Skeletal Polyhedra, Complexes, and Symmetry

Egon Schulte
Northeastern University, Boston

Algebraic Graph Theory International Webinar

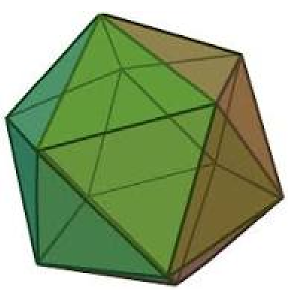
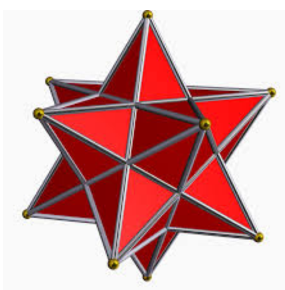
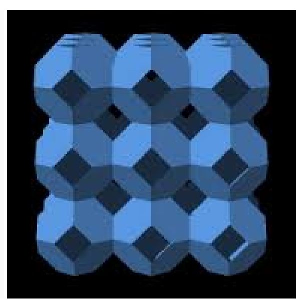
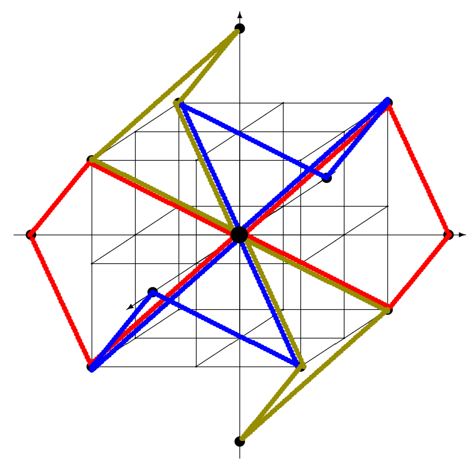
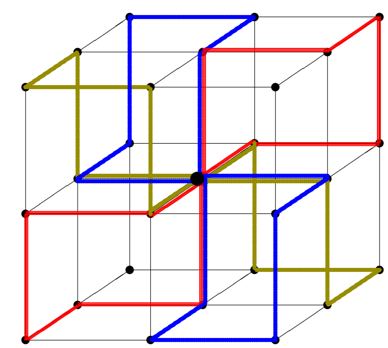
February 21, 2023

Polyhedra

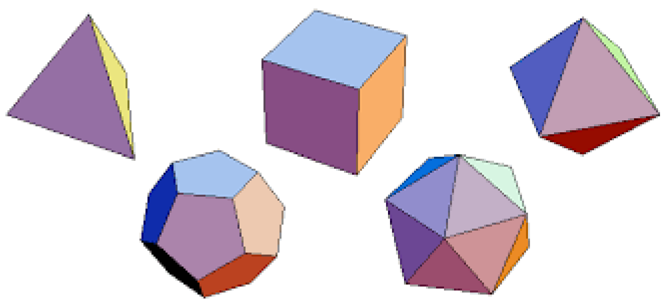
- Ancient history (Greeks), closely tied to symmetry.

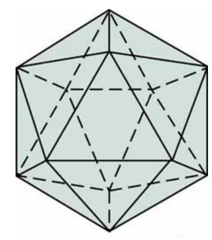
- Over time, many changes in point of view about polyhedral structures and their symmetry. Many different definitions!

So what's a polyhedron?

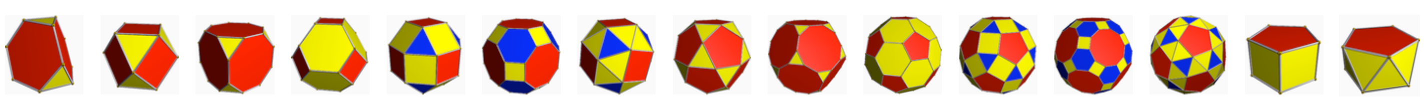
Five Platonic solids (solids, convexity)



