The cage problem for cubic graphs

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For $d \geq 3$:

$$n(d,g) \geq M(d,g) = \begin{cases} \dfrac{d(d-1)^{(g-1)/2} - 2}{d-2}, & g \text{ odd;} \\[2ex] \dfrac{2(d-1)^{g/2} - 2}{d-2}, & g \text{ even.} \end{cases}$$

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g	Bound	Best	%	g	Bound	Best	%
3	4	4*	100.0	18	1022	2560	250.5
4	6	6*	100.0	19	1534	4324	281.9
5	10	10*	100.0	20	2046	5376	262.8
6	14	14*	100.0	21	3070	16028	522.1
7	22	24*	109.1	22	4094	16206	395.8
8	30	30*	100.0	23	6142	49326	803.1
9	46	58*	126.1	24	8190	49608	605.7
10	62	70*	112.9	25	12286	108906	886.4
11	94	112*	119.1	26	16382	109200	666.6
12	126	126*	100.0	27	24574	285852	1163.2
13	190	272	143.2	28	32766	415104	1266.9
14	254	384	151.2	29	49150	1141484	2322.4
15	382	620	162.3	30	65534	1143408	1744.8
16	510	960	188.2	31	98302	3649794	3712.8
17	766	2176	284.1	32	131070	3650304	2785.0

Table 1: Smallest known cubic graphs. Source: `http://combinatoricswiki.org/`

Hypergraphs

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So a cubic graph can be thought of as a 3-regular, 2-uniform hypergraph.

A d-regular, r-uniform hypergraph with n vertices and m hyperedges satisfies the equality:

$$nd = mr.$$

Cycles in hypergraphs

A *Berge cycle* of length k in a hypergraph is a sequence $v_0, e_0, v_1, e_1, \ldots, v_{k-1}, e_{k-1}, v_0$ such that each v_i is contained in e_{i-1} and e_i (mod k), all v_i are unique **and** all e_i are unique.

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Other notions of cycle and girth are available.

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Let H be a d-regular, r-uniform hypergraph of girth $g = 2k + 1$.

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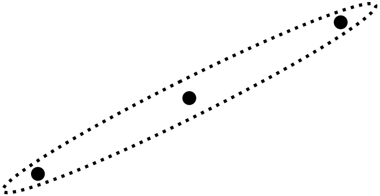

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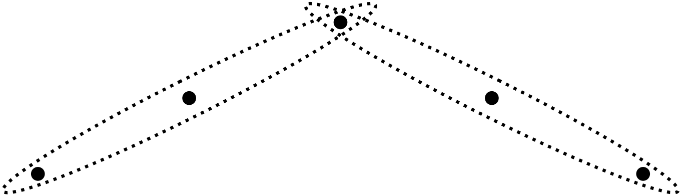

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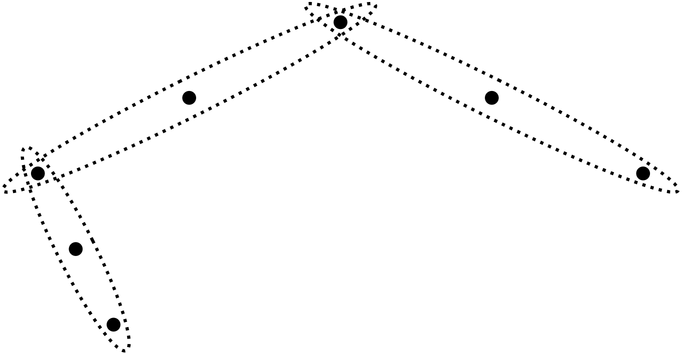

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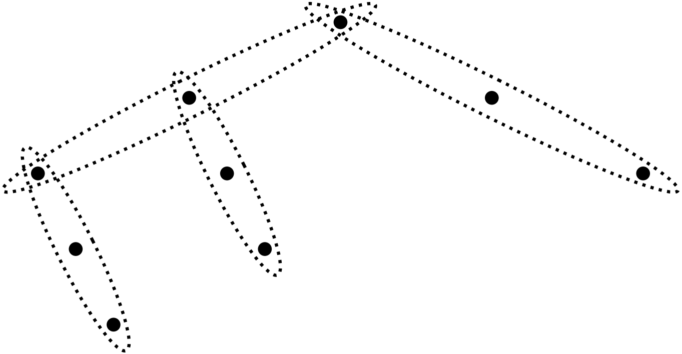

