbb{Z}G) = \mathbb{Z}G$ and hence every S-ring over G is central.
- The central S-rings over G are in one-to-one correspondence with the supercharacters of G (Hendrickson, 2010).
- The Schur-Wielandt theory for central S-rings (Chen-Muzychuk-Ponomarenko, 2016).
- Results on central S-rings over projective special linear groups (Humphries-Wagner, 2017).
- Results on automorphism groups of central S-rings over almost simple groups (Ponomarenko-Vasil'ev, 2017, Guo-Guo-R.-Vasil'ev, 2022).

Generalized Schur groups

A group G is defined to be generalized Schur if every central S-ring over G is schurian.

- G is generalized Schur and H is a normal subgroup of G.
- G/H is generalized Schur.
- In general, H is not generalized Schur.
- If every conjugacy class of H is also a conjugacy class of G, then H is generalized Schur.
- A_5 is generalized Schur.
- There exist infinitely many nonabelian generalized Schur as well as not generalized Schur groups.

Problem

Determine all generalized Schur groups.

Generalized Schur groups

Theorem 1

Let p be a prime.

- If a noncyclic p-group is generalized Schur, then $p \in \{2, 3\}$.
- If $p \in \{2, 3\}$, then a p-group with a maximal cyclic subgroup is generalized Schur.

Theorem 2

Let $n \geq 1$ be an integer. The dihedral group of order $2n$ is generalized Schur if and only if n belongs to one of the following families of integers:

$$p^k, pq^k, 2pq^k, pqr, 2pqr,$$

where p, q, r are primes and $k \geq 0$ is an integer.

Proposition

A dihedral group of order $2n$ is generalized Schur if and only if the cyclic group of order n is Schur.

Generalized Schur groups

- G is a Camina group if G there is $\{e\} < H \triangleleft G$ such that each H-coset distinct from H is contained in a conjugacy class of G. The pair (G, H) is a Camina pair.
- Frobenius and extraspecial groups are Camina groups.
- Any Camina group is a generalized B-group (Burnside group) (Chen-Muzychuk-Ponomarenko, 2016).

Theorem 3

Let (G, H) be a Camina pair. If H and G/H are generalized Schur groups, then so is G. In particular, a Frobenius group with generalized Schur kernel and complement is generalized Schur.

Corollary

Let p and q are primes such that $q \equiv 1 \pmod{p}$. Then the nonabelian group of order pq is generalized Schur.


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