Prisms Aren't Boring

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Joint work with G. Cunningham and M. Mixer

The most important thing…

SIGMAP 2022 is in Fairbanks, Alaska July 10-16.

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Warning

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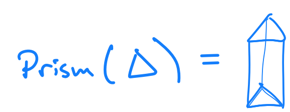
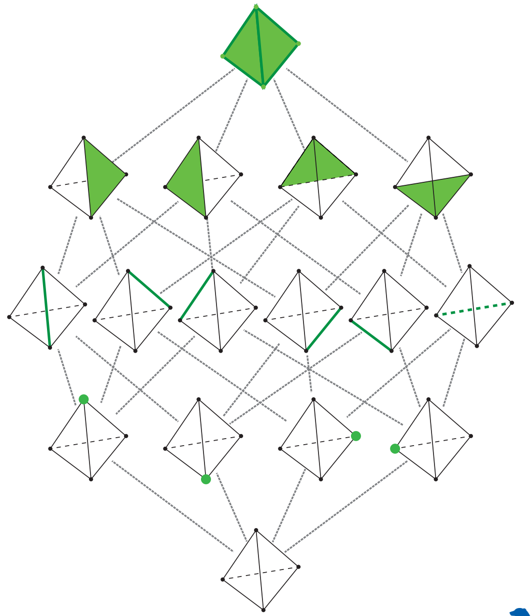
What is a prism?

- Geometrically: 

- Abstractly:
  - Prism($\mathscr{P}$) $= \mathscr{P} \times \mathscr{I} =$ $\{(f,g) : \mathscr{P}_{-1} \neq f \in \mathscr{P}, \varnothing \neq g \in \mathscr{I}\} \cup \{(\mathscr{P}_{-1}, \varnothing)\}$ where $\mathscr{I} = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$.

Posets as Polytopes



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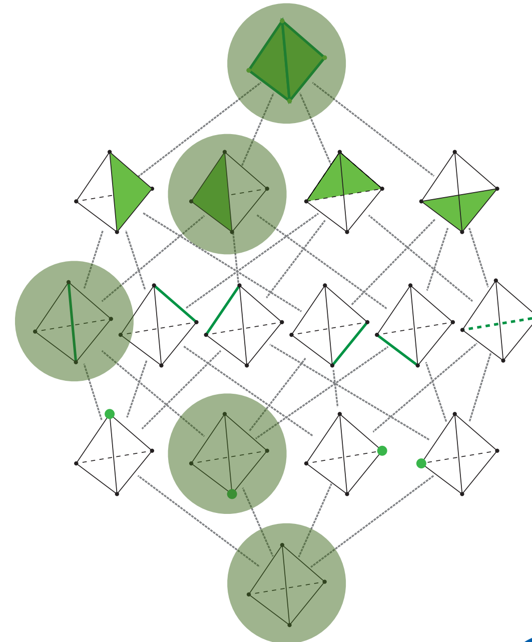

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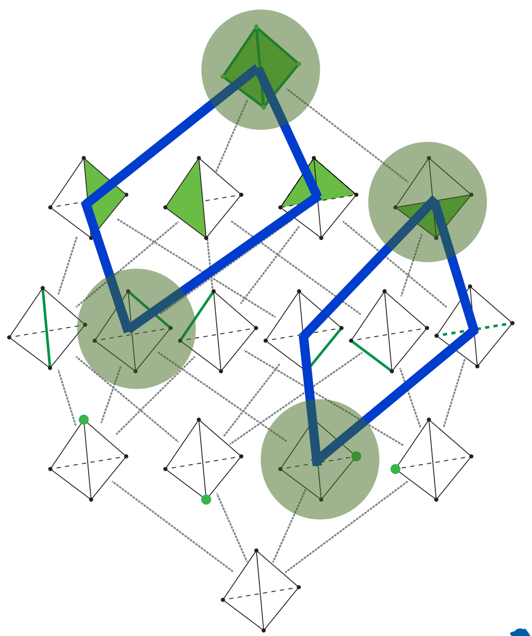