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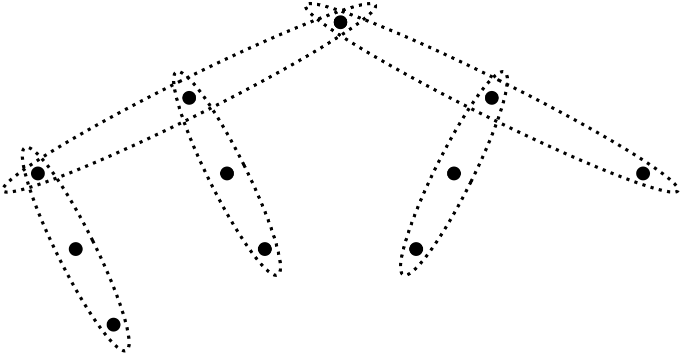

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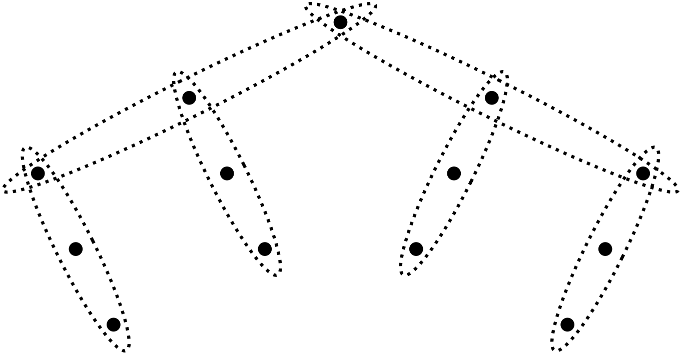

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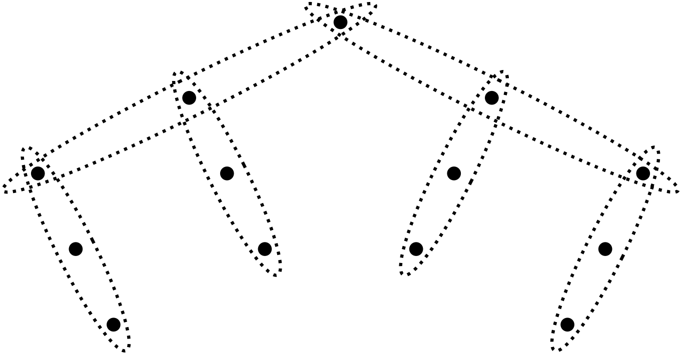


$$|V(H)| \geq M(d, r, k) = 1 + d(r-1)\frac{(d-1)^k(r-1)^k - 1}{(d-1)(r-1) - 1}$$

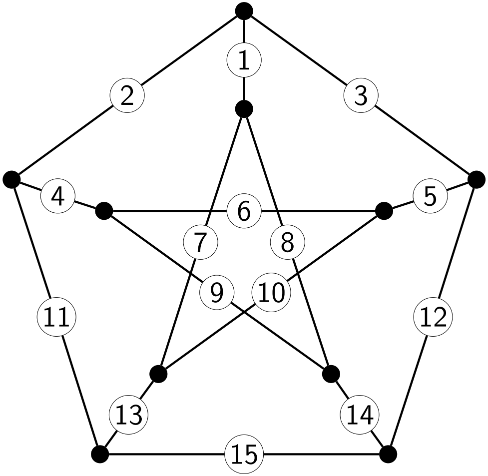
A first observation

For a 2-regular, 3-uniform hypergraph of girth 5 the Moore bound is 13. We can get a hypergraph of order 15 as follows.

Label the edges of the Petersen graph from 1 to 15. These are the vertices of the hypergraph.

The hyperedges are the labels of the edges which meet at a vertex of the graph.

For example, $\{1, 2, 3\}$, $\{3, 5, 12\}$.



