Maps and maximal subgroups

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February 14, 2022

Outline of the talk

The modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\} \cong \mathrm{C}_3 * \mathrm{C}_2$ is important in several areas of mathematics, such as number theory, Riemann surfaces, hyperbolic geometry, etc.

Much is known about its subgroups of finite index, but less seems to be known about its subgroups of infinite index.

Planar maps give simple constructions of uncountably many conjugacy classes of maximal subgroups of infinite index in Γ and in other groups, generalising results of Neumann, Magnus and others.

Bernhard Neumann

Wilhelm Magnus

In turn these yield results and conjectures about the realisation of groups as automorphism groups of maps and hypermaps.

Actions of Γ

$\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acts by Möbius transformations

$$t \mapsto \frac{at+b}{ct+d} \quad (a,b,c,d \in \mathbb{Z}, ad-bc=1)$$

on

- the upper half plane $\mathcal{U} \subset \mathbb{C}$ (a model of the hyperbolic plane),
- the real projective line $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ (the boundary of $\mathcal{U}$),
- the rational projective line $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\} \subset \mathbb{P}^1(\mathbb{R})$.

The action of Γ on $\mathbb{P}^1(\mathbb{Q})$ is transitive; the subgroup Γ_∞ fixing ∞ is the infinite cyclic group $P = \langle Z : t \mapsto t+1 \rangle$.

A non-identity element of Γ is parabolic if it has a single fixed point in $\mathbb{P}^1(\mathbb{Q})$, or equivalently has trace $a+d=\pm 2$; the parabolic elements of Γ are the conjugates of the powers Z^i $(i \neq 0)$ of Z.

Triangle groups

For $p, q, r \in \mathbb{N} \cup \{\infty\}$ define the triangle group $\Delta = \Delta(p, q, r)$ generated by rotations X, Y, Z through $2\pi/p$, $2\pi/q$, $2\pi/r$ around the vertices of a triangle T with internal angles π/p, π/q, π/r.

Δ acts on a surface S^2, $\mathbb{C}$ or $\mathbb{H}$ (the hyperbolic plane) as

$$p^{-1} + q^{-1} + r^{-1} > 1, \; = 1 \;\text{ or }\; < 1,$$

where $\infty^{-1} := 0$, and 'angle 0' means 'parallel', meeting at infinity.

Δ has a presentation

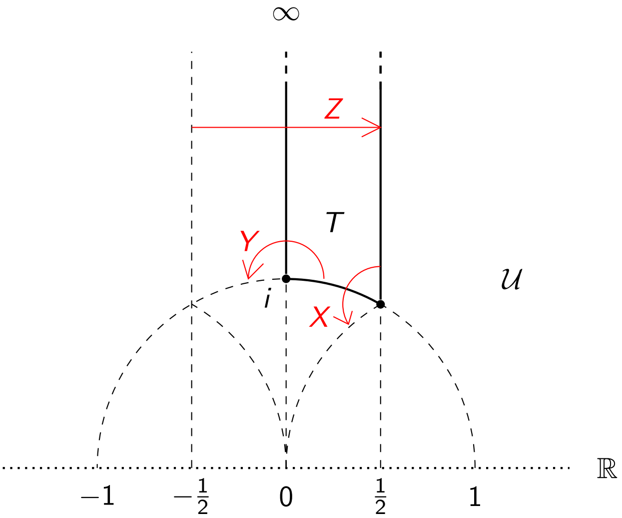
$$\Delta(p, q, r) = \langle X, Y, Z \mid X^p = Y^q = Z^r = XYZ = 1 \rangle.$$

Ignore any relations $W^\infty = 1$, so if $r = \infty$ eliminating Z gives

$$\Delta(p, q, \infty) = \langle X, Y \mid X^p = Y^q = 1 \rangle \cong \mathrm{C}_p * \mathrm{C}_q \quad \text{(free product)}.$$

Example $\Gamma \cong \Delta(3, 2, \infty) \cong \mathrm{C}_3 * \mathrm{C}_2$; X and Y have orders 3 and 2.

Generators X, Y, Z of Γ acting on $\mathbb{H} = \mathcal{U} \subset \mathbb{C}$



X, Y are rotations of orders 3 and 2, Z is a translation $t \mapsto t + 1$.

