Constructing k-orbit polytopes from their automorphism groups

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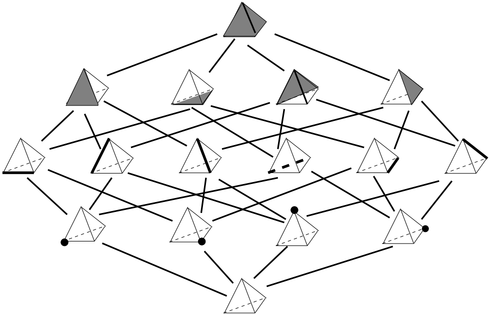

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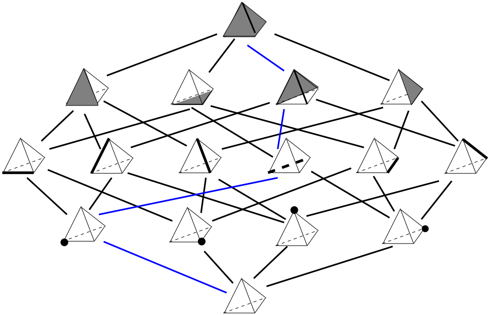

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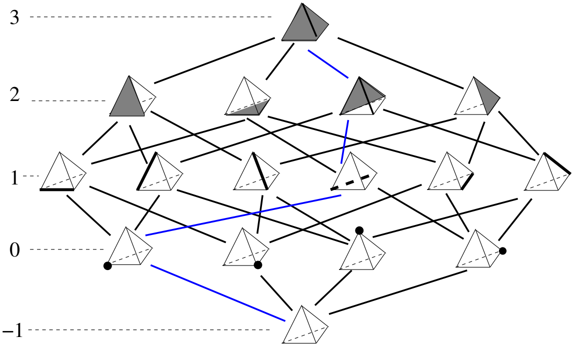

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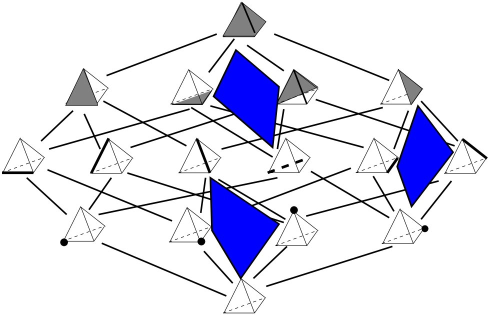

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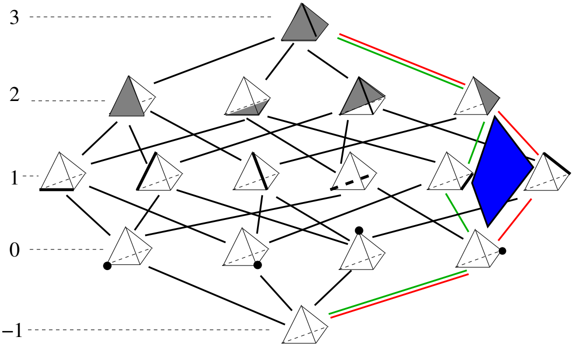

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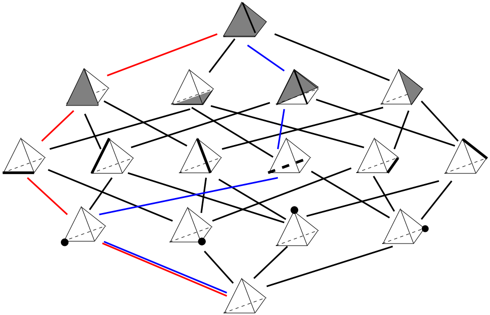

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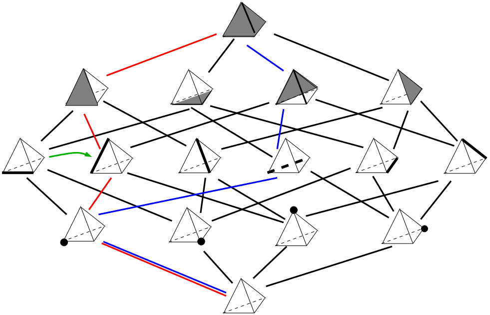

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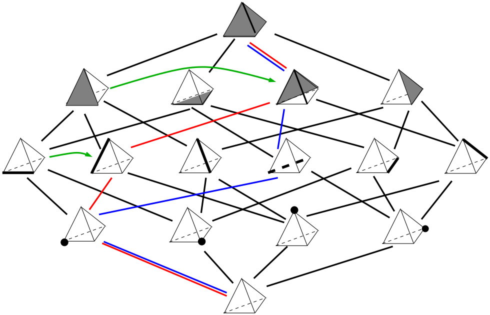

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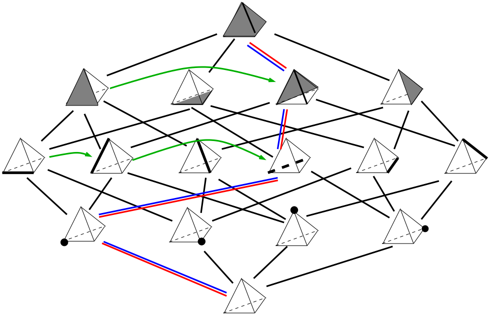

Regular polytopes

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If $\Gamma(\mathcal{P})$ acts transitively (and hence regularly) on the flags of $\mathcal{P}$ we say that $\mathcal{P}$ is regular.

