Finite 3-orbit skeletal polyhedra

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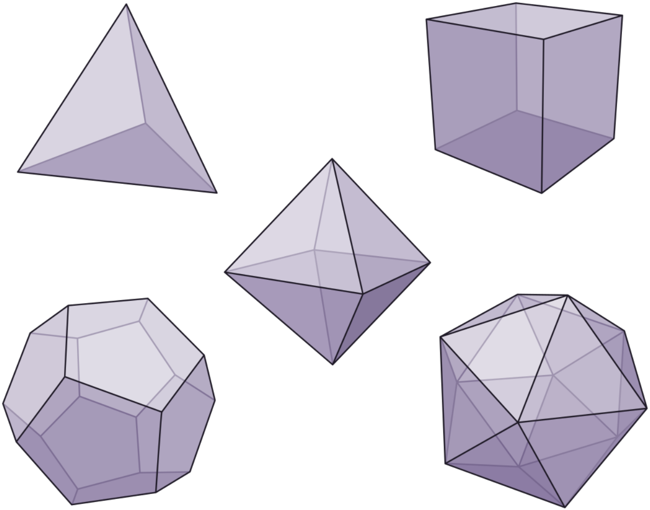
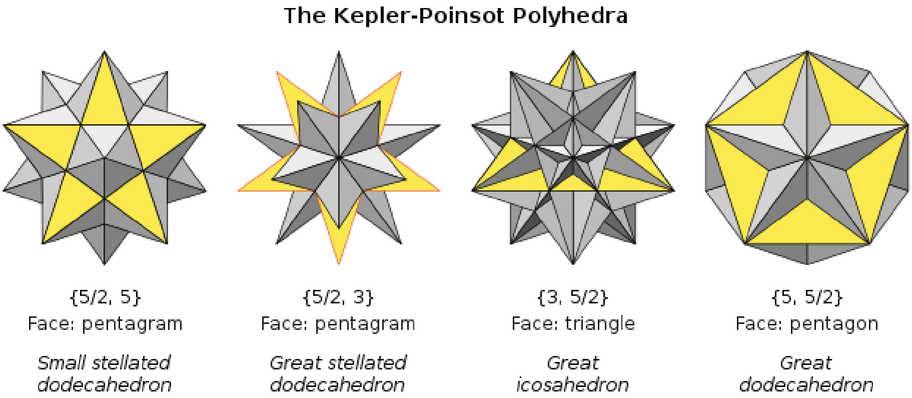
Platonic solids

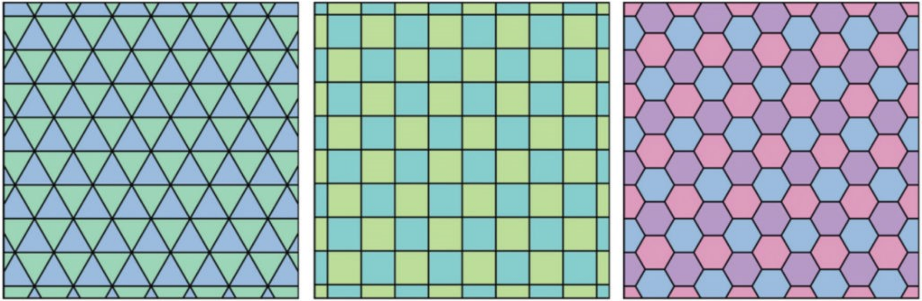


Kepler-Poinsot polyhedra



Johannes Kepler recognized the first two as regular in 1619.
Louis Poinsot recognized the second two as regular in 1809.

Tilings of the plane



Skeletal polyhedra

Regular polyhedra—old and new by Branko Grünbaum (1977):
What if we stop thinking of polygons as flat pieces and just think of them as edge-circuits?

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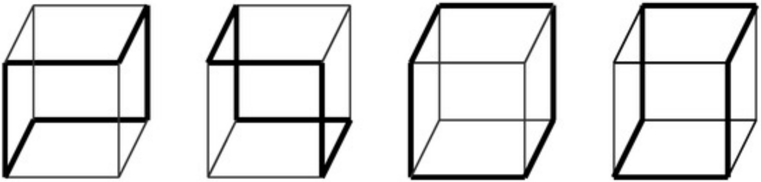
- The *vertex-figures* are connected. (Roughly: for every vertex v, the faces that contain v can be arranged in a cyclic order where adjacent faces share an edge on v.)
- P is *discrete* (each compact subset of $\mathbb{E}^3$ meets only finitely many faces of P).