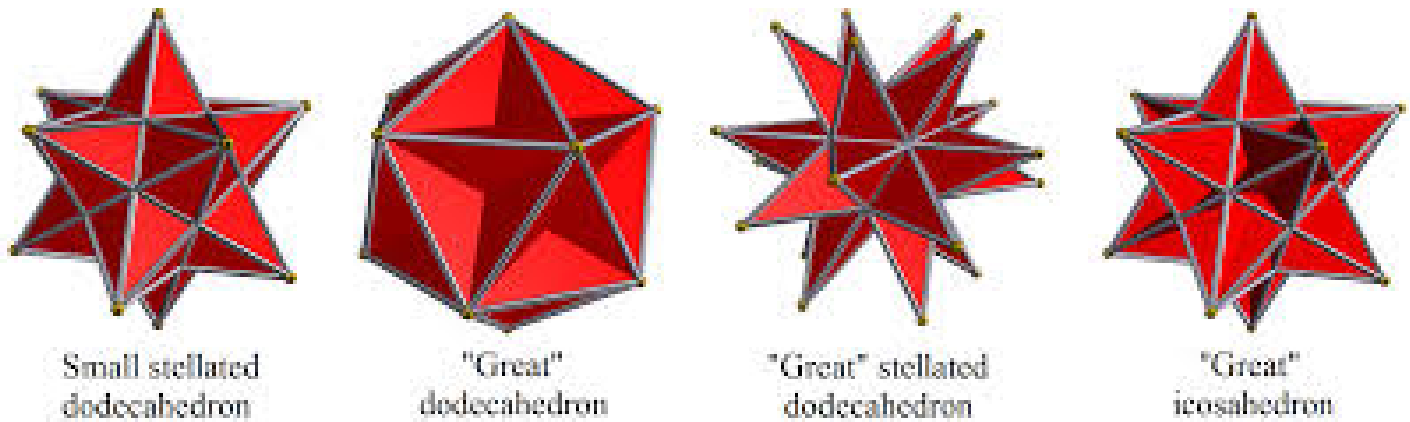


$\{3, 5\}$

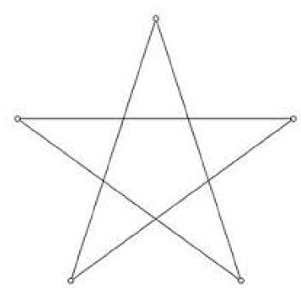
Thirteen Archimedean solids, plus prisms and antiprisms



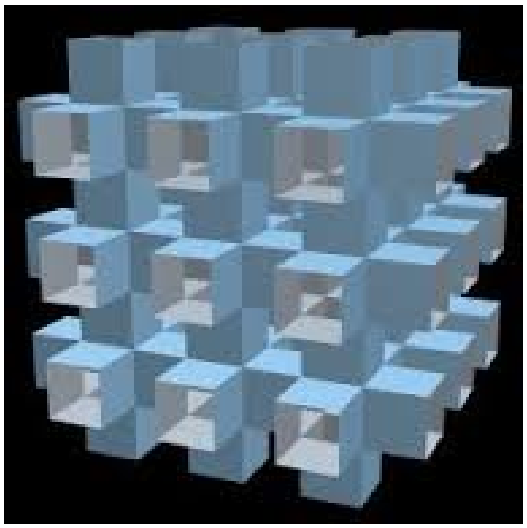
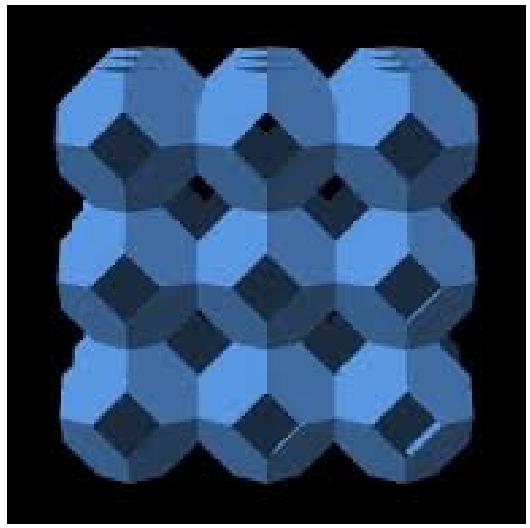
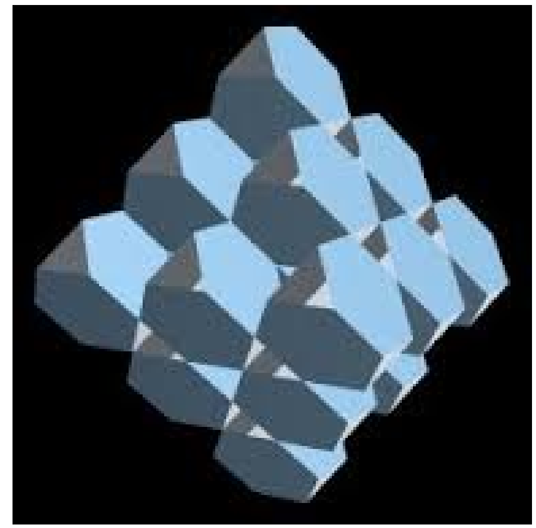
Four Kepler-Poinsot (star) polyhedra



Faces and vertex-figures can be star-polygons (pentagrams).