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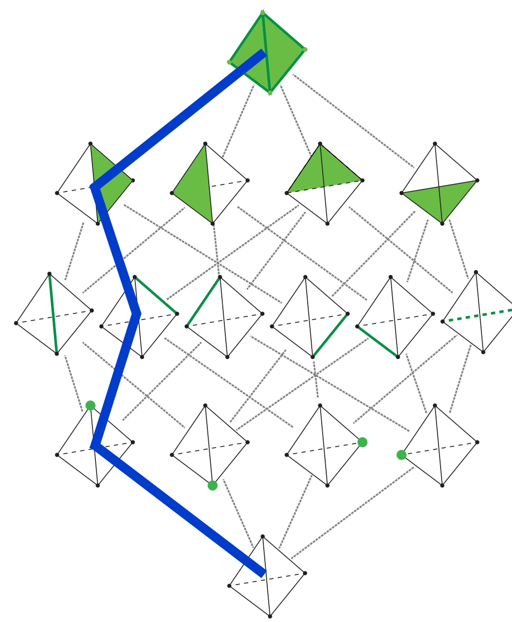

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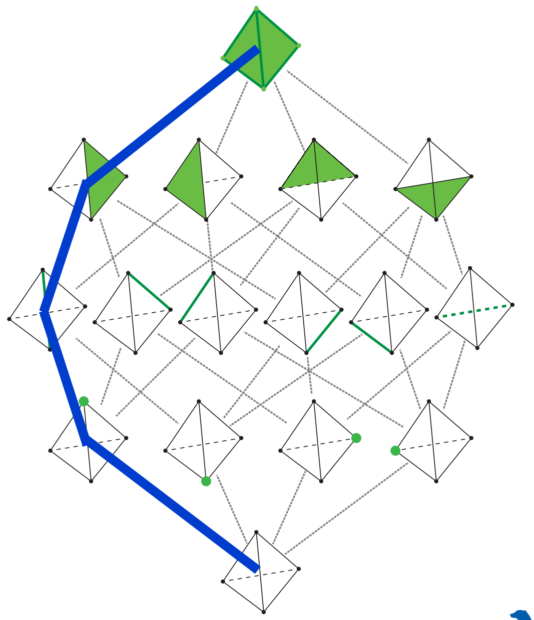

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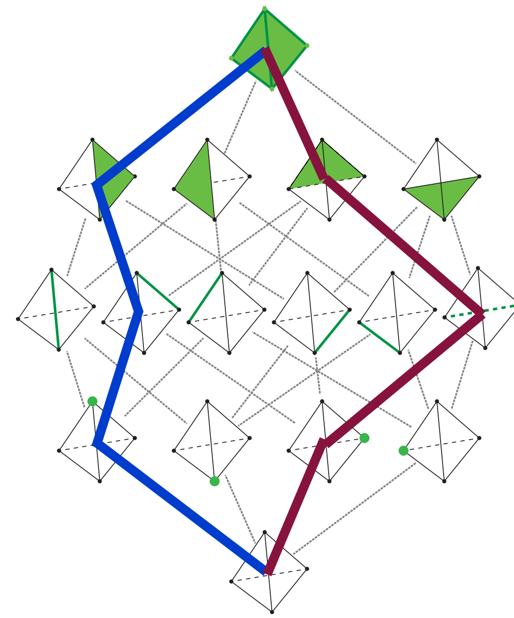

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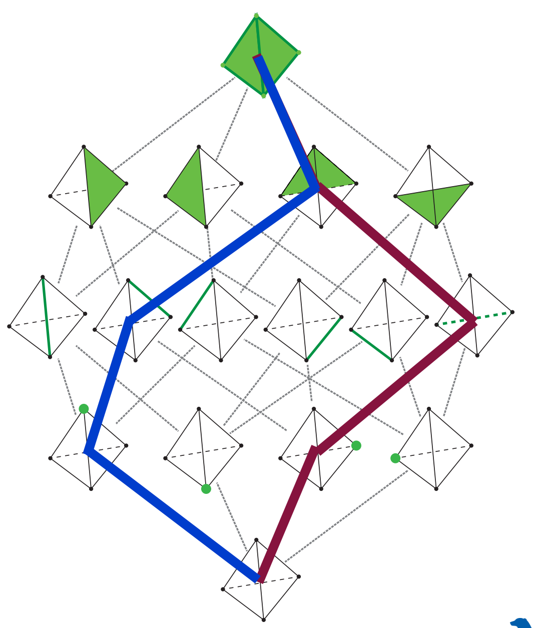

