Regular maps with primitive automorphism group

Gareth Jones

University of Southampton

Work in progress with Martin Mačaj, Comenius University, Bratislava

September 20, 2022

Action on vertices

Definition A map $\mathcal{M}$ is regular if $\mathrm{Aut}\,\mathcal{M}$ acts transitively on flags.

$\mathcal{M}$ is orientably regular if it is orientable and the orientation-preserving automorphism group $\mathrm{Aut}^+\mathcal{M}$ acts transitively on arcs.

Each of these conditions implies transitivity on vertices (and faces), but what about primitivity (a stronger condition)?

Definition A permutation group is primitive if it preserves no non-trivial equivalence relations.
Equivalently the point stabilisers are maximal subgroups.

Examples Every doubly transitive group is primitive.

A dihedral group D_n, acting naturally on the vertices of an n-gon, is primitive if and only if n is prime.

The automorphism groups of the tetrahedron and cube are primitive (2-transitive) and imprimitive (antipodality) on vertices.

Fact: each normal subgroup ($\neq 1$) of a primitive group is transitive.

The orientably regular case

Theorem (J, 2013) Let $\mathcal{M}$ be orientably regular. Then $G := \mathrm{Aut}^+\mathcal{M}$ acts primitively and faithfully on the vertices of $\mathcal{M}$ if and only if $\mathcal{M}$ is a generalised Paley map.

This is a Cayley map with vertex set $V = \mathbb{F}_q$ (finite field, $q = p^d$), with generating set a subgroup $G_0 \leq \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$ acting on V as an irreducible subgroup of $\mathrm{GL}_d(p)$, so $G = V \rtimes G_0 \leq \mathrm{AGL}_d(p)$.

The neighbours of 0 are $1, c, c^2, \ldots$ where c generates G_0.

Examples If $G_0 = \mathbb{F}_q^*$ then $\mathcal{M}$ is a Biggs embedding of a complete graph K_q.

If $q \equiv 1 \bmod (4)$ and the index $|\mathbb{F}_q^* : G_0| = 2$ then $\mathcal{M}$ is an embedding of a Paley graph.

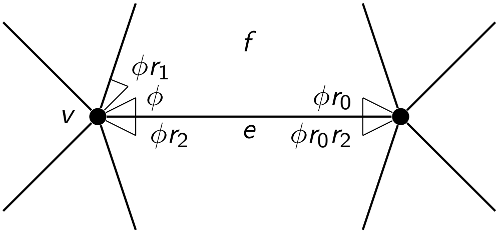
Deleting 'faithfully' from the theorem allows just regular cyclic covering of these maps, branched over the vertices.

Regular maps (without boundary)

$G \cong \mathrm{Aut}\,\mathcal{M}$ for a regular map $\mathcal{M}$ iff $G = \langle r_0, r_1, r_2 \rangle$ where

$$r_i^2 = 1 \quad \text{and} \quad r_0 r_2 = r_2 r_0.$$

Each r_i is a reflection, changing the i-dimensional component of an incident vertex-edge-face flag $\phi = (v, e, f)$.



Then $r_1 r_2$, $r_2 r_0$ and $r_0 r_1$ are rotations leaving v, e and f invariant.

They have orders n, 2 and m where $\mathcal{M}$ has type $\{m, n\}$.

Vertex-primitive regular maps

$G = \mathrm{Aut}\,\mathcal{M}$ for a regular map $\mathcal{M}$ iff $G = \langle r_0, r_1, r_2 \rangle$ where

$$r_i^2 = 1 \quad \text{and} \quad r_0 r_2 = r_2 r_0.$$

The stabilisers of v, e and f in G are the dihedral subgroups

$$G_v = \langle r_1, r_2 \rangle, \quad G_e = \langle r_2, r_0 \rangle, \quad G_f = \langle r_0, r_1 \rangle.$$

G acts primitively on the vertices if and only if $D := G_v$ is a maximal subgroup of G, as we will assume from now on.