Automorphism groups of regular polytopes

Theorem (Danzer and Schulte,1982)

A group Γ *is the automorfism group of a regular polytope of rank* n *iff it is generated by* n *involutions* $\rho_0, \rho_1, \ldots, \rho_{n-1}$ *satisfying that* $(\rho_i \rho_j)^2 = 1$ *whenever* $|i - j| > 1$ *and that:*

$$\langle \rho_i : i \in I \rangle \cap \langle \rho_i : i \in J \rangle = \langle \rho_i : i \in I \cap J \rangle,$$

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$\Leftarrow$: They construct a poset in the following way...

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Schulte and Danzer prove that $\mathcal{P}(\Gamma)$ is a regular polytope with automorphism group Γ.

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Describe a way to build a general polytope as a coset geometry of its automorphism group, given a distinguished set of generators.

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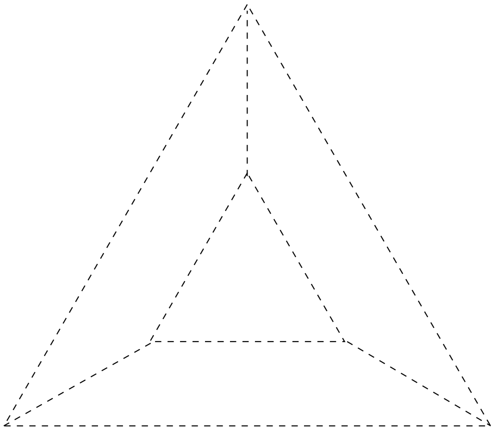

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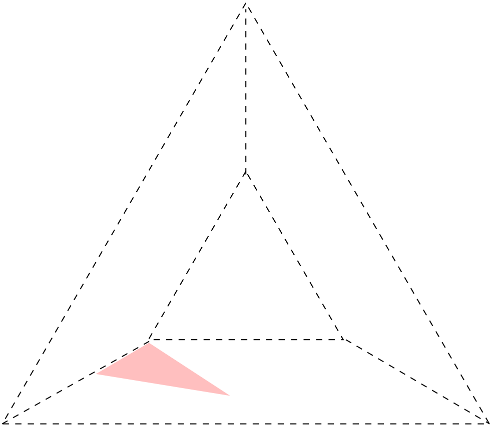

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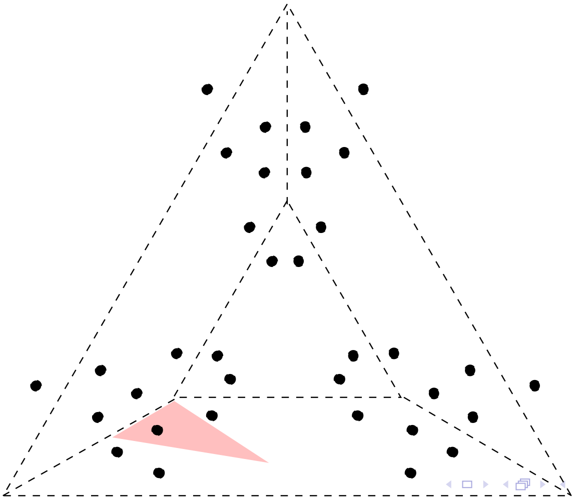

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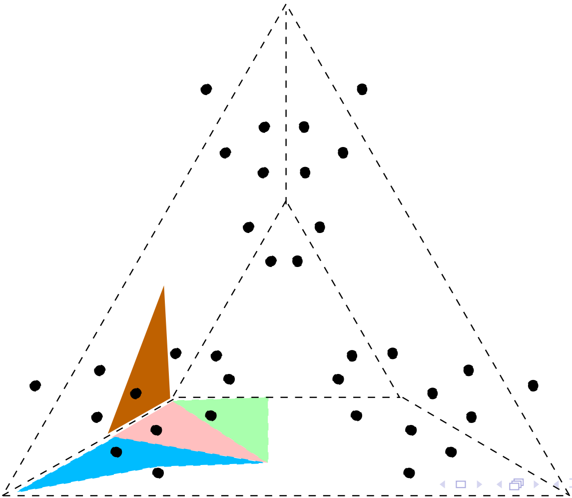

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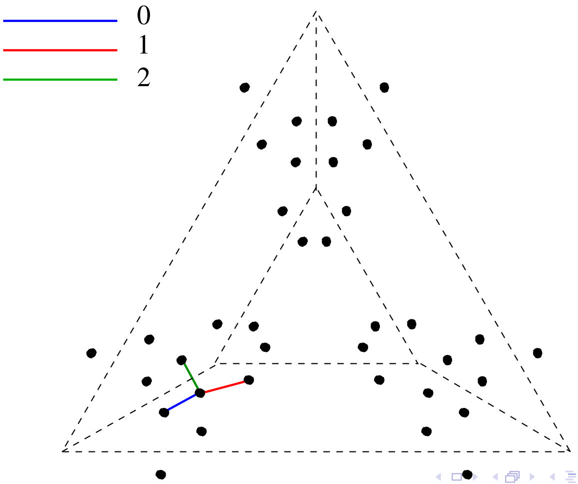