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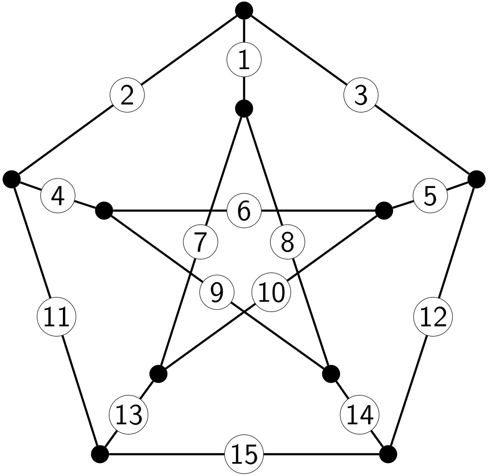
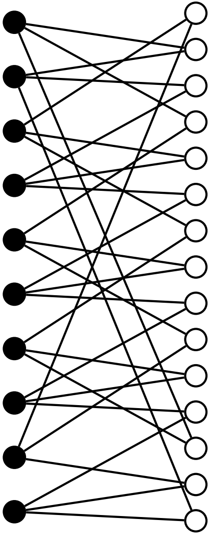


In fact since $nd = mr$, 15 is the best we can do.

Incidence graphs and duality



Given a hypergraph H, its *incidence graph* I has vertex set $V(H)$ (black vertices) and $E(H)$ (white vertices). There is an edge from v to e in I if and only if $v \in e$ in H.

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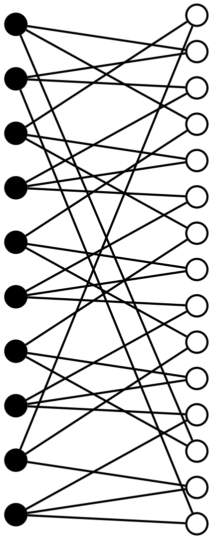

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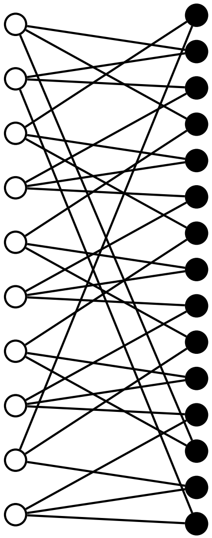

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By swapping the colour classes in I we get the incidence graph I^* of the *dual* hypergraph H^*. This is r-regular and d-uniform.

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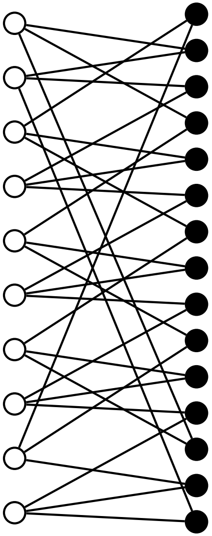
This is exactly what the previous construction was doing.



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Observation

$\mathrm{girth}(H) = \mathrm{girth}(H^*)$.

This simple observation has some interesting consequences.
A d-regular, r-uniform hypergraph of girth g and order n is dual to an r-regular, d-uniform hypergraph or girth g and order $\frac{d}{r}n$.

Cubic graphs and 3-regular, 3-uniform hypergraphs

A 3-regular, 3-uniform hypergraph of girth g has an incidence graph which is a cubic bipartite graph of girth $2g$. So the smallest hypergraphs can be determined from the list of smallest known cubic graphs of even girth.

$2g$	Graph
6	14
8	30
10	70
12	126
14	384
16	960
18	2,560

$2g$	Graph
20	5,376
22	16,206
24	49,608
26	109,200
28	415,104
30	1,143,408
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$2g$	Graph	Hypergraph
6	14	7
8	30	15
10	70	35
12	126	63
14	384	192
16	960	480
18	2,560	1,280

$2g$	Graph	Hypergraph
20	5,376	2,688
22	16,206	8,103
24	49,608	24,804
26	109,200	54,600
28	415,104	207,552
30	1,143,408	571,704
32	3,650,304	1,825,152

Cubic graphs and 2-regular, 3-uniform hypergraphs

A cubic graph has the same girth as its dual, which is a 2-regular, 3-uniform hypergraph.