Let R be the set of involutions in $C_G(r_2) \setminus D$ ($C_G :=$ centraliser). If $r \in R$ then $r^2 = 1$, $rr_2 = r_2 r$ and $\langle r, r_1, r_2 \rangle = G$, so taking $r_0 = r$ gives a regular map $\mathcal{M}$ with $\mathrm{Aut}\,\mathcal{M} \cong G$ and $G_v = D$.

Choices $r, r' \in R$ give $\mathcal{M} \cong \mathcal{M}'$ iff $r \mapsto r'$ under an automorphism of G fixing D, so the number of maps is $|R|/|C_A(D)|$, $A := \mathrm{Aut}\,G$.

Properties of $\mathcal{M}$

$\mathcal{M}$ is orientable iff each $r_i \in G \setminus G^+$ for some $G^+ < G$ of index 2.

$\mathcal{M}$ has extended type $\{m, n\}_l$ where n, m, l (valencies of vertices, faces and Petrie polygons) are the orders of $r_1 r_2$, $r_0 r_1$ and $r_0 r_1 r_2$.

If $r \in R$ then $r' := r r_2 \in R$; replacing r_0 with $r_0 r_2$ gives the Petrie dual $\mathcal{M}' = P(\mathcal{M})$ of $\mathcal{M}$, so these maps $\mathcal{M}$ form Petrie dual pairs, with the same G, D and action on the vertices, but different faces.

$\mathcal{M}$ has Euler characteristic

$$\chi = \frac{|G|}{2}\left(\frac{1}{m} - \frac{1}{2} + \frac{1}{n}\right),$$

and therefore has genus

$$g = 1 - \frac{\chi}{2} \quad \text{or} \quad 2 - \chi$$

as $\mathcal{M}$ is orientable or not.

Which groups G arise?

The O'Nan–Scott Theorem ($\sim$ 1980) divides finite primitive permutation groups G into a small number of classes.

The most important classes (here and in general) are:

- affine groups $G = V \rtimes G_0 \leq \mathrm{AGL}_d(p)$ with $V \cong \mathbb{F}_p^d$ acting regularly and $G_0 \leq \mathrm{GL}_d(p)$ acting irreducibly on V;
- almost simple groups G with $T \leq G \leq \mathrm{Aut}\, T$ for some nonabelian finite simple group T, e.g. $G = \mathrm{S}_n$ or A_n, $n \geq 5$.

Other examples (without dihedral stabilisers, so not arising here):

- $G = T \times T$, T as above, acting on T by $(t_1, t_2) : t \mapsto t_1^{-1} t t_2$;
- $G = \mathrm{S}_n \wr \mathrm{S}_d = (\mathrm{S}_n \times \cdots \times \mathrm{S}_n) \rtimes \mathrm{S}_d$ (wreath product) acting on N^d, $N = \{1, \ldots, n\}$ (automorphisms of a Hamming graph);
- further variations on these examples.

Maps with almost simple groups

Building on work of Li (2009) we have:

Theorem

An almost simple primitive group G has point stabilisers $D \cong \mathrm{D}_n$ if and only if one of the following occurs:

1. $G \cong \mathrm{PSL}_2(q)$ *for odd* $q > 11$, *with* $n = (q-1)/2$;.
2. $G \cong \mathrm{PGL}_2(q)$ *for odd* $q > 5$, *with* $n = q - 1$;
3. $G \cong \mathrm{PSL}_2(q) = \mathrm{SL}_2(q)$ *for* $q = 2^e \geq 4$, *with* $n = q - 1$;
4. $G \cong \mathrm{PSL}_2(q)$ *for odd* $q > 9$, *with* $n = (q+1)/2$;
5. $G \cong \mathrm{PGL}_2(q)$ *for odd* $q > 5$, *with* $n = q + 1$;
6. $G \cong \mathrm{PSL}_2(q) = \mathrm{SL}_2(q)$ *for* $q = 2^e \geq 4$, *with* $n = q + 1$;
7. $G \cong \mathrm{Sz}(q)$ *for* $q = 2^e$ *and odd* $e > 1$, *with* $n = q - 1$.

In each case G has a single conjugacy class of maximal subgroups $D \cong \mathrm{D}_n$ for the specified value of n.

$\mathrm{Sz}(q)$ is a Suzuki group, a simple subgroup of $\mathrm{Sp}_4(q)$.