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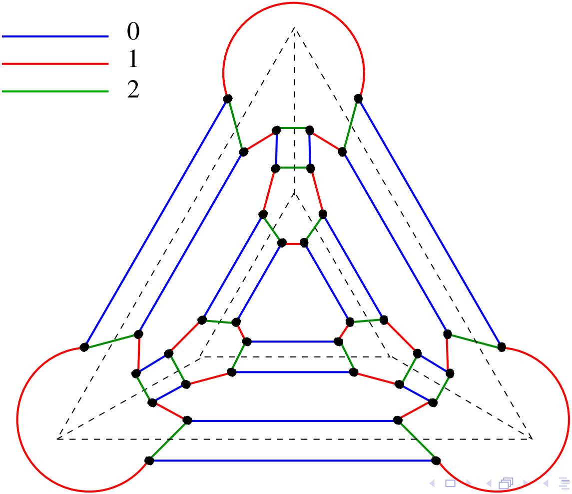

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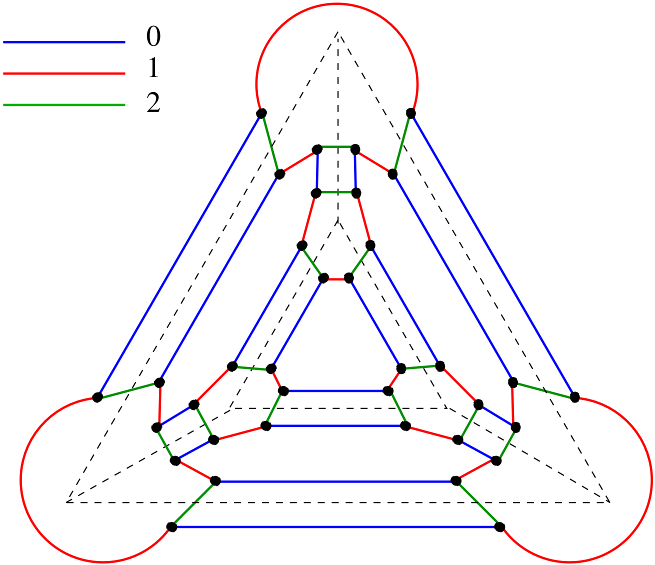

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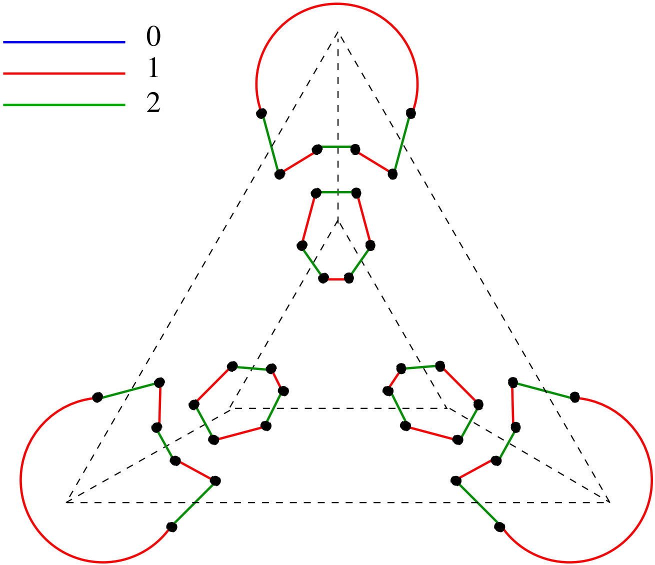

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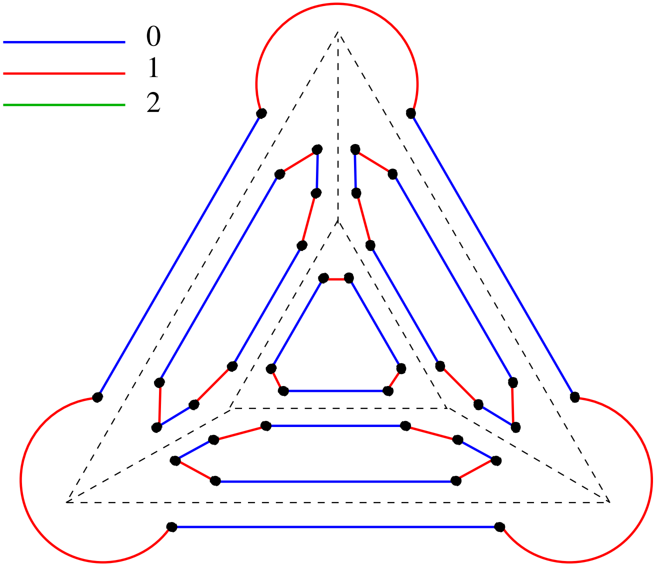

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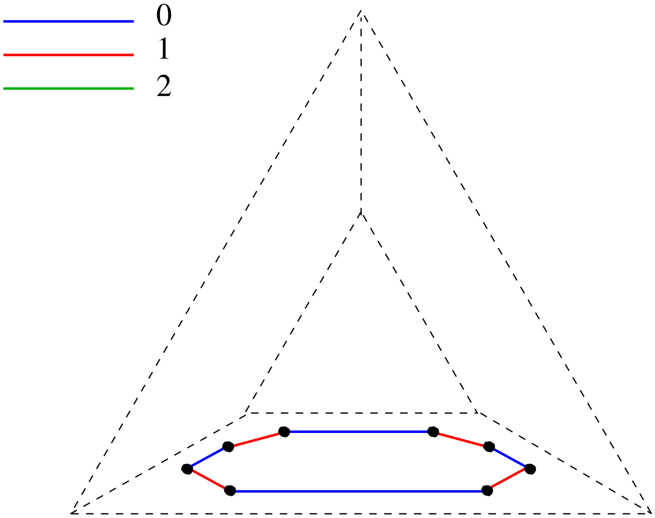