g	Graph
3	4
4	6
5	10
6	14
7	24
8	30
9	58
10	70
11	112
12	126
13	272
14	384
15	620
16	960
17	2,176

g	Graph
18	2,560
19	4,324
20	5,376
21	16,028
22	16,206
23	49,326
24	49,608
25	108,906
26	109,200
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g	Graph	Hypergraph
3	4	6
4	6	9
5	10	15
6	14	21
7	24	36
8	30	45
9	58	87
10	70	105
11	112	168
12	126	189
13	272	408
14	384	576
15	620	930
16	960	1,440
17	2,176	3,264

g	Graph	Hypergraph
18	2,560	3,840
19	4,324	6,486
20	5,376	8,064
21	16,028	24,042
22	16,206	24,309
23	49,326	73,989
24	49,608	74,412
25	108,906	163,359
26	109,200	163,800
27	285,852	428,778
28	415,104	622,656
29	1,141,484	1,712,226
30	1,143,408	1,715,112
31	3,649,794	5,474,691
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Cubic graphs from hypergraphs

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There are hypergraph analogues of Cayley graphs.

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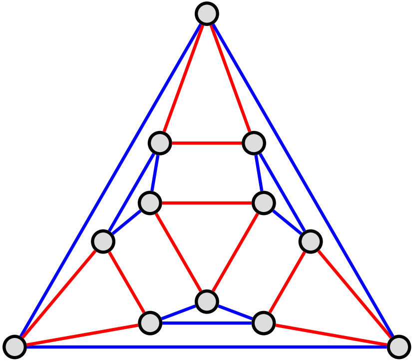

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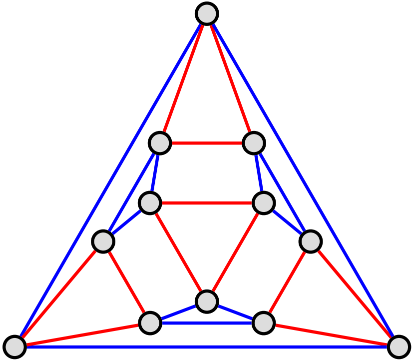

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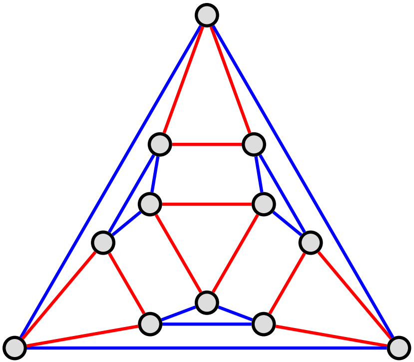

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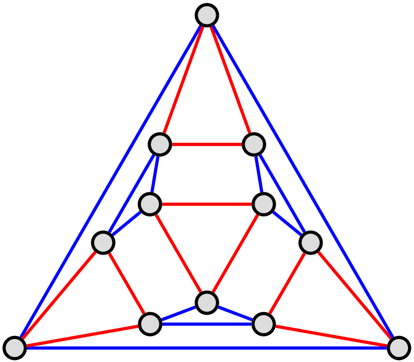

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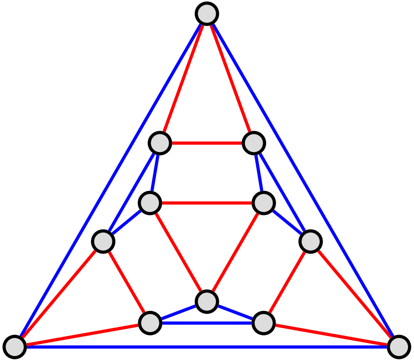
These graphs are highly symmetric, since (left) multiplication of all vertices by any element of the group induces a graph automorphism.



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Let G be a finite group, $S \subseteq G \setminus \{1\}$ and let $t \geq 2$. The t-*Cayley hypergraph* $t\text{-}\mathrm{Cay}(G, S)$ has vertex set G and hyperedge set

$$\{\{g, gs, \ldots, gs^{t-1}\} : g \in G, s \in S\}$$

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Note that if $t = 2$, then $2\text{-Cay}(G,S) = \text{Cay}(G, S \cup S^{-1})$.

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Other definitions of Cayley hypergraph are possible, but this is the most useful one for our needs.

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The basic idea is to find a 2-regular 3-Cayley hypergraph of large girth.