Example of a skeletal polyhedron

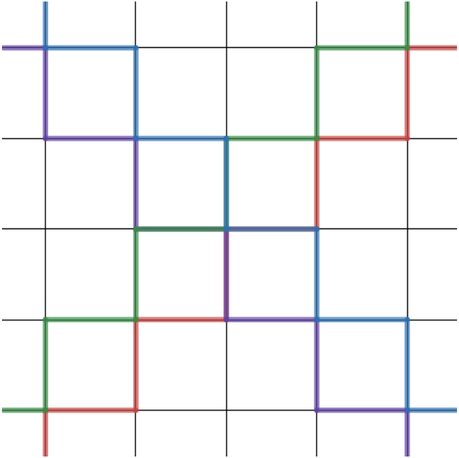


These 4 skew hexagons can be glued together to form the *Petrie dual* of the cube.

Petrie dual

- A *Petrie polygon* of a polyhedron is an edge-cycle where every two consecutive edges share a face, but no three consecutive edges do.
- The Petrie dual of a polyhedron P is the polyhedron with the same graph, but where the faces are the Petrie polygons of P.

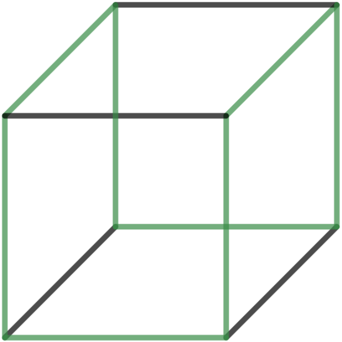
Petrie dual of the tiling by squares



Regularity

What should regularity mean for skeletal polyhedra?

Regular polygons in $\mathbb{E}^3$



The green octagon is equilateral and equiangular, but the vertical edges are distinguishable from the horizontal edges.

Regular skeletal polyhedra

A *flag* of a polyhedron is an incident vertex-edge-face triple.

A skeletal polyhedron is *regular* if its symmetry group acts transitively on its flags.

Grünbaum-Dress polyhedra

There are 48 regular skeletal polyhedra in $\mathbb{E}^3$:

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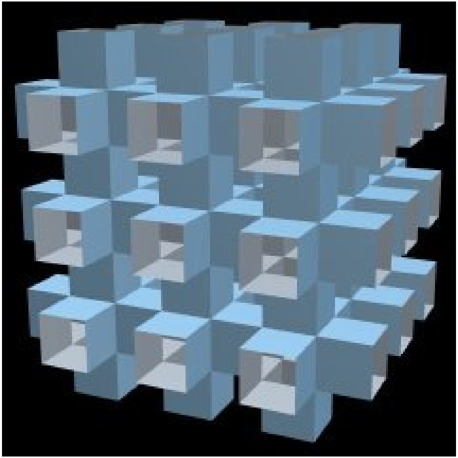
47 were found by Grünbaum in his 1977 paper. Andrea Dress found the last one and proved that the classification was complete in 1985.

One of the pure polyhedra



This is the *Petrie-Coxeter polyhedron* $\{4, 6|4\}$, discovered in the 1920s.

Measuring regularity

How can we characterize "highly-symmetric" polyhedra that are not regular?

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A *k-orbit polyhedron* is one where the action of the symmetry group on the flags has k orbits.

Chiral skeletal polyhedra

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The chiral skeletal polyhedra in $\mathbb{E}^3$ were classified by Egon Schulte in 2004-2005. They are all infinite.

There are six other classes of 2-orbit polyhedra. Someone in this room might be working on classifying 2-orbit skeletal polyhedra...

2-orbit skeletal polyhedra

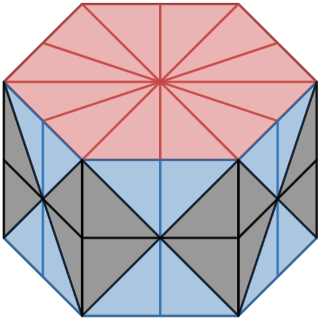
There are six other classes of 2-orbit polyhedra. Someone in this room might be working on classifying 2-orbit skeletal polyhedra...