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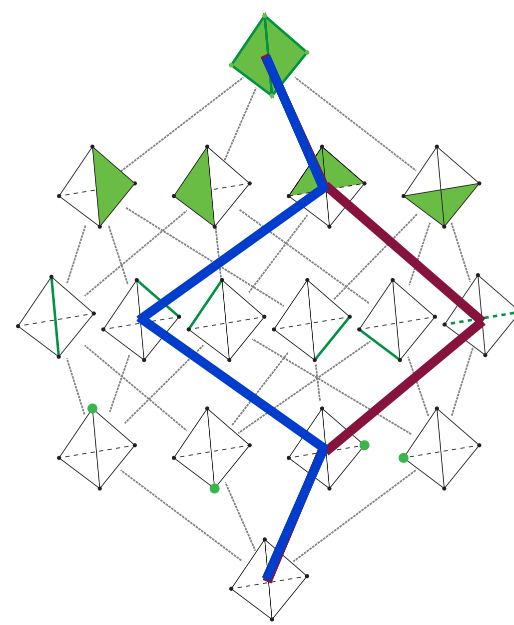

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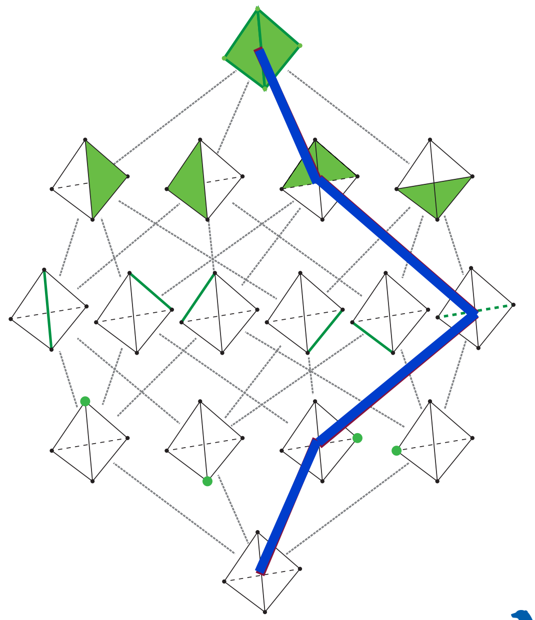

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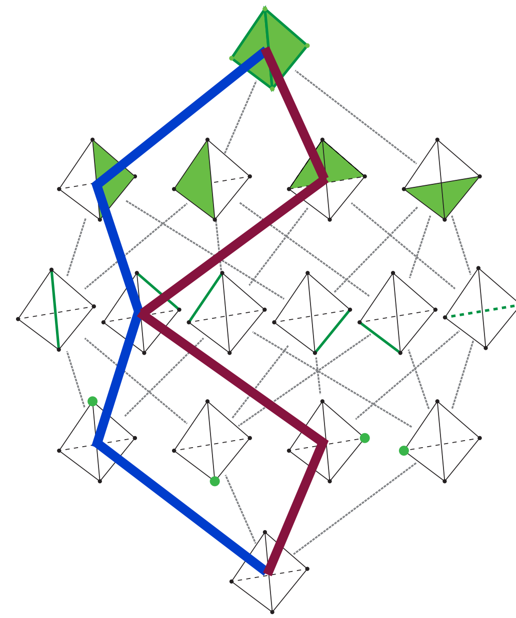

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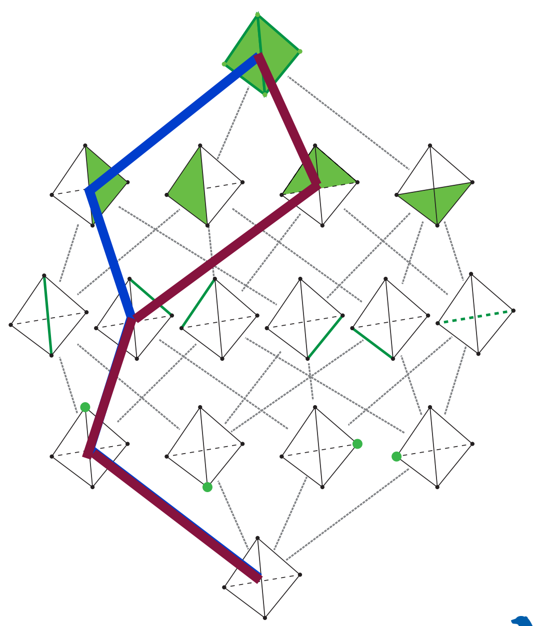