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The range of orders of interest goes up to about 2M. We find as many groups as we can, including groups generated by two random elements of a suitable symmetric group. Then take all possible direct products, provided the resulting group is still (3,3)-generated.

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- A way to view the cubic graph H^* is as a bipartite graph with partitions the left cosets of $\langle a \rangle$ and $\langle b \rangle$, with an edge from $x\langle a \rangle$ to $y\langle b \rangle$ whenever $x\langle a \rangle \cap y\langle b \rangle \neq \emptyset$.

Results

g	Current	Our best	Group
4	6	6	$\mathbb{Z}_3 \times \mathbb{Z}_3$
6	14	14	$\mathbb{Z}_7 \rtimes \mathbb{Z}_3$
8	30	40	$PSL(2,5)$
10	70	112	$\mathbb{Z}_2^3 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$
12	126	162	$(\mathbb{Z}_3 \times (\mathbb{Z}_3^2 \rtimes \mathbb{Z}_3)) \rtimes \mathbb{Z}_3$
14	384		$(\mathbb{Z}_{13} \rtimes \mathbb{Z}_3) \times SL(2,3)$
16	960		$\mathbb{Z}_3^2 \times PSL(2,7)$
18	2,560		$A_4 \times SL(2,7)$
20	5,376		$\mathbb{Z}_3 \times A_4 \times PSL(2,8)$
22	16,206		$((\mathbb{Z}_3^3 \rtimes \mathbb{Z}_2^2) \times (\mathbb{Z}_2^2 \rtimes (\mathbb{Z}_9 \rtimes \mathbb{Z}_3))) \rtimes \mathbb{Z}_3$
24	49,608		$(\mathbb{Z}_3^3 \rtimes \mathbb{Z}_3) \times PSL(2,11)$
26	109,200		$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times PSL(2,25)$
28	415,104		$(\mathbb{Z}_2^4 \rtimes SL(2,5)) \times SL(2,3) \times A_4$
30	1,143,408		$(\mathbb{Z}_7^3 \cdot PSL(3,2)) \times (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$
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Table 2: Smallest cubic graphs of even girth g

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g	Current	Our best	Group
4	6	6	$\mathbb{Z}_3 \times \mathbb{Z}_3$
6	14	14	$\mathbb{Z}_7 \rtimes \mathbb{Z}_3$
8	30	40	$PSL(2,5)$
10	70	112	$\mathbb{Z}_2^3 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$
12	126	162	$(\mathbb{Z}_3 \times (\mathbb{Z}_3^2 \rtimes \mathbb{Z}_3)) \rtimes \mathbb{Z}_3$
14	384	624	$(\mathbb{Z}_{13} \rtimes \mathbb{Z}_3) \times SL(2,3)$
16	960	1,008	$\mathbb{Z}_3^2 \times PSL(2,7)$
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So this is an example of a *coset graph*, and such objects have been used before in the cage problem.

At girth 26, our method produces a graph of order 109,200 — exactly matching the previous best graph found by Bray, Parker and Rowley. This is not a coincidence.

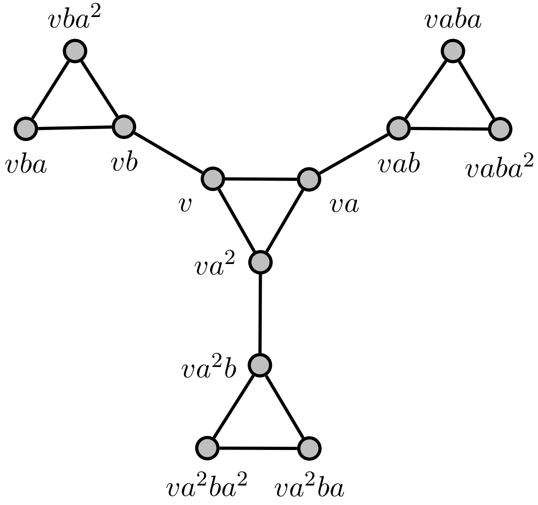
Other coset graphs — the Bray/Parker/Rowley construction

Let $G = \langle a, b \mid a^3, b^2, \ldots \rangle$.