In cases $i = 1, \ldots, 7$ we enumerated and described the maps $\mathcal{M}$. The number $\nu_i(q)$ of them is positive for all relevant i and q. They are all non-orientable.

Example: Case (6), $G = \mathrm{SL}_2(q)$ for $q = 2^e \geq 4$, with $n = q + 1$.

$$\nu_6(q) = \frac{(q-2)\phi(q+1)}{2e}.$$

If $q = 4$, so that $G = \mathrm{SL}_2(4) \cong \mathrm{A}_5$, we obtain two maps:

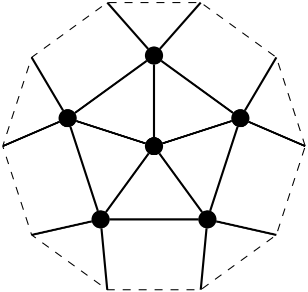
- the antipodal quotient of the icosahedron, of type $\{3,5\}_5$, and
- its Petrie dual N5.3 (in Marston's list), the antipodal quotient of the great dodecahedron, of type $\{5,5\}_3$ and genus 5.

If $q = 8$ we obtain six maps:

- N16.1 of type $\{3,9\}_7$ & its Petrie dual N64.3 of type $\{7,9\}_3$,
- a Petrie dual pair N64.4 and N64.5 of type $\{7,9\}_7$, and
- a dual pair N72.9 of type $\{9,9\}_9$.

If $q = 16$ or 32 we obtain 28 or 60 maps, and so on.

Example for case (6)



Identify opposite boundary points to give the antipodal quotient of the icosahedron, a non-orientable regular embedding of K_6, with automorphism group $G = \mathrm{SL}_2(4) \cong \mathrm{A}_5$ acting primitively (in fact 2-transitively, as $\mathrm{PSL}_2(5)$) on the vertices.

Example: Case (7), $G = \mathrm{Sz}(q)$, $q = 2^e$ (odd $e \geq 3$), $n = q - 1$.

$$\nu_7(q) = \frac{(q-2)\phi(q-1)}{2e},$$

the same as for $\mathrm{PGL}_2(q)$ and $\mathrm{SL}_2(q)$ with $n = q - 1$.

If $q = 8$ ($|G| = 29\,120$) we have six maps, each of genus $g \geq 2290$.

A computer search shows that they have the following types:
$\{5,7\}_{13}$, $\{13,7\}_5$, $\{7,7\}_7$ (two maps) and $\{13,7\}_{13}$ (two maps).

If $q = 32$ or 128 there are 90 or 1134 maps, and so on.

Maps with affine groups

Jajcay, Li, Širáň and Wang (2019) considered quasiprimitive groups $\mathrm{Aut}\,\mathcal{M}$ (all non-trivial normal subgroups transitive). They are more general than primitive groups, but equivalent for affine groups.

In this case $G = V \rtimes D$ with $V \cong \mathbb{F}_p^d$ and $D = G_0 = \langle r_1, r_2 \rangle \cong \mathrm{D}_n$ acting on the vertex set V as an irreducible subgroup of $\mathrm{GL}_d(p)$.

Here $d = 2e$ where $e := \min\{i \geq 1 \mid p^i \equiv \pm 1 \bmod (n)\}$.

Case A: $p^e \equiv -1$ mod (n), so p has order d in $\mathbb{Z}_n^*$.
Then V is an irreducible module for $D^+ := \langle r_1 r_2 \rangle \cong \mathrm{C}_n \leq \mathbb{F}_{p^d}^*$, which acts as a Singer subgroup inverted by elements of $D \setminus D^+$.

Case B: $p^e \equiv 1$ mod (n), so p has order e in $\mathbb{Z}_n^*$.
Then V is a reducible D^+-module $V_1 \oplus V_2$, with V_i irreducible e-dimensional D^+-modules transposed by elements of $D \setminus D^+$.

In each case $G \rhd_2 G^+ := V \rtimes D^+$, and the Petrie dual pairs consist of an orientable map $\mathcal{M}^+$ and a non-orientable map $\mathcal{M}^-$.