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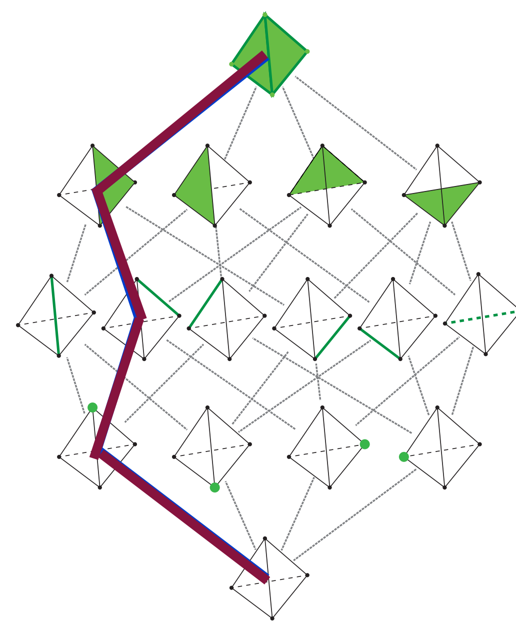
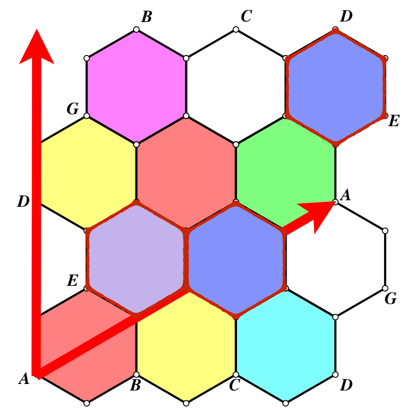


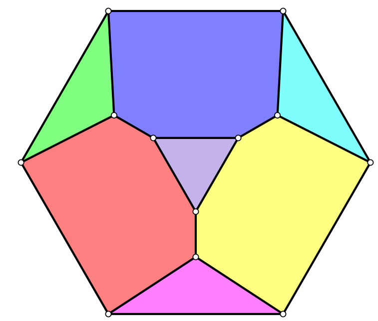
Quick Facts About Abstract Polytopes

- A polytope is *regular* if its automorphism group acts transitively on the flags.
- The *connection group* is Con$(\mathscr{P}) = \langle r_0, r_1, \ldots, r_{d-1} \rangle$
- A polytope $\mathscr{P}$ is a *cover* of a polytope $\mathscr{Q}$ if there exists a rank and adjacency preserving map $\pi : \mathscr{P} \to \mathscr{Q}$. We write $\mathscr{P} \searrow \mathscr{Q}$.
- A cover $\mathscr{R}$ of a polytope $\mathscr{P}$ is *minimal* if $\mathscr{R} \neq \mathscr{P}$ and $\mathscr{R} \searrow \mathscr{Q} \searrow \mathscr{P}$ implies $\mathscr{R} = \mathscr{Q}$ or $\mathscr{Q} = \mathscr{P}$.

A Regular Cover for the Truncated Tetrahedron



$\{6,3\}_{(2,2)}$



(3.6.6)

Quick Facts About Abstract Polytopes

- A *string group generated by involutions* (*sggi*) is a group generated by involutions $\rho_0, \rho_1, \ldots, \rho_{d-1}$ satisfying $(\rho_j \rho_k)^2 = 1$ if $|j - k| > 1$.
- A *string C-group* is an sggi satisfying the intersection condition:

$$\langle \rho_k \,|\, k \in I \rangle \cap \langle \rho_k \,|\, k \in J \rangle = \langle \rho_k \,|\, k \in I \cap J \rangle$$

- There is a 1-1 correspondence between regular abstract polytopes and string C-groups. The automorphism group of a regular abstract polytope is always a string C-group.