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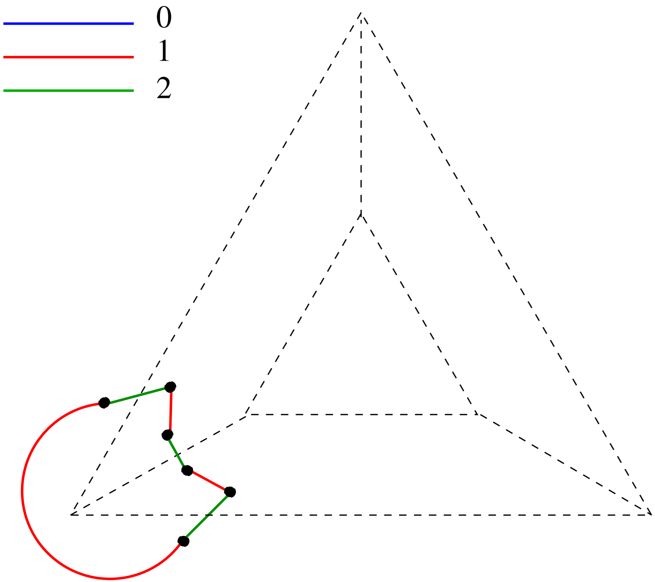

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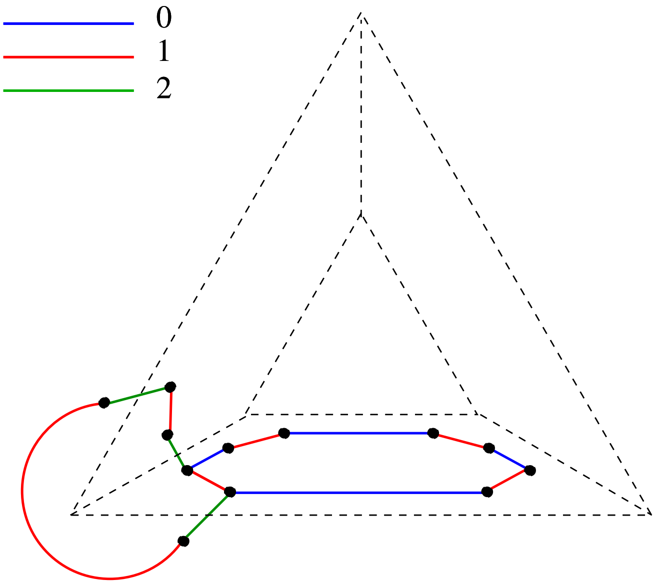

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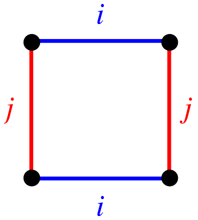

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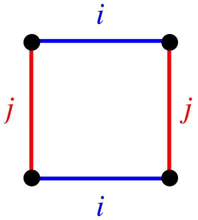

The vertices of a maniplex are often refered to as flags.

Symmetry type graph

Definition

Let $\mathcal{M}$ be a maniplex and let $\Gamma(\mathcal{M})$ be its automorphism group. Let $H \leq \Gamma(\mathcal{M})$. We define the symmetry type graph (STG) of $\mathcal{M}$ with respect to H to be the quotient $\mathcal{T}(\mathcal{M}, H) := \mathcal{M}/H$.

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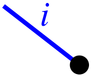

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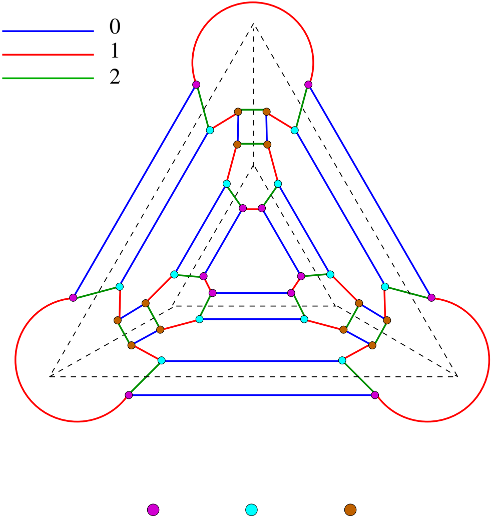

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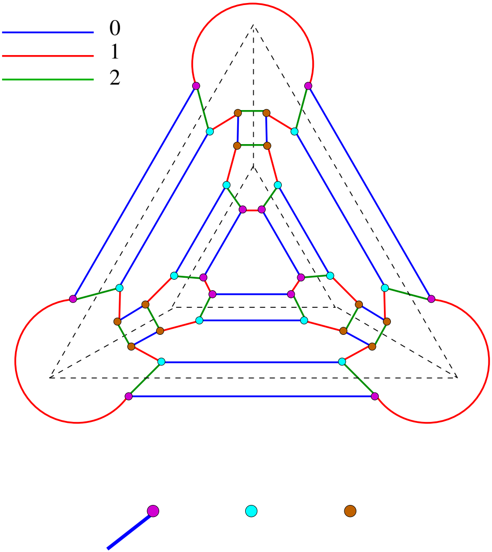