Let $\Gamma = \mathrm{Cay}(G, \{a, b\})$.

Locally, a vertex v lives in a triangle induced by $\langle a \rangle$ and is joined to one other triangle by the generator b.

The triangles represent (left) cosets of $\langle a \rangle$.



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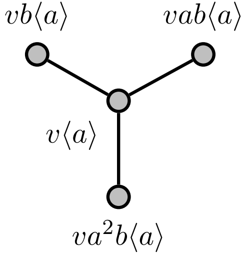
Form a new graph by collapsing each triangle to a single vertex.

The new graph has vertex set the cosets of $\langle a \rangle$.

These graphs are vertex-transitive but not necessarily Cayley.



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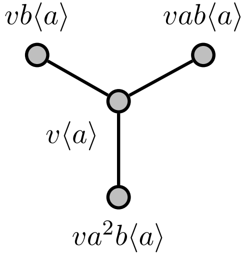
Our coset graph of girth 26 turns out to be isomorphic to the known graph produced in this way.



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Theorem (N. Biggs, 1998)

Let Γ *be a cubic graph of girth* $g \geq 4$ *and order* n*, and let* $r = \lfloor \frac{g}{4} \rfloor$*. Then there exists a cubic graph* Γ' *of order* $n - \epsilon$ *and girth* $g - 1$*, where*

$$\epsilon = \begin{cases} 2^{r+1} - 2, & g \equiv 0, 1 \pmod 4; \\ 3 \times 2^r - 2, & g \equiv 2, 3 \pmod 4. \end{cases}$$

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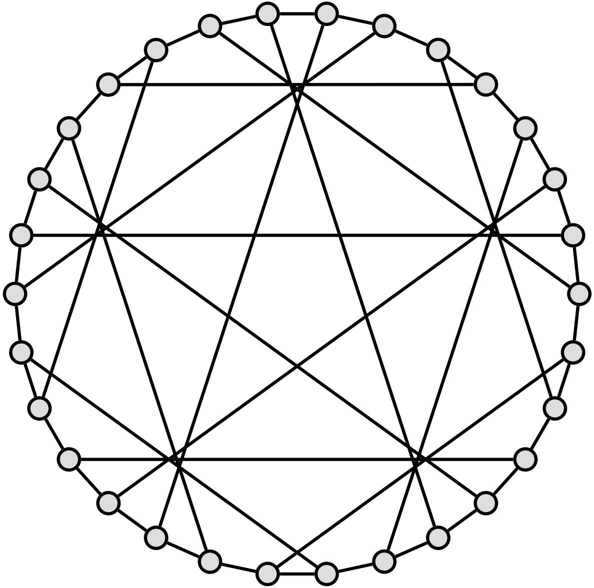
Idea: *Excise a tree of depth $r - 1$ from an edge, or a tree of depth r rooted at a vertex. Then join back up the vertices of valency 2 thus created.*

Observation: *This gives a lower bound for the number of vertices which can be excised. It is frequently possible to do better by carrying out the initial excision then attempting to chop out smaller trees from the resulting graph.*

Excision example

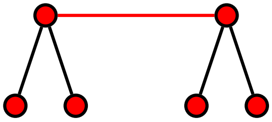
Begin with a cubic graph of girth 8 with order 30.

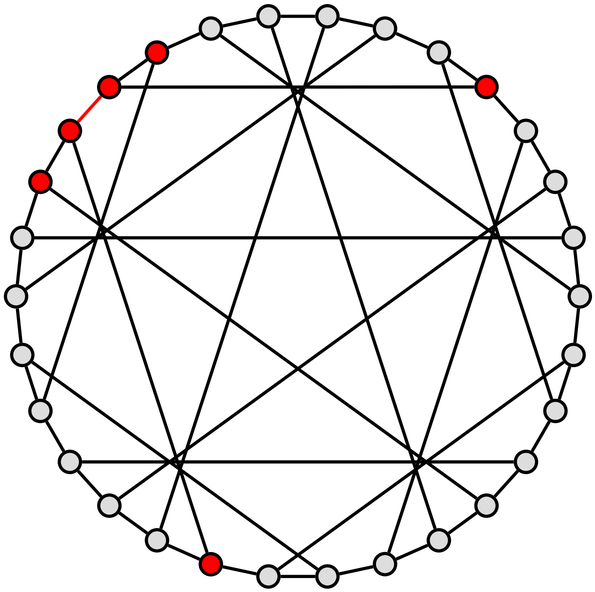
This is the *Tutte-Coxeter graph* or *Tutte 8-cage*.