Polytopal Regular Covers

(Monson, Pellicer, W. 2012) The Tomotope has infinitely many distinct minimal regular covers.

(Monson, Pellicer, W. 2014)

- If the connection group of a polytope $\mathscr{P}$ is a string C-group, then $\mathscr{P}$ has a unique minimal regular cover.
  - We will say $\mathscr{P}$ is *C-connected.*
- The connection group of every abstract polyhedron is a string C-group, i.e., every polyhedron has a unique minimal regular cover.

Prisms and Covers

Turns out, they aren't boring:

- G. Cunningham, M. Mixer and G.W. have been developing a package library for GAP called RAMP (**R**esearch **A**ssistant for **M**aniplexes and **P**olytopes) to make working with maps, maniplexes and abstract polytopes *much* easier.
- We've been using RAMP to develop conjectures.
- There are also LOTS of 4-polytopes $\mathscr{P}$ that ***are*** C-connected. However…

But…

- Let $\mathscr{P} = \{\{6,3\}_{(2,0)},2\}$, then Con(Prism($\mathscr{P}$)) is ***not*** a string C-group.
- Let $\mathscr{P} = \{3,3,2\}$, then Con(Prism($\mathscr{P}$)) is ***not*** a string C-group.
- Let $\mathscr{P} = \{3,3,4\}$, then Con(Prism($\mathscr{P}$)) is ***not*** a string C-group.
- We also tested more than 1600 small regular polyhedra, and in each case, Prism($\mathscr{P}$) was C-connected. Hmm…

In fact…

Main Theorem (Cunningham, Mixer, W.) Let $\mathscr{B}$ be a polyhedron. Then Prism($\mathscr{B}$) is C-connected.

Showing a Group is String C

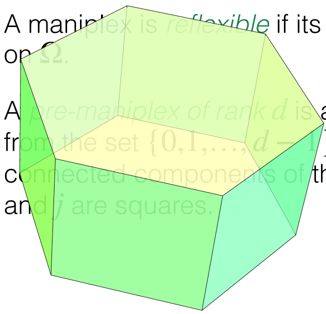
An approach:

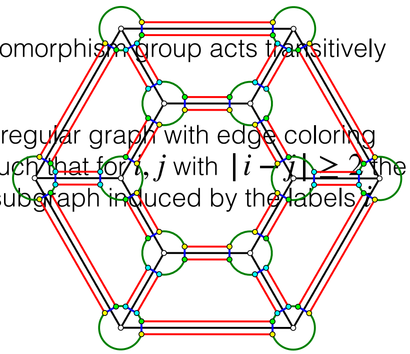
(ARP 2E17) Let Γ be an sggi, Δ a string C-group and $\pi : \Gamma \searrow \Delta$, where $\pi(\rho_i) = \sigma_i, \forall i$ and that is one-to-one on $\langle \rho_0, \ldots, \rho_{d-2} \rangle$ or $\langle \rho_1, \ldots, \rho_{d-1} \rangle$. Then Γ is also a string C-group.

(Pellicer, W. 2018) Let $\mathscr{P}$ be a polytope and $H \leq$ Aut($\mathscr{P}$) such that $\mathscr{Q} := \mathscr{P}/H$ is a pre-maniplex with the property that Con($\mathscr{Q}$) is a string C-group. Let $L := \{l \in \text{Con}(\mathscr{P}) \,|\, \forall \Phi \in \mathscr{F}(\mathscr{P}), \exists h \in H \text{ s.t. } l\Phi = \Phi h\}$. Finally, suppose that if $h \in H$ fixes an incident vertex-facet pair, then $h = 1$. Then $L \lhd$ Con($\mathscr{P}$), Con($\mathscr{P}$)$/L \cong$ Con($\mathscr{Q}$), and Con($\mathscr{P}$) is a string C-group.

Maniplexes???