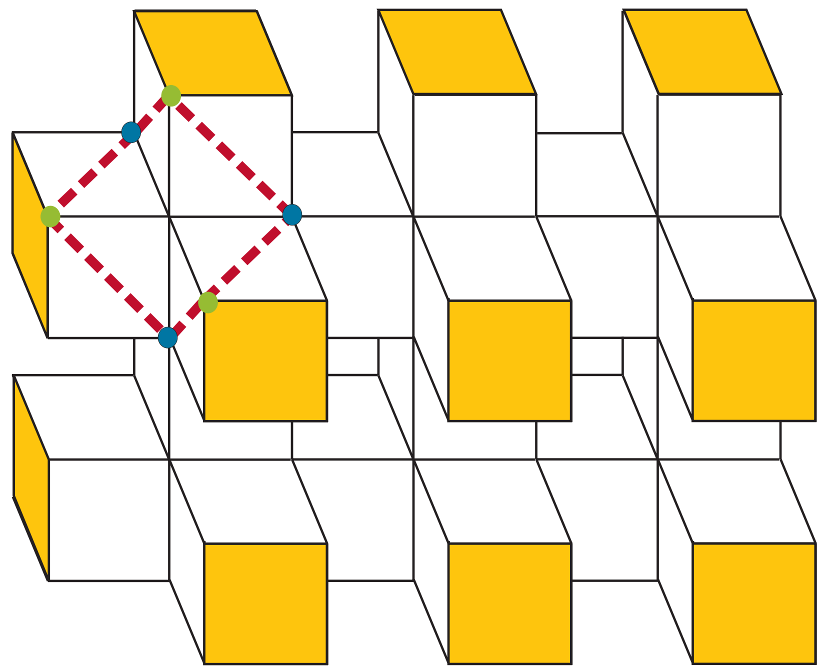
Three Petrie-Coxeter polyhedra (sponges)

Infinite polyhedra (apeirohedra)! Faces still convex polygons! Vertex-figures are skew (non-planar) polygons! Periodic!

Vertex-figure at vertex x: joins the vertices adjacent to x in the order in which they occur around x

Vertex-figure of the Petrie-Coxeter polyhedron $\{4, 6|4\}$



Vertex-figures skew hexagons! Faces squares! All regular!

Can build a new polyhedral structure from this by putting together all vertex-figures taken at every other vertex!

Skeletal Polyhedra

• Graph-theoretical (skeletal) approach initiated by Grünbaum (1970's).

• Faces and vertex-figures allowed to be skew!
• Faces are cycles or path of edges! Allowed to be zigzags or helical polygons!
• No membranes spanned into faces! Focus on skeleton!

• Skeletal regular polyhedra in ordinary space?
Grünbaum-Dress Polyhedra

• Symmetry groups are reflection groups generated by reflections R_0, R_1, R_2 in points, lines, or planes. Accounts for *skew* faces and vertex-figures!