Neumann's results

In 1933 B. H. Neumann, responding to a question from Schmidt about the foundations of geometry, used permutations to construct uncountably many subgroups of $\mathrm{SL}_2(\mathbb{Z})$ which act regularly on the primitive elements of $\mathbb{Z}^2$, those $(u, v) \in \mathbb{Z}^2$ with u and v coprime.

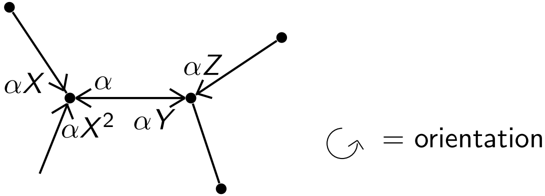
As pointed out by Magnus (1973), and in his book *Noneuclidean Tesselations and their Groups* (1974), the images of these subgroups in the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ are maximal nonparabolic subgroups, that is, maximal with respect to containing no parabolic elements.

Magnus conjectured that Neumann had constructed all the maximal nonparabolic subgroups of Γ, but further examples were subsequently found by Tretkoff (1975) and by Brenner and Lyndon (1983, 1984). They all used complicated constructions, involving permutation representations of Γ, but simple diagrams are better.

The modular group and cubic maps

Let $\mathcal{M}$ be a map (a connected graph, embedded without crossings, and with simply connected faces) on an oriented surface. Assume that $\mathcal{M}$ is cubic (all vertex valencies divide 3), and allow free edges (only one vertex). Let Ω be the set of arcs (directed edges) of $\mathcal{M}$.

Since $\Gamma = \langle X, Y \mid X^3 = Y^2 = 1 \rangle$, one can define a transitive action of Γ on Ω by letting X rotate arcs around their incident vertices (following the orientation), and Y reverse arcs, so 1-valent vertices and free edges give fixed points of X and Y.



Vertices, edges and faces correspond to cycles of X, Y and Z.

Conversely, given any transitive action of Γ one can define an oriented cubic map: the vertices, edges and faces are the cycles of X, Y and Z, with incidence given by non-empty intersection, and cyclic order within cycles giving the orientation.

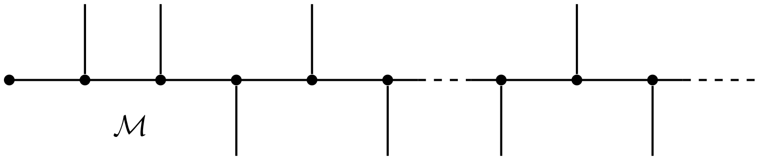
For any map $\mathcal{M}$ (cubic, oriented), the map subgroups

$$M = \Gamma_\alpha = \{g \in \Gamma \mid \alpha g = \alpha\} \quad (\alpha \in \Omega)$$

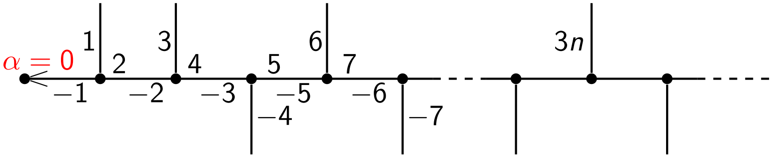
are mutually conjugate. They have index $|\Gamma : M| = |\Omega|$, and are maximal if and only if Γ acts primitively on Ω, i.e. preserves no non-trivial equivalence relations on Ω.

Proposition

The map subgroups for the following map $\mathcal{M}$ are maximal in Γ and are non-parabolic. (Hence they are maximal non-parabolic.)

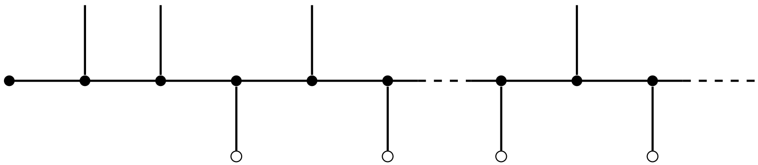


Proof. Here is $\mathcal{M}$, with α the left-most arc. There is a unique face, so Z has a single cycle on Ω, with each arc $\alpha Z^i \in \Omega$ labelled $i \in \mathbb{Z}$.