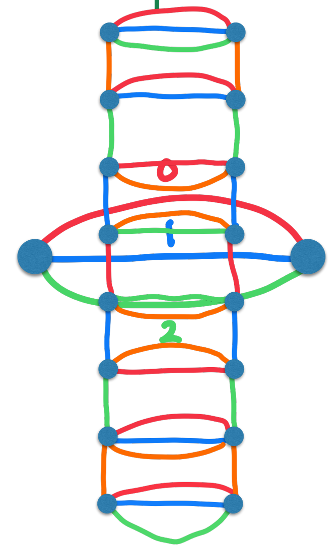
- A *maniplex $\mathcal{M}$ of rank d* is an ordered pair $(\Omega, [r_0, r_1, \ldots, r_{d-1}])$ where Ω is a set whose elements are called *flags*, and r_i is a fixed-point-free involution on Ω satisfying
  - Con$(\mathcal{M}) := \langle r_0, r_1, \ldots, r_{d-1} \rangle$ acts transitively on Ω
  - If $|i - j| \geq 2$ then $r_i r_j = r_j r_i$
  - If $i \neq j$ then r_i and r_j have no transpositions in common
- A maniplex is *reflexible* if its automorphism group acts transitively on Ω.
- A *pre-maniplex of rank d* is a d-regular graph with edge coloring from the set $\{0, 1, \ldots, d-1\}$, such that for i, j with $|i - j| \geq 2$ the connected components of the subgraph induced by the labels i and j are squares.





A Useful Pre-Maniplex

- $\mathcal{G}_3$ is the pre-maniplex with two flags of rank 3.
- $Q :=$ Prism$(\mathcal{G}_3)$
  - Con$(\mathcal{Q})$ is a string C-group



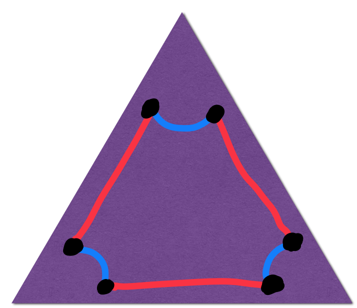
Stratified Operations

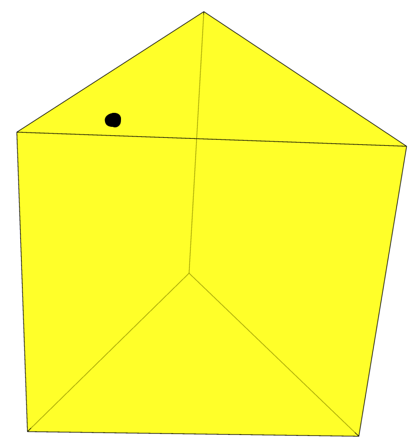
(From Cunningham, Pellicer, W.)

Let M_d be the set of all maniplexes of rank d.

An operation $F : M_n \to M_m$ is *stratified* if there is a set A (the *strata*) such that

- If Flags$(\mathcal{M}) = \Omega$, then Flags$(F(\mathcal{M})) \subseteq A \times \Omega$ s.t. the canonical projections are surjective.
- The universal string Coxeter group W_m has a well defined action on A.
- The action of W_m can be described *nicely* in terms of the natural action of W_n on Ω and the action of W_m on A.





Stratified Operations

A stratified operation F is *fully* stratified if Flags$(F(\mathscr{M})) = A \times \Omega$.

It is *cover-preserving* if, whenever $\mathscr{M} \searrow \mathscr{L}$, then $F(\mathscr{M}) \searrow F(\mathscr{L})$.

Proposition (Cunningham, Pellicer, W.) Fully-stratified operations are cover-preserving.

Proposition (CPW) Let $\mathscr{R}$ be a minimal regular cover of $\mathscr{P}$, then Con$(F(\mathscr{R})) \cong$ Con$(F(\mathscr{P}))$.

Proof for Prisms Over Polyhedra

We show the following:

1. Let $\mathscr{B}$ be an orientable regular polyhedron, then Con(Prism$(\mathscr{B})$) is a string C-group.
   - Proof uses fact Prism$(\mathscr{B}) \searrow$ Prism$(\mathscr{G}_3)$.
2. Let $\mathscr{M}$ be a non-orientable reflexible 3-maniplex and let $\mathscr{L}$ be its orientable double cover. Then the prism with base $\mathscr{L}$ is the same as the orientable double cover of the prism with base $\mathscr{M}$.
   - Proof uses fact that ODC is cover preserving.