(Spoiler: It's Isabel)

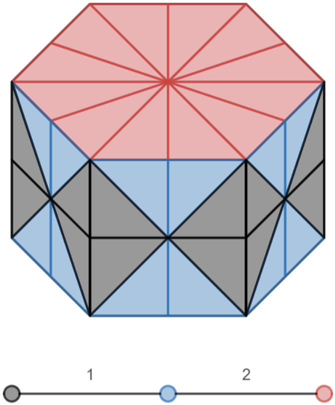
3-orbit polyhedra

What can we say about 3-orbit polyhedra?

A familiar 3-orbit polyhedron



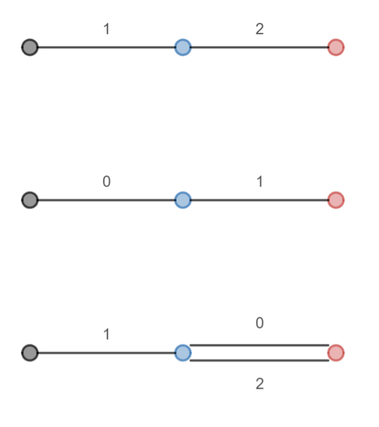
Symmetry type graph of a prism



Symmetry type graphs of 3-orbit polyhedra

Theorem (C., Del Rio-Francos, Hubard, Toledo, 2015)

Every 3-orbit polyhedron has one of the following symmetry type graphs:



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3-orbit polyhedra in $\mathbb{E}^2$

Let's classify the finite 3-orbit polyhedra in $\mathbb{E}^2$.

3-orbit polyhedra in $\mathbb{E}^2$

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No! There is only a single point with a nontrivial stabilizer – not enough to make a polyhedron.

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So, a 3-orbit polyhedron P in $\mathbb{E}^2$ must have dihedral symmetry group D_n of order $2n$, generated by two reflections.

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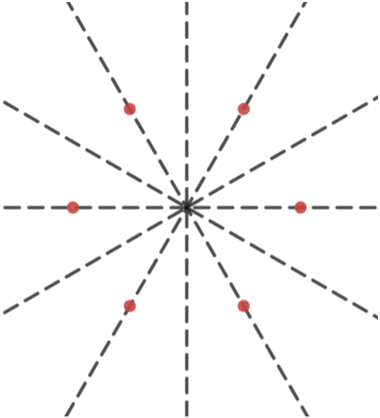
Furthermore,

- P must be vertex-transitive, with one orbit of n vertices.
- Each vertex must be 3-valent.

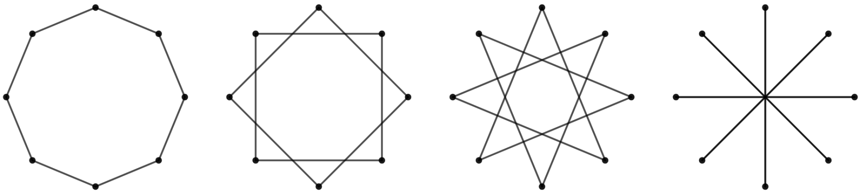
Vertices of a 3-orbit polyhedron



Determining the edges

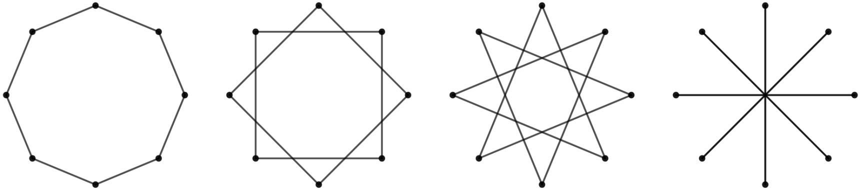
Now that we know what the vertices are, let's determine the edges.

Edge-orbits



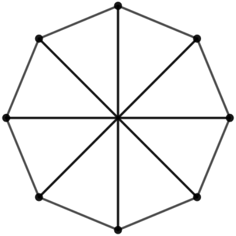
We need one “large” orbit and one “small” orbit.

Edge-orbits

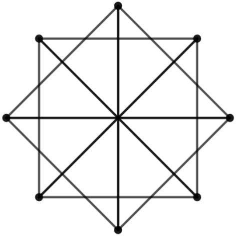


We need one "large" orbit and one "small" orbit. There is only one small orbit of edges!