Any non-trivial Γ-invariant equivalence relation $\equiv$ on Ω must be invariant under Z, which acts on labels by $i \mapsto i+1$, so $\equiv$ must be congruence mod (n) on $\mathbb{Z}$ for some $n \geq 2$. However, Y transposes 0 and -1, and fixes $3n$, so it both moves and preserves the congruence class $[0] = [3n]$, a contradiction. Hence Γ acts primitively on Ω, so the map subgroups M are maximal in Γ. Since Z has no finite cycles on Ω, Z^i has no fixed points in Ω for $i \neq 0$, so each subgroup M is non-parabolic. $\square$

One can modify this construction to give $2^{\aleph_0}$ conjugacy classes of non-parabolic maximal subgroups M of Γ by adding 1-valent vertices to an arbitrary set of free edges 'below the horizontal axis', as indicated by the white vertices:



This changes the labelling of arcs with labels $i < -3$ (those below the axis), but the proof given earlier is still valid. There are $2^{\aleph_0}$ choices for the set of new vertices, giving $2^{\aleph_0}$ non-isomorphic maps; these give $2^{\aleph_0}$ inequivalent primitive actions of Γ and hence $2^{\aleph_0}$ conjugacy classes of non-parabolic maximal subgroups M.

Each M is a free product of copies of C_3 and C_2, one for each fixed point of X or Y, i.e. each 1-valent vertex or free edge.

Factorisations and regular subgroups

Let (G, Ω) be a transitive permutation group, $\alpha \in \Omega$. A subgroup $R \leq G$ is regular on Ω if and only if $G = G_\alpha R$ and $G_\alpha \cap R = 1$.

If G preserves a binary relation on Ω we get a graph $\mathcal{G}$ with $G \leq \mathrm{Aut}\,\mathcal{G}$. Then $\mathcal{G}$ is a Cayley graph for R.

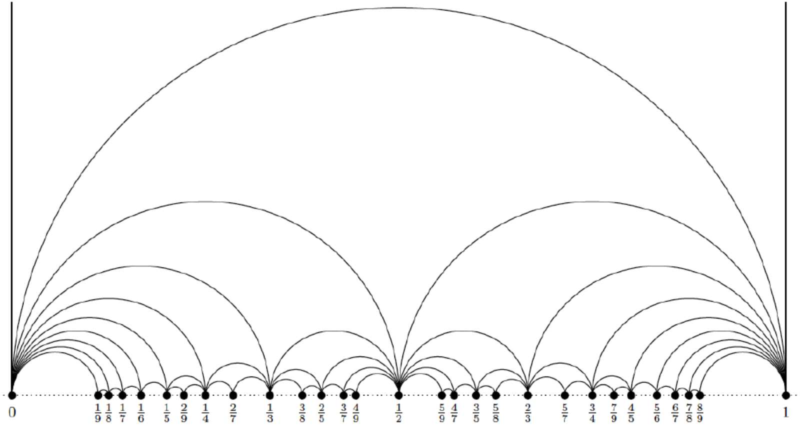
In the construction of maximal subgroups $M \leq \Gamma$, the subgroup $P := \langle Z \rangle$ is regular on Ω, so $\Gamma = MP$ and $M \cap P = 1$.

By symmetry $\Gamma = PM$ and $P \cap M = 1$. Since $P = \Gamma_\infty$ in the action of Γ on $\mathbb{P}^1(\mathbb{Q})$, it follows that each M is regular on $\mathbb{P}^1(\mathbb{Q})$.

Γ preserves the Farey graph $\mathcal{F}$ on $\mathbb{P}^1(\mathbb{Q})$, so each of these maximal subgroups $M \leq \Gamma$ has $\mathcal{F}$ as a Cayley graph.

They form $2^{\aleph_0}$ conjugacy classes in Γ and $\aleph_0$ isomorphism classes.

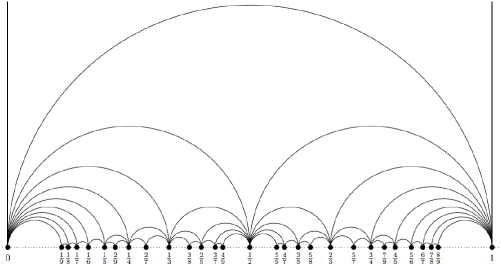
The Farey graph $\mathcal{F}$



The Farey graph $\mathcal{F}$ has vertex set $\mathbb{P}^1(\mathbb{Q})$, with a/b and c/d adjacent (by a hyperbolic geodesic) if and only if $ad - bc = \pm 1$.