Proof Steps

3. Recall (CunPelWil) that if F is a parallel product then F is cover-preserving and connection-preserving.

   - Also: Con$(\mathscr{M}) \cong$ Con(src$(\mathscr{M})$)

4. Show that the orientable cover operation is a parallel product operation on maniplexes.

Proof Steps

5. Show that if $\mathscr{M}$ is a non-orientable 3-maniplex and $\mathscr{L} =$ odc$(\mathscr{M})$, then src(Prism$(\mathscr{M})$) $=$ src(Prism$(\mathscr{L})$).

6. Complete the proof:
   - We have the case when $\mathscr{B}$ is orientable and reflexible.
   - By 5, holds for all reflexible polyhedra.
   - Note Con$(\mathscr{B})$ is a string C-group for any polyhedron.
   - Use result about Prism being a parallel product to generalize to all polyhedra.

Something Interesting About Cubes

- Observation 1: Let $\mathscr{B}$ be a regular polyhedron with automorphism group $\Gamma = \langle \rho_0, \rho_1, \rho_2 \rangle$, and let $\mathscr{P} = \text{Prism}(\mathscr{B})$. Then we may represent $\text{Con}(\mathscr{P}) = \langle s_0, s_1, s_2, s_3 \rangle \leq S_8 \wr \text{Aut}(\mathscr{B})$ with generators:
  - $s_0 := ((4,5), [\rho_0, \rho_0, \rho_0, e, e, \rho_0, \rho_0, \rho_0])$
  - $s_1 := ((3,4)(5,6), [\rho_1, \rho_1, e, e, e, e, \rho_1, \rho_1])$
  - $s_2 := ((2,3)(6,7), [\rho_2, e, e, \rho_1, \rho_1, e, e, \rho_2])$
  - $s_3 := ((1,2)(7,8), [e, e, \rho_2, \rho_2, \rho_2, \rho_2, e, e])$

Something Interesting About Cubes

- The interesting bit:
  - $(s_0 s_1)^4 = ((), [(\rho_0\rho_1)^4, (\rho_0\rho_1)^4, e, e, e, e, (\rho_0\rho_1)^4, (\rho_0\rho_1)^4])$
  - $(s_1 s_2)^3 = ((), [(\rho_1\rho_2)^3, e, e, e, e, e, e, (\rho_1\rho_2)^3])$
  - $(s_2 s_3)^3 = ((), [e, e, e, (\rho_1\rho_2)^3, (\rho_1\rho_2)^3, e, e, e])$
- So if $|(\rho_0\rho_1)| = 4$ and $|(\rho_1\rho_2)| = 3$ then $\mathscr{P}$ is the 4-cube (γ_4 in Coxeter's notation).

Something About Cubes

- Observation 2: Let $\mathscr{G}_d$ be the pre-maniplex with two flags of rank d. Then we can show that $\text{Con}(\text{Prism}(\mathscr{G}_d)) = \text{Con}(\gamma_d)$.
- Observation 3: Let $\mathscr{B}$ be a polyhedron, then $\text{Con}(\mathscr{B} \diamond \gamma_3)$ ***is*** a string C-group.
- Observation 4: (Theorem) Let $\mathscr{M}$ be a reflexible 3-maniplex, and let $\mathscr{P} = \text{Prism}(\mathscr{M})$. Then $\text{src}(\mathscr{P}) \searrow \gamma_4$ and $\text{src}(\mathscr{P}) \diamond \gamma_4 = \text{src}(\mathscr{P})$.

Something About Cubes

- Observation 5: For every *examined* $\mathscr{B}$ of rank>3: if $\text{Con}(\text{Prism}(\mathscr{B}))$ is ***not*** a string C-group *then* $\text{Con}(\mathscr{B} \diamond \gamma_d)$ is not a string C-group!

Conjecture: Let $\mathscr{B}$ be an abstract polytope of rank d, and let γ_d be the d-cube. Then $\text{Con}(\text{Prism}(\mathscr{B}))$ is a string C-group iff $\text{Con}(\mathscr{B} \diamond \gamma_d)$ is a string C-group.

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