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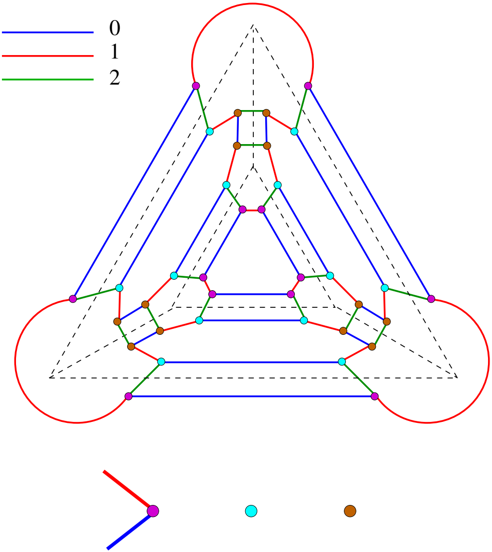

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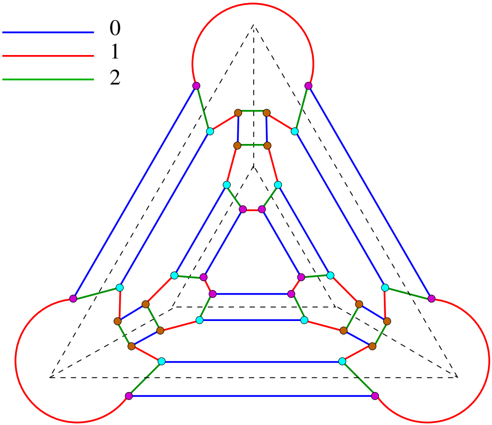

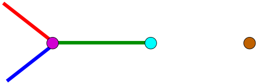

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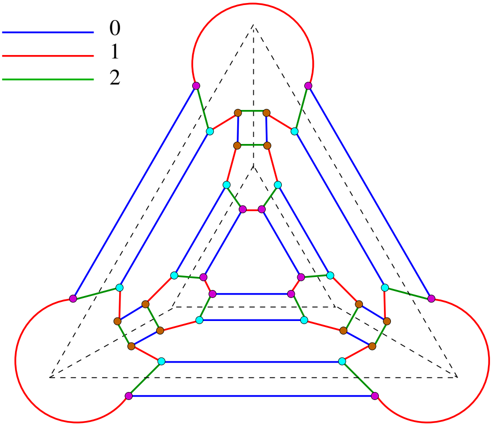

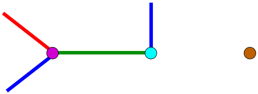

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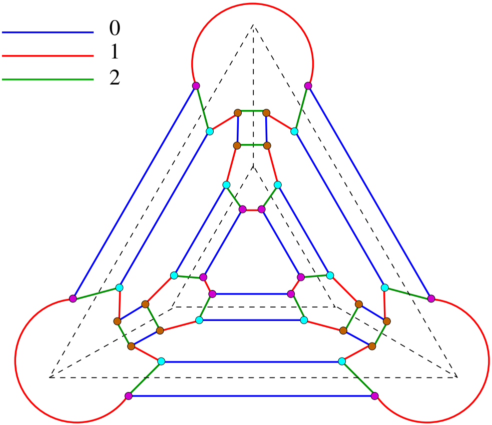

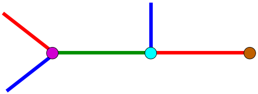

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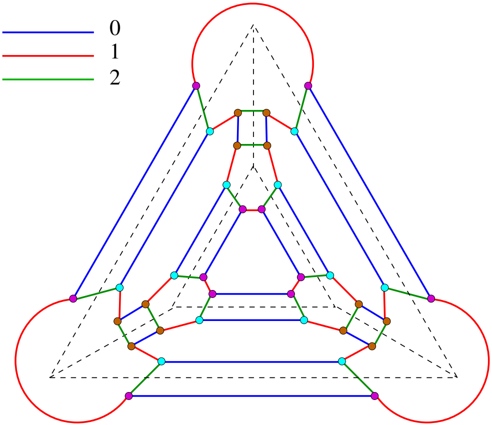

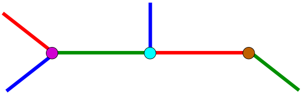

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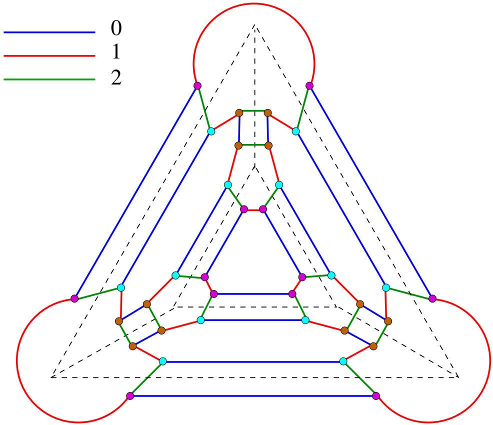

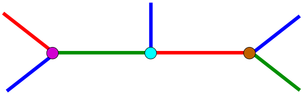

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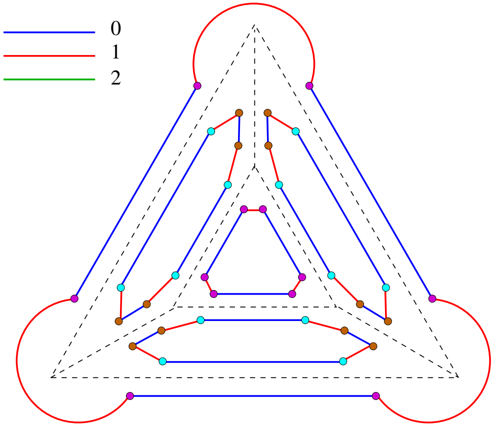