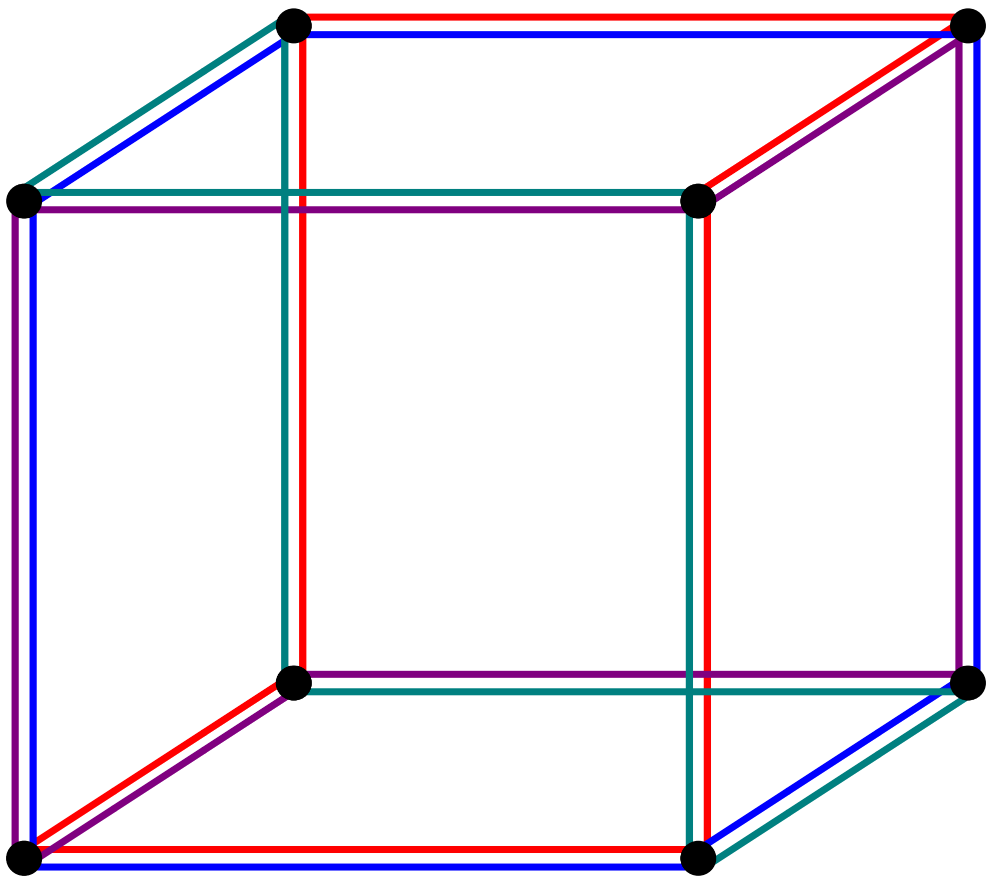
Skeletal Polyhedron

Polygon: connected *graph*, only vertices of valency 2.

A polyhedron P in $\mathbb{E}^3$ is a finite or infinite family of simple polygons, called *faces*, such that

- each edge of a face is an edge of just one other face,
- all faces incident with a vertex form one circuit,
- P is connected,
- each compact set meets only finitely many faces (discreteness).

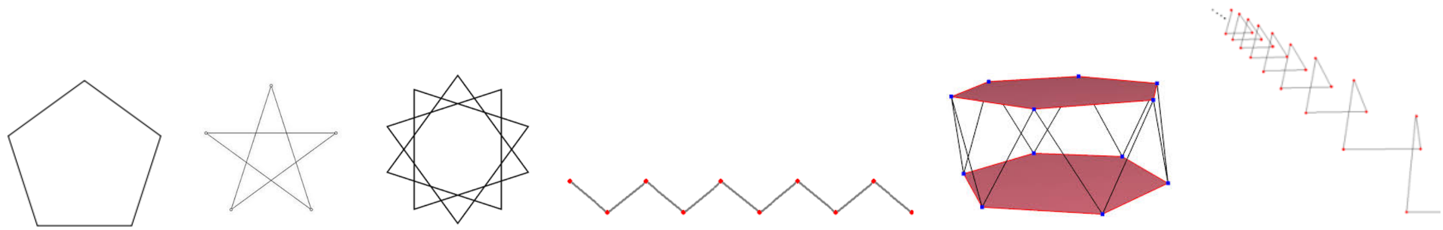
All traditional polyhedra are skeletal polyhedra.



The Petrie dual (Petrial) of the cube. A regular polyhedron with 8 vertices, 12 edges, 4 skew hexagonal faces. Type $\{6,3\}$.

Highly symmetric skeletal polyhedra

• Faces finite (flat or skew) or infinite (helical or zig-zags)! Vertex-figures finite (flat or skew)!

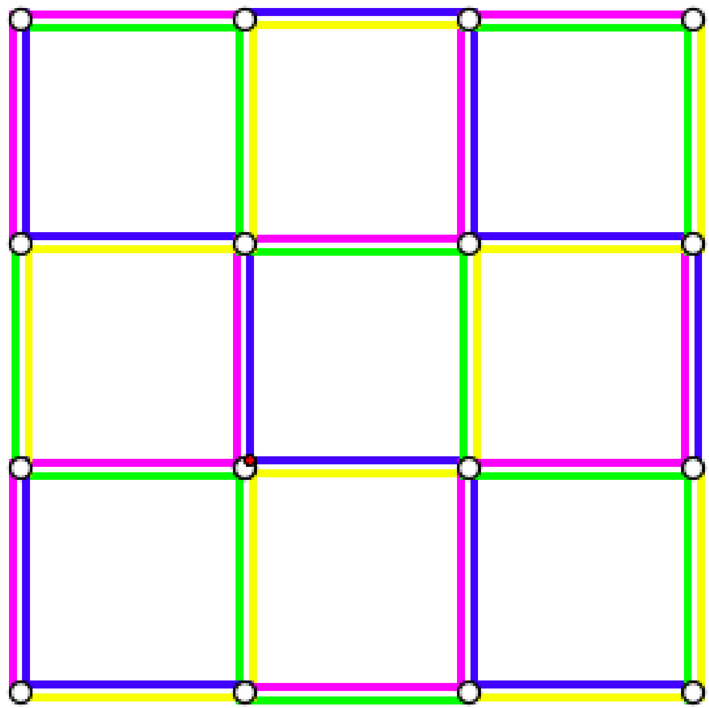


• P called regular if the symmetry group $G(P)$ is transitive on the flags.