The above pattern repeats in both directions, with period 1.
The vertical edges 'meet' at $\infty = 1/0$. (Diagram by Jan Karabás.)

The Farey graph $\mathcal{F}$



Example $\infty = 1/0$ is adjacent to the vertices $c/1 = c \in \mathbb{Z}$.

Example $0 = 0/1$ is adjacent to their inverses $1/d$ for $d \in \mathbb{Z}$.

$\mathrm{Aut}\,\mathcal{F} = \mathrm{PGL}_2(\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Z})/\{\pm I\}$ contains Γ with index 2.

The embedding of $\mathcal{F}$ in $\mathcal{U} \cup \mathbb{P}^1(\mathbb{Q})$ is the universal triangular map; it has every triangular map as a quotient. Its dual is the universal cubic map, which has every cubic map as a quotient.

Generalisation to other triangle groups

Theorem (J, 2019)

If $p \geq 3$ and $q \geq 2$ then the triangle group $\Delta(p, q, \infty) \cong C_p * C_q$ has $2^{\aleph_0}$ conjugacy classes of non-parabolic maximal subgroups.

Outline proof. If $q = 2$ then the construction is similar to that for the modular group (where $p = 3$), but using p-valent planar maps.

If $p, q \geq 3$ a similar but more complicated construction is required, using bipartite planar maps with black and white vertices of valencies dividing p and q; in this case, the generators X and Y of order p and q permute the set Ω of edges of the map, rotating them around their incident black and white vertices.

In all cases the map used has a single face, so Z has a single cycle which can be identified with $\mathbb{Z}$, allowing a proof of primitivity. □

The Realisation Problem

Given a group A and class $\mathfrak{C}$ of mathematical objects, is A isomorphic to $\mathrm{Aut}_{\mathfrak{C}}\,\mathcal{O}$ for some object $\mathcal{O} \in \mathfrak{C}$?

Theorem (Frucht, 1939)

Every finite group is isomorphic to the automorphism group of a finite graph.

Theorem (Sabidussi, 1960)

Every group is isomorphic to the automorphism group of a graph.

There are similar results for many other classes of objects, e.g. Riemann surfaces, fields, hyperbolic manifolds, polytopes, etc.

Theorem (Cori and Machì, 1982)

Every finite group is isomorphic to the automorphism group of a finite oriented map.

Can one extend this result to infinite groups?

Theorem (J, 2019)

If $p \geq 3$ then given any countable group A there are $2^{\aleph_0}$ non-isomorphic p-valent oriented maps $\mathcal{M}$ with $\mathrm{Aut}\,\mathcal{M} \cong A$.

Proof p-valent oriented maps $\mathcal{M}$ correspond to permutation representations $\Delta \to G \leq S := \mathrm{Sym}(\Omega)$ of $\Delta = \Delta(p, 2, \infty)$, or equivalently to conjugacy classes of subgroups $M \leq \Delta$. Then

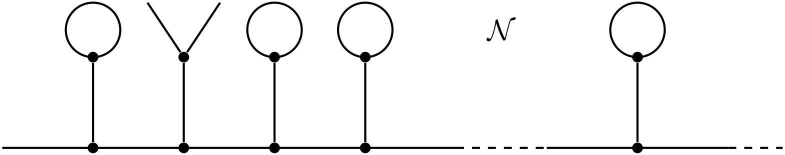
$$\mathrm{Aut}\,\mathcal{M} \cong C_S(G) \cong N_\Delta(M)/M,$$

where C and N denote centraliser and normaliser.

Therefore, to realise a group A as $\mathrm{Aut}\,\mathcal{M}$ for such a map $\mathcal{M}$ it is sufficient to find a subgroup $M \leq \Delta$ with $N_\Delta(M)/M \cong A$.

The simplest case is when $p = 3$, with Δ the modular group

$$\Gamma = \Delta(3, 2, \infty) = \mathrm{PSL}_2(\mathbb{Z}).$$



The map subgroups N for this map $\mathcal{N}$ are maximal in Γ, and are isomorphic to $C_2 * C_2 * C_2 * C_\infty * C_\infty * \cdots = C_2 * C_2 * C_2 * F_\infty$ (the cyclic free factors correspond to the fixed points of Y and Z).