Excision example

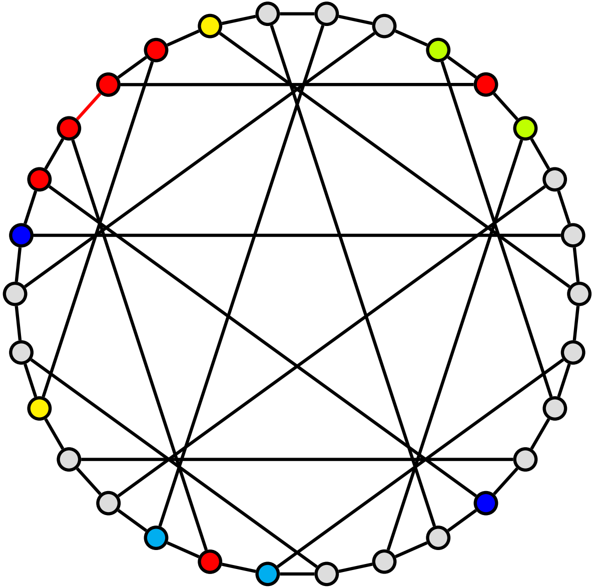
Choose an edge and mark a tree of depth 1 from that edge.





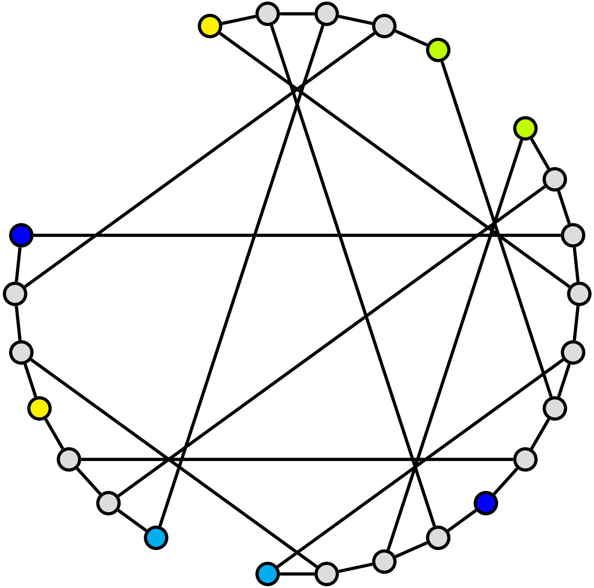
Excision example

Identify the two other neighbours of all the leaf nodes in the tree to be excised.



Excision example

Remove the vertices in the tree.

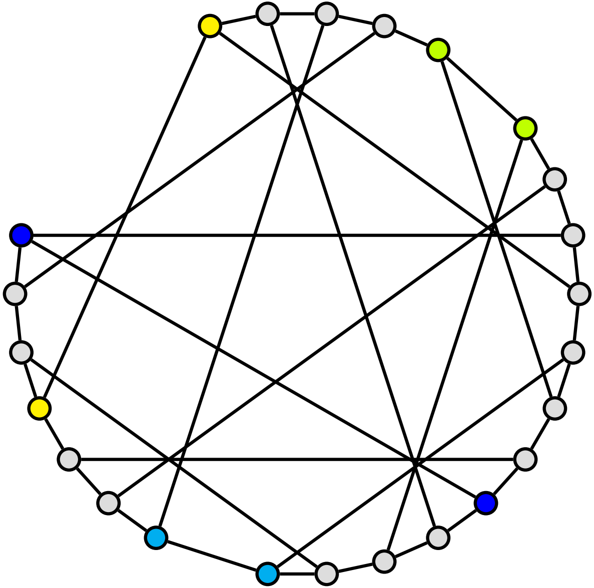


Excision example

Finally, join up the identified vertices of valency 2. These were at distance 2 in the original graph.

The result is a cubic graph of order 24 and girth 7.