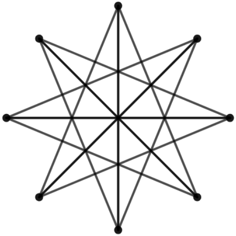
Possible graphs with 8 vertices



Möbius ladder M_8

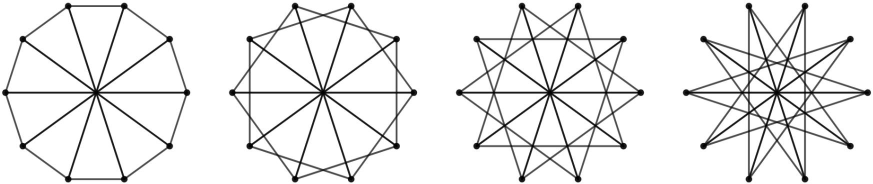


Disconnected!



M_8

Possible graphs with 10 vertices



M_{10} Prism graph Y_5 M_{10} Y_5

Possible graphs with n vertices

In general, the only possible connected graphs are the Möbius ladder M_n and the prism graph $Y_{n/2}$ if $n/2$ is odd.

Going abstract

The embeddings of M_n and $Y_{n/2}$ are fully symmetric in the sense that every graph automorphism is represented by a geometric symmetry. So let's forget the embeddings for a moment and just work with the graph.

Abstract polyhedra

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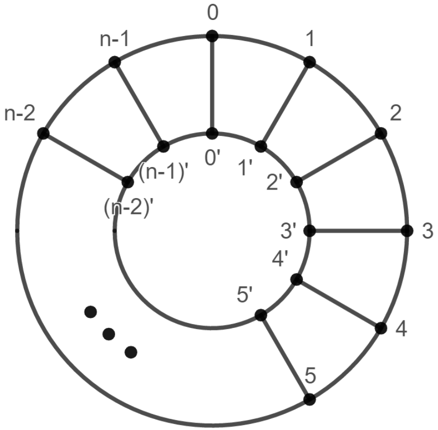
- Each edge is part of exactly 2 faces
- The *vertex-figures* are connected.
- ~~P is *discrete*.~~

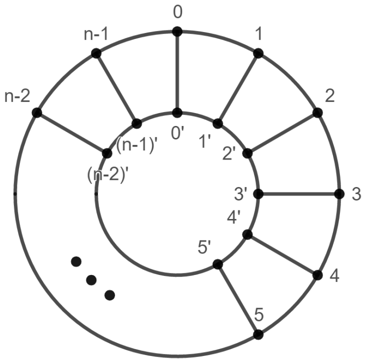
Automorphisms and regularity

An *automorphism* of an abstract polyhedron is a graph automorphism that preserves the face structure.

Abstract 3-orbit polyhedra with graph Y_n

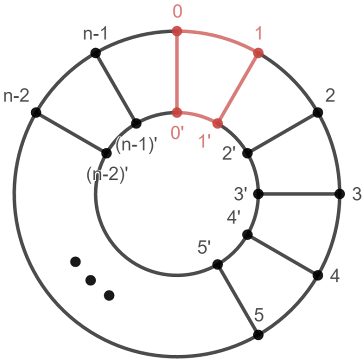


Case 1: Non-face-transitive



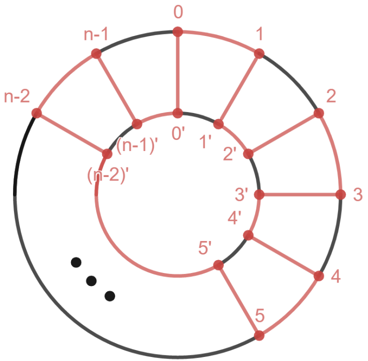
One type of face uses only the large edge orbit.
The other type alternates edge-orbits.