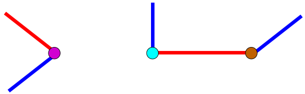

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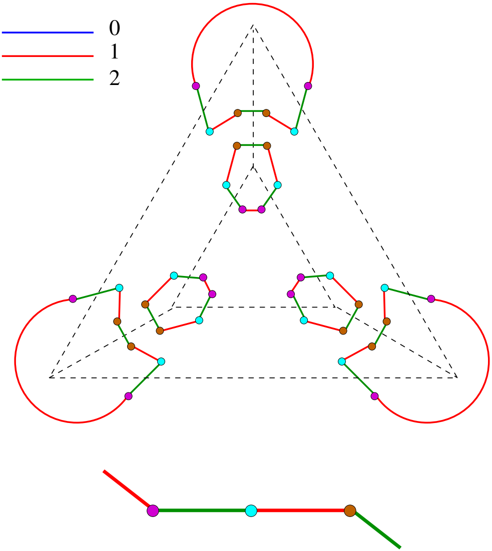

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- $\mathcal{T}(\mathcal{M}, H)$ is an n-valent connected graph with a proper edge coloring $c : E \to \{0, 1, \ldots, n-1\}$.
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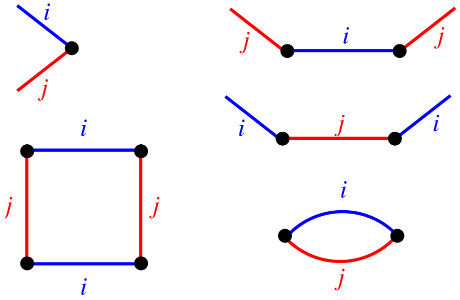
Connected components of the graph induced by edges of colors colors i and j are one of the following:



Voltages

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Given a vertex x in X, we denote the (fundamental) group of closed paths based at x as $\Pi^x(X)$.

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Let X be a graph. A voltage assignment is a (groupoid) anti-morphism $\xi : \Pi(X) \to \Gamma$ where Γ is a group. In this case, Γ is called the voltage group.

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We only need to assign voltage to the darts of X to define the voltage of all its paths. The voltage of a path is the product of the voltages of its darts in reverse order.

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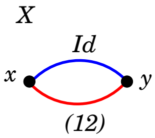

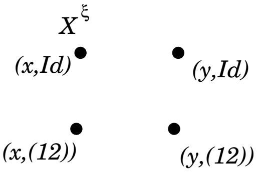

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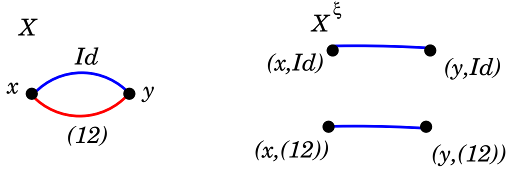

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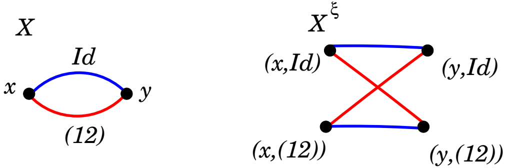

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There's always a voltage assignment which recovers $\mathcal{M}$ from $\mathcal{T}(\mathcal{M}, H)$.

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Let $p : X^\xi \to X$ be the natural projection $(x, \gamma) \mapsto x$.

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The end point of $\widetilde{W}$ is $(y, \xi(W)\gamma)$.

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- X^ξ *is a maniplex* $\Leftrightarrow$ *both previous conditions hold, and whenever* $|i - j| > 1$, *the voltage of any path of length 4 that alternates colors between i and j is trivial.*

Polytopality of maniplexes

Definition

A maniplex satisfies the *strong path intersection property* *(SPIP) if whenever two flags* Φ *and* Ψ *are connected by some path with colors in a set* I *and another path with colors in* J*, then they are connected by a path with colors in* $I \cap J$*.*

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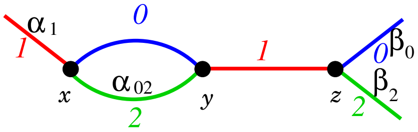

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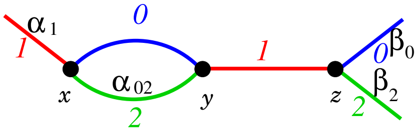

- Connected: $\Gamma = \langle \alpha_1, \alpha_{02}, \beta_0, \beta_2 \rangle$.

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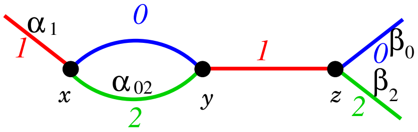

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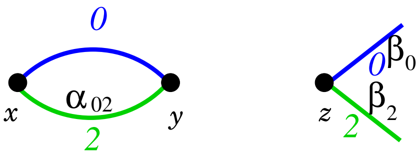

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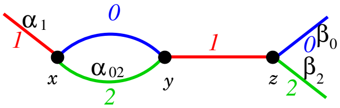

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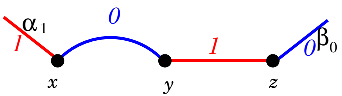

- $\xi(\Pi^x_{\{0,1\}}) = \langle \alpha_1, \beta_0 \rangle$.