In fact this is the McGee graph which is the unique cage of girth 7.



New table

g	Graph	Description	g	Graph	Description
3	4	K_4	18	$2,560$	Exoo
4	6	$K_{3,3}$	19	$4,324$	Hoare
5	10	Petersen	20	$5,376$	Exoo
6	14	Heawood	21	$16,028$	Exoo
7	24	McGee	22	$16,206$	Biggs/Hoare
8	30	Tutte	23	$35,446$	* NEW *
9	58	Brinkmann/McKay/Saager	24	$35,640$	* NEW *
10	70	O'Keefe/Wong	25	$108,906$	Exoo
11	112	McKay/Myrvold; Balaban	26	$109,200$	Bray/Parker/Rowley
12	126	Benson	27	$285,852$	Bray/Parker/Rowley
13	272	McKay/Myrvold; Hoare	28	$368,640$	* NEW *
14	384	McKay; Exoo	29	$805,746$	* NEW *
15	620	Biggs	30	$806,736$	* NEW *
16	960	Exoo	31	$1,440,338$	* NEW *
17	$2,176$	Exoo	32	$1,441,440$	* NEW *

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The smallest possible value would be $\frac{1}{2}$, but our best known infinite family constructions* only get us to $\frac{3}{4}$. If $\frac{3}{4}$ or something close to it is a hard limit, this explains why the best known graphs in the table are so far from the Moore bound.

* Biggs/Hoare, The sextet construction for cubic graphs, Combinatorica 3 (1983) 153-165

Can the $3/4$ barrier be breached?

Cayley graphs of $PSL(2, q)$ often perform well in the cubic cage problem if the generators are carefully chosen.

These groups are relatively easy to work with and it is known that non-solvable groups have a higher chance of success.

We know the highest girth Cayley graphs up to $q = 251$, but it seems difficult to discern any pattern in the generator sets which give these good graphs.

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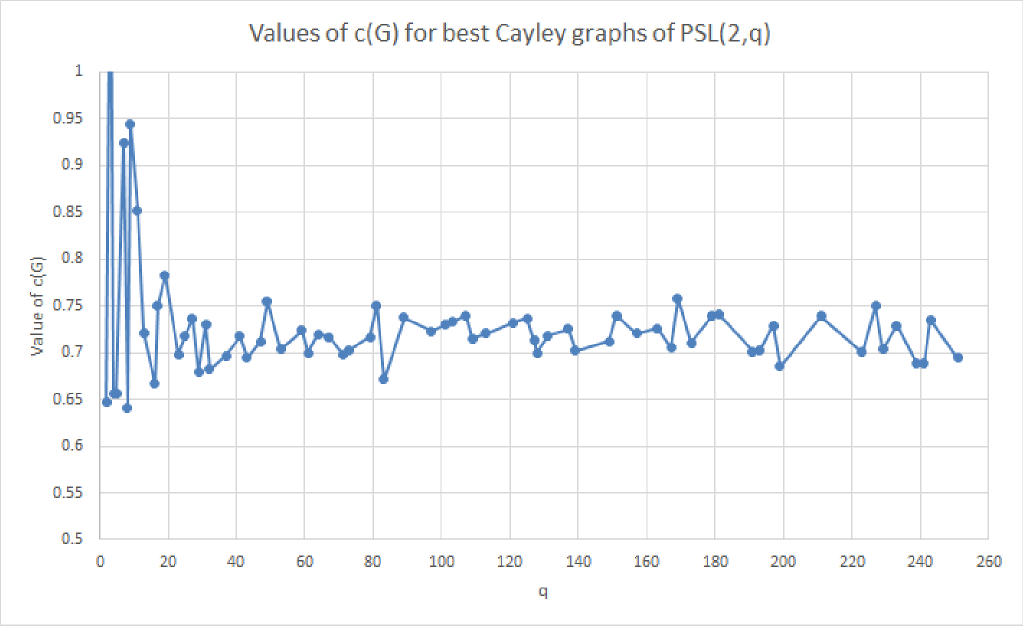

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Reference: G. Erskine and J. Tuite. *Small graphs and hypergraphs of given degree and girth.* `https://arxiv.org/abs/2201.07117`.