Case 1a: Non-face-transitive



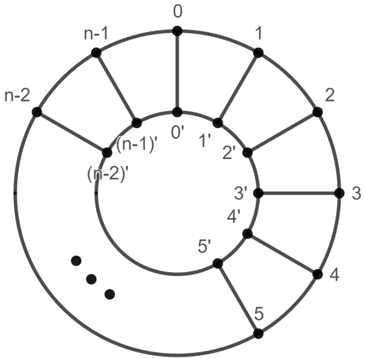
Possibility 1: an abstract prism

Case 1b: Non-face-transitive



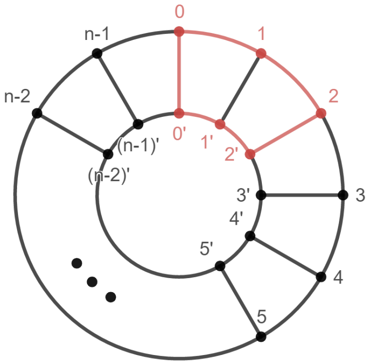
Possibility 2

Case 2: Face-transitive



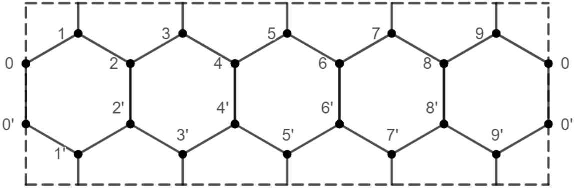
Now we want only one type of face, where every third edge is "vertical". (Connecting some i to i'.)

Case 2: Face-transitive

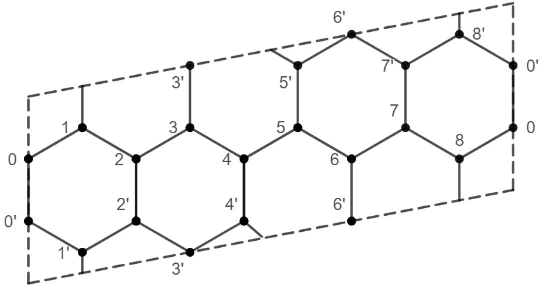


Possibility 3: Hexagonal faces, meeting 3 at each vertex.

Case 2: Face-transitive

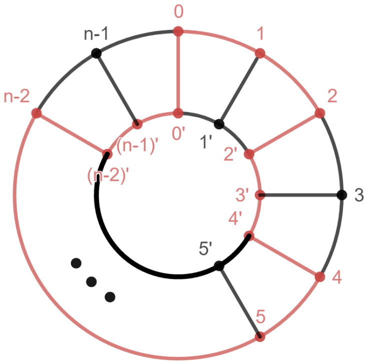


$n = 10$:



$n = 9$:

Case 2: Face-transitive



Possibility 4: n is divisible by 4 and we get 4 $3n/2$-gons. (This is the Petrie dual of a prism!)

Back to geometry!

We now know all of the abstract 3-orbit polyhedra that use the graph of a prism.

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All of the (combinatorially) 3-orbit polyhedra with graph Y_n (n odd) can be realized as (geometrically) 3-orbit polyhedra in $\mathbb{E}^2$.

3-orbit polyhedra in $\mathbb{E}^2$

Then we perform similar analysis for the graph M_n...

3-orbit polyhedra in $\mathbb{E}^2$

Graph	Polyhedron	Class	Restrictions
$Y_{n/2}$	Prism over $n/2$-gon	$3^{1,2}$	$n/2$ odd
$Y_{n/2}$	$\{6,3\}_{(k,1),(-1,2)}$	3^1	$n/2 = 2k+1$, $k \geq 1$
$Y_{n/2}$	$\{6,3\}_{\backslash n/2,1\backslash}$	3^1	$n/2$ odd
M_n	Hemi-prism over n-gon	$3^{1,2}$	n even, $n \geq 4$
M_n	Petrial of hemi-prism over n-gon, type $\{\frac{3n}{4}, 3\}$	3^1	$n \equiv 4 \pmod 8$, $n \geq 4$
M_n	$\{6,3\}_{(k,1),(-1,2)}$	3^1	$n = 4k+2$, $k \geq 1$
M_n	Petrial of $\{6,3\}_{(k,1),(-1,2)}$	$3^{1,2}$	$n = 4k+2$, $k \geq 1$
M_n	$\{6,3\}_{\backslash n/2,1\backslash}$	3^1	$n \equiv 0 \pmod 4$
M_6	$\{6,3\}_{(1,1)}$	$3^{1,2}$	

Table 2: The 3-orbit polyhedra in $\mathbb{E}^2$

Next steps

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- 7 symmetry groups + 7 infinite families of symmetry groups
- 8 classes of vertex-orbits
- Then look at all of the edge-orbits
- *Then* consider possible faces...

A preview

Theorem (C. and Pellicer, 2021)

The 3-orbit skeletal polyhedra in $\mathbb{E}^3$ are... (...including 170 with icosahedral symmetry...)

Thank you!