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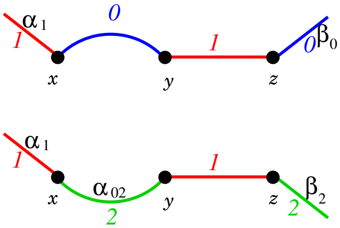

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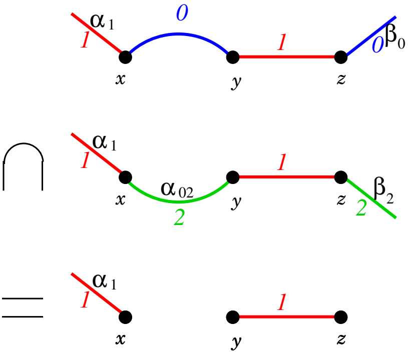

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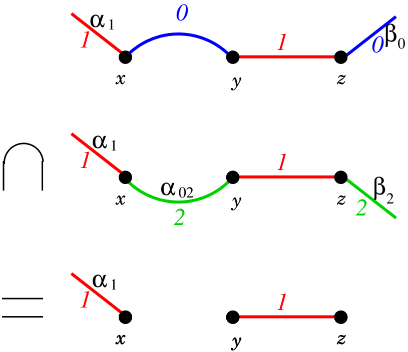

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- $\therefore \langle \alpha_1, \beta_0 \rangle \cap \langle \alpha_1, \beta_2^{\alpha_{02}} \rangle = \langle \alpha_1 \rangle$.

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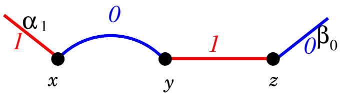

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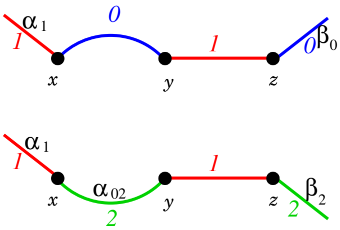

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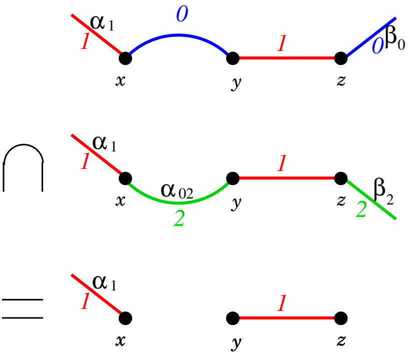

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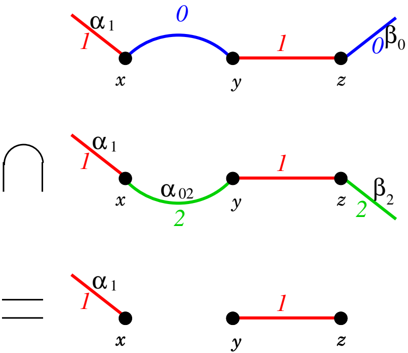

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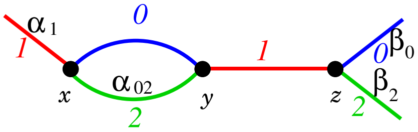
By checking all possibilities for k, m, a and b we get:



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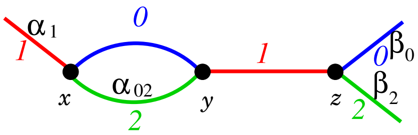

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Theorem

Let H be a group. There exists a polytope $\mathcal{P}$ such that $H \leq \Gamma(\mathcal{P})$ and $\mathcal{T}(\mathcal{P}, H)$ is the graph from the example iff H is generated by four involutions $\alpha_1, \beta_0, \beta_2, \alpha_{02}$ such that $(\beta_0\beta_2)^2 = 1$ and the previous intersection properties hold.

Construction as coset geometry

Problem (Cunningham and Pellicer, 2018)

Describe a way to build a general polytope as a coset geometry of its automorphism group, given a distinguished set of generators.

Construction as coset geometry

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Let X be a premaniplex and let $\xi : \Pi(X) \to \Gamma$ be a voltage assignement such that X^{ξ} is polytopal. Let the poset $\mathcal{P}(\Gamma)$ be the poset defined as follows:

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- *Define the following order relation:*

$$\begin{aligned}\xi(\Pi^a(C))\sigma < \xi(\Pi^{a'}(C'))\tau \quad \Leftrightarrow \quad & \exists b \in C \cap C' : \\ & \xi(\Pi^{a,b}(C))\sigma \cap \xi(\Pi^{a',b}(C'))\tau \neq \emptyset.\end{aligned}$$

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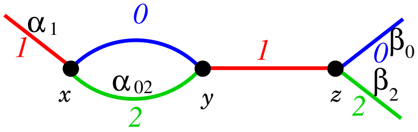
Then $\mathcal{P}(\Gamma)$ is an abstract polytope and $\mathcal{T}(\mathcal{P}(\Gamma), \Gamma) = X$.

Example



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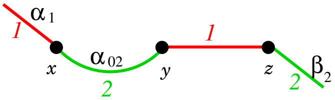

- Vertices: Right cosets of $\xi(\Pi^x_{\{1,2\}}) = \langle \alpha_1, \beta_0^{\alpha_{02}} \rangle =: \Gamma_0$.