There is an epimorphism $N \to F_\infty$, and hence an epimorphism $\theta : N \to A$ for every countable group A.

If $A \neq 1$ there are $2^{\aleph_0}$ such epimorphisms θ with kernels M not normal in Γ (so $N_\Gamma(M) = N$) and mutually non-conjugate in Γ.

These subgroups M correspond to $2^{\aleph_0}$ non-isomorphic oriented cubic maps $\mathcal{M}$ (coverings of $\mathcal{N}$) with $\mathrm{Aut}\,\mathcal{M} \cong N/M \cong A$.

Similar arguments deal with the cases $A = 1$ and $p > 3$. $\square$

Cocompact triangle groups

The preceding proofs of primitivity, maximality and realisation depend on a generator Z of Δ having infinite order. What about cocompact triangle groups $\Delta = \Delta(p, q, r)$, with finite p, q and r?

If $p^{-1} + q^{-1} + r^{-1} \geq 1$ then Δ acts on the sphere or euclidean plane, and is abelian-by-finite with at most $\aleph_0$ subgroups, all well-understood. Hence assume that Δ is hyperbolic, that is,

$$p^{-1} + q^{-1} + r^{-1} < 1.$$

Example The finite quotients of $\Delta(3, 2, 7)$ are the Hurwitz groups, those attaining Hurwitz's bound $|G| \leq 84(g - 1)$ for $G = \mathrm{Aut}\,\mathcal{S}$, where $\mathcal{S}$ is a compact Riemann surface of genus $g \geq 2$.

What about their maximal subgroups of infinite index?

In 1980 Marston Conder constructed finite primitive permutation representations of $\Delta(3,2,7)$ to show that alternating groups A_n are Hurwitz groups for all sufficiently large n (in fact, $n \geq 168$). In 1981 he generalised this to $\Delta(3,2,r)$ for each $r \geq 7$.

Extending his method (of sewing coset diagrams together) gives $2^{\aleph_0}$ primitive representations of $\Delta(3,2,r)$ of infinite degree, and hence $2^{\aleph_0}$ conjugacy classes of maximal subgroups of infinite index.

Lifting back by an obvious epimorphism gives the same results for $\Delta(p,q,r)$ provided one of p,q,r is even, another is divisible by 3, and the third is at least 7 (satisfied by 121/216 of all triples).

Corollary If $r \geq 7$ each countable group A is the automorphism group of $2^{\aleph_0}$ non-isomorphic maps of type $\{r,3\}$ (or dually $\{3,r\}$). If A is finite then $\aleph_0$ of them are finite.

Proof The maps $\mathcal{M}$ have map subgroups $M = \ker(N \twoheadrightarrow A) \ntriangleleft \Delta$, N maximal in Δ, so that $\mathrm{Aut}\,\mathcal{M} \cong N_\Delta(M)/M = N/M \cong A$.

Conjecture Every finitely generated non-elementary (not cyclic or dihedral) Fuchsian group Γ has $2^{\aleph_0}$ conjugacy classes of maximal subgroups.

'Proof' Everitt (1999) extended Conder's results on $\Delta \to \mathrm{A}_n$ to all such groups Γ, using similar methods, so adapt them as before for triangle groups Δ to give $2^{\aleph_0}$ primitive representations of Γ.

'Corollary' Each countable group A is the automorphism group of $2^{\aleph_0}$ non-isomorphic hypermaps of any hyperbolic type (p, q, r). If A is finite then $\aleph_0$ of them are finite.

(A hypermap is a 'map with q-valent edges'; maps have $q = 2$.

More precisely it is a bipartite map in which the black and white vertices and the faces have valencies with gcds p, q and $2r$.

It is of hyperbolic type if $p^{-1} + q^{-1} + r^{-1} < 1$.)

References

For background on the modular group and its actions, see:

W. Magnus, *Noneuclidean Tesselations and their Groups*, Academic Press, New York and London, 1974.

For details and references on maximal subgroups, see:

G. A. Jones, Maximal subgroups of the modular and other groups, *J. Group Theory* 22 (2019), 277–296; arXiv:1806.03871 [math.GR].

For details and references on realisation of groups, see:

G. A. Jones, Realisation of groups as automorphism groups in permutational categories, *Ars Math. Contemp.*, 21 (2021), 1–22; arXiv:1807.00547 [math.GR].

THANK YOU FOR LISTENING!