Local actions and eigenspaces of vertex-transitive graphs

Gabriel Verret

AGTIW, 23 November 2021

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A graph Γ is G-vertex-transitive (G-arc-transitive) if $G \leq \mathrm{Aut}(\Gamma)$ acts transitively on the the vertex-set (arc-set) of Γ.

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$|G_v|$ is bounded by a constant, so $|G|$ grows at most linearly in terms of $|\mathrm{V}(\Gamma)|$.

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Theorem (Potočnik, Spiga, V 2017)

The number of 3-valent arc-transitive graphs of order at most n is

$$\sim n^{c \log n}.$$

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For example, $(\mathrm{W}_m, \mathrm{Aut}(\mathrm{W}_m))$ is locally-D_4.

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Gardiner's result: $\mathrm{Alt}(4)$ and $\mathrm{Sym}(4)$ are graph-restrictive.

Which groups are graph-restrictive?

Conjecture (Potočnik, Spiga, V 2011)

Semiprimitive $\iff$ *graph-restrictive.*

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Many other results in the other direction (Tutte, Gardiner, Weiss, Trofimov, Stellmacher, Spiga, Morgan...)

A recent application

Theorem (Potočnik, Toledo, V 2021+)

Let $d \geq 3$. For every $i \geq 1$, there exists a Cayley graph Γ_i of valency d such that $|\mathrm{V}(\Gamma_i)|$ is at least a tower of exponentials in i while the exponent of $\mathrm{Aut}(\Gamma_i)$ is at most $2^i(f(d))$.

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Let G be the "free" group generated by d involutions $a_1, \ldots, a_d$, and consider the series $G = G_0 \geq G_1 \geq G_2 \geq \cdots$ where $G_{i+1} = (G_i)^2$ for $i \geq 0$.

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We also know of some groups with polynomial (non-constant) growth, such as D_n for n even, $n \geq 6$.

Main problem

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Find the growth rate of every permutation group.

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Theorem (Hujdurović, Potočnik, V 2021)

The remaining 3 *groups have exponential growth.*

Example: $\mathrm{Sym}(4)$ transitive on 6 points

We have

$$L := \mathrm{Sym}(4) \cong \mathbb{Z}_2^2 \rtimes \mathrm{Sym}(3) < \mathbb{Z}_2^3 \rtimes \mathrm{Sym}(3) = \mathbb{Z}_2 \wr \mathrm{Sym}(3) \leq \mathrm{Sym}(6),$$

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The 1-eigenspace over $\mathbb{Z}_2$ of the 3-valent graph turns out to be relevant!

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More results of this type?

Open problems

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Characterize groups of polynomial graph-type.