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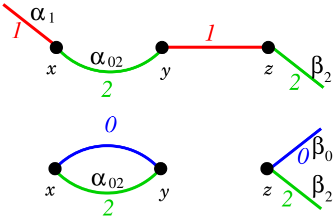
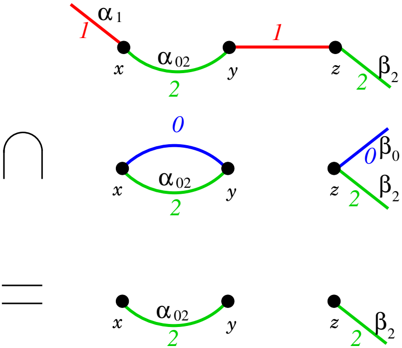


- Vertices: Right cosets of $\xi(\Pi^x_{\{1,2\}}) = \langle \alpha_1, \beta_0^{\alpha_{02}} \rangle =: \Gamma_0$.
- Edges: Right cosets of $\xi(\Pi^x_{\{0,2\}}) = \langle \alpha_{02} \rangle =: \Gamma_1$ and $\xi(\Pi^z_{\{0,2\}}) = \langle \beta_0, \beta_2 \rangle =: \Gamma'_1$.

Incidence



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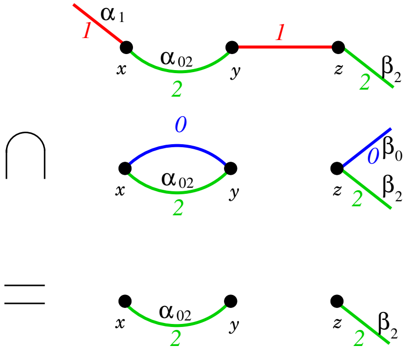

- $\Gamma_0\sigma < \Gamma_1\tau$ iff
  $\xi(\Pi^{x}_{\{1,2\}})\sigma \cap \xi(\Pi^{x}_{\{0,2\}})\tau = \Gamma_0\sigma \cap \Gamma_1\tau \neq \emptyset$ or
  $\xi(\Pi^{x,y}_{\{1,2\}})\sigma \cap \xi(\Pi^{x,y}_{\{0,2\}})\tau = \alpha_{02}\Gamma_0\sigma \cap \Gamma_1\tau \neq \emptyset$.

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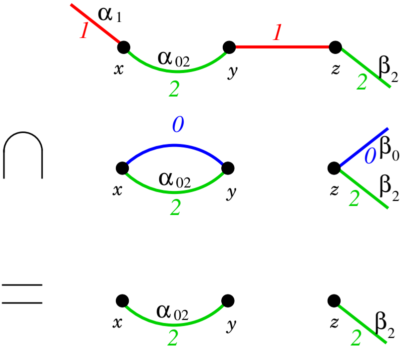

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  $\xi(\Pi^{x,y}_{\{1,2\}})\sigma \cap \xi(\Pi^{x,y}_{\{0,2\}})\tau = \alpha_{02}\Gamma_0\sigma \cap \Gamma_1\tau \neq \emptyset$.
- $\Gamma_0\sigma < \Gamma_1'\tau$ iff
  $\xi(\Pi^{x,z}_{\{1,2\}})\sigma \cap \xi(\Pi^{z}_{\{0,2\}})\tau = \alpha_{02}\Gamma_0\sigma \cap \Gamma_1'\tau \neq \emptyset$.

Example

Theorem

Let $\Gamma = \langle \alpha_1, \alpha_{02}, \beta_0, \beta_1 \rangle$ *be the a group satisfying the intersection properties of a previous theorem. Let* $\mathcal{P}(\Gamma)$ *be the set of right cosets of* $\Gamma_0 := \langle \alpha_1, \beta_2^{\alpha_{02}} \rangle$, $\Gamma_1 := \langle \alpha_{02} \rangle$, $\Gamma_1' := \langle \beta_0, \beta_2 \rangle$ *and* $\Gamma_2 = \langle \alpha_1, \beta_0 \rangle$ *with the order relation given by:*

$$
\begin{array}{lll}
\Gamma_0\sigma < \Gamma_1\tau & \text{if and only if} & \Gamma_0\sigma \cap \Gamma_1\tau \neq \emptyset; \\
\Gamma_0\sigma < \Gamma_1'\tau & \text{if and only if} & \alpha_{02}\Gamma_0\sigma \cap \Gamma_1'\tau \neq \emptyset; \\
\Gamma_0\sigma < \Gamma_2\tau & \text{if and only if} & (\Gamma_0\sigma \cup \alpha_{02}\Gamma_0\sigma) \cap \Gamma_2\tau \neq \emptyset; \\
\Gamma_1\sigma < \Gamma_2\tau & \text{if and only if} & \Gamma_1\sigma \cap \Gamma_2\tau \neq \emptyset; \\
\Gamma_1'\sigma < \Gamma_2\tau & \text{if and only if} & \Gamma_1'\sigma \cap \Gamma_2\tau \neq \emptyset.
\end{array}
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Then $\mathcal{P}(\Gamma)$ *is an abstract polytope and* $\mathcal{T}(\mathcal{P}, \Gamma) = X$.

Applications

- If $[0, n] \setminus I$ has at most two elements, there are 2-orbit polytopes of symmetry type 2^n_I.

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Applications

- If $[0, n] \setminus I$ has at most two elements, there are 2-orbit polytopes of symmetry type 2^n_I.
- If X is a caterpillar then, almost for sure, there are polytopes with symmetry type X and Boolean automorphism group.
- (Ongoing work with A. Montero and G. Cunningham) If K is an abstract polytope of rank n and X is an $(n+1)$-premaniplex consisting of a copy of K plus some extra edges of color n, then by giving the "most general" voltage possible to the edges of color n while giving trivial voltage to the other edges, we get a polytope as the derived graph. This is the Universal extension of K of type X.

Thank you!