In case B $\mathcal{M}^+$ is the regular cover of a chiral pair $\mathcal{M}_i = \mathcal{M}^+/V_i$.

Theorem (Jajcay, Li, Širáň and Wang; Mačaj & J)

(1) *If* $n \equiv 0$ (4) *and* $p > 2$, *or if* n *is odd and* $p = 2$, $\mathcal{M}$ *is*

- *an orientable map* $\mathcal{M}^+$ *of type* $\{n, n\}_{2p}$ *and genus* $g^+ = 1 + p^d(n-4)/4$, *or*
- *its Petrie dual* $\mathcal{M}^-$, *a non-orientable map of type* $\{2p, n\}_n$ *and genus* $g^- = 2 + p^{d-1}(np - n - 2p)/2$;

$\mathcal{M}^+$ *is self-dual unless* $4 < n \equiv 0$ (4), *when it is in a dual pair.*

(2) *If* $n \equiv 2$ (4) *and* $p > 2$, $\mathcal{M}$ *is*

- *an orientable map* $\mathcal{M}^+$ *of type* $\{n/2, n\}_{2p}$ *and genus* $g^+ = 1 + p^d(n-6)/4$, *or*
- *its Petrie dual* $\mathcal{M}^-$, *a non-orientable map of type* $\{2p, n\}_{n/2}$ *and genus* $g^- = 2 + p^{d-1}(np - n - 2p)/2$.

(3) *There are* $\phi(n)/d$ *Petrie dual pairs* $\mathcal{M}^\pm$ *(hence* $\phi(n)/e$ *maps), with the* $\mathcal{M}^+$ *equivalent under hole operations* H_j, j *coprime to* n.

(4) *Each* $\mathcal{M}_i$ *has genus* $1 + (g^+ - 1)/p^e$ *and the same type as* $\mathcal{M}$.

Examples

There are no regular maps $\mathcal{M}$ associated with G unless either

- n is odd and $p = 2$, or
- n is even and $\gcd(n, p) = 1$.

First consider odd n, so that $p = 2$.

Example: $n = 3$

Since $2^1 \equiv -1 \bmod (3)$ use case A with $e = 1$ and $d = 2$, so $G = V \rtimes D \cong \mathbb{F}_4 \rtimes \mathrm{D}_3 \cong \mathrm{S}_4$ with $D^+ \cong \mathrm{C}_3$ acting irreducibly.

There are $\phi(3)/e = 2/1 = 2$ regular maps $\mathcal{M}$:

- the tetrahedron $\mathcal{M}^+ = \{3, 3\}$, of genus 0 and type $\{3, 3\}_4$,
- its Petrie dual $\mathcal{M}^-$ of genus 1 and type $\{4, 3\}_3$, the antipodal quotient of the cube.

Both maps are regular embeddings of the complete graph K_4.

Example: $n = 7$

Since 2 has order $e = 3 \bmod (7)$ and $2^i \not\equiv -1 \bmod (7)$ for any i, use case B with $d = 6$, $G \cong (\mathbb{F}_8 \oplus \mathbb{F}_8) \rtimes \mathrm{D}_7$.

There are $\phi(7)/3 = 6/3 = 2$ regular maps $\mathcal{M}$:

- an orientable map $\mathcal{M}^+ = $ R49.57, of type $\{7,7\}_4$, and
- its non-orientable Petrie dual $\mathcal{M}^- = $ N50.5, of type $\{4,7\}_7$.

$\mathcal{M}^+$ is the minimal regular cover of the chiral pair $\mathcal{M}_i = $ C7.2 of Edmonds embeddings of the complete graph K_8, each of type $\{7,7\}_4$ with automorphism group $\mathrm{AGL}_1(8) \cong \mathbb{F}_8 \rtimes \mathrm{C}_7$.

Even n

If n is even then $\gcd(p, n) = 1$, so that $n \mid p^d - 1$ for some d.

Example: $n = 4$

Each prime $p \equiv 1 \bmod (4)$ has order $e = 1$ in $\mathbb{Z}_4^*$, so taking $d = 2$ in case B we have $V = V_1 \oplus V_2 \cong \mathbb{F}_p \oplus \mathbb{F}_p$ and $G \cong V \rtimes \mathrm{D}_4$ of order $8p^2$, with $D^+ \cong \mathrm{C}_4$ acting irreducibly on each V_i.

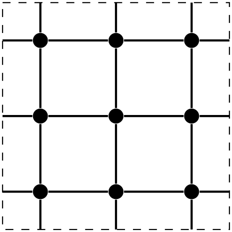
For primes $p \equiv -1 \bmod (4)$ we have $e = 1$ in case A, with $d = 2$ again, so $G \cong \mathbb{F}_{p^2} \rtimes \mathrm{D}_4$ with $D^+ \cong \mathrm{C}_4$ acting irreducibly on V.

For each $p > 2$ there are $\phi(4)/e = 2/1 = 2$ regular maps:

- the torus map $\mathcal{M}^+ = \{4,4\}_{p,0}$ of type $\{4,4\}_{2p}$, with chiral quotients $\mathcal{M}_i = \{4,4\}_{a,b}$, $\{4,4\}_{b,a}$ if $p = a^2 + b^2 \equiv 1$ (4),
- its Petrie dual $\mathcal{M}^-$, a non-orientable map of genus $(p-1)^2 + 1$ and type $\{2p, 4\}_4$.

When $p = 3, 5, 7$ or 11, $\mathcal{M}^-$ is N5.2, N17.2, N37.1 or N101.3 in Marston's list.

Example: $n = 4, p = 3$



Identify opposite sides of the square to give the torus map $\{4, 4\}_{3,0}$, a regular embedding of the Paley graph P_9, with automorphism group $G = \mathbb{F}_9 \rtimes \mathrm{D}_4$ acting primitively on the vertices.

$n = 10$ or 12, $p = 11$

If $n = 10$ then $11^1 \equiv 1 \bmod (10)$ and no $11^i \equiv -1 \bmod (10)$, so take $e = 1$, $d = 2$ in case B, with $G \cong (\mathbb{F}_{11} \oplus \mathbb{F}_{11}) \rtimes \mathrm{D}_{10}$.

There are $\phi(10)/e = 4/1 = 4$ regular maps $\mathcal{M}$:

- $\mathcal{M}^+ =$ R122.3 and R122.4 of type $\{5, 10\}_{22}$,
- $\mathcal{M}^- =$ N431.1 and N431.2 of type $\{22, 10\}_5$.

The maps $\mathcal{M}^+$ are the minimal regular covers of the chiral pairs $\mathcal{M}_i =$ C12.1 and C12.2 of orientably regular embeddings of K_{11}.

If $n = 12$ then $11^1 \equiv -1 \bmod (12)$, so take $e = 1$, $d = 2$ in case A, with $G \cong \mathbb{F}_{11^2} \rtimes \mathrm{D}_{12}$.

There are $\phi(12)/e = 4/1 = 4$ regular maps $\mathcal{M}$:

- a dual pair $\mathcal{M}^+ =$ R243.16 of type $\{12, 12\}_{22}$,
- $\mathcal{M}^- =$ N541.6 and N541.7 of type $\{22, 12\}_{12}$.

Generalisations

Non-faithful primitive action: As in the orientably regular case, the only possibilities are regular cyclic coverings, branched over the vertices. (See Li & Širáň, 2005, for general non-faithful actions.)

Primitivity on faces: we just get the duals of the maps already considered, with the same groups and surfaces.

Hypermaps: Here almost everything is as for maps, with dihedral stabilisers implying that G is either almost simple or affine. However we lose the relation $r_0 r_2 = r_2 r_0$, so instead of looking for involutions $r_0 \in C_G(D) \setminus D$ we look for them in $G \setminus D$, giving many more hypermaps than maps, not yet closely studied.

Example: almost simple case (6), $q = 4$, $n = 5$, $G = \mathrm{SL}_2(4) \cong \mathrm{A}_5$; there are two maps and eight proper hypermaps.

Also, there are affine examples with n and p both odd, e.g. RPH6.1, of type $(3, 5, 3)$, has $n = 3, p = 5$ and $G \cong \mathbb{F}_{5^2} \rtimes \mathrm{D}_3$.

Thank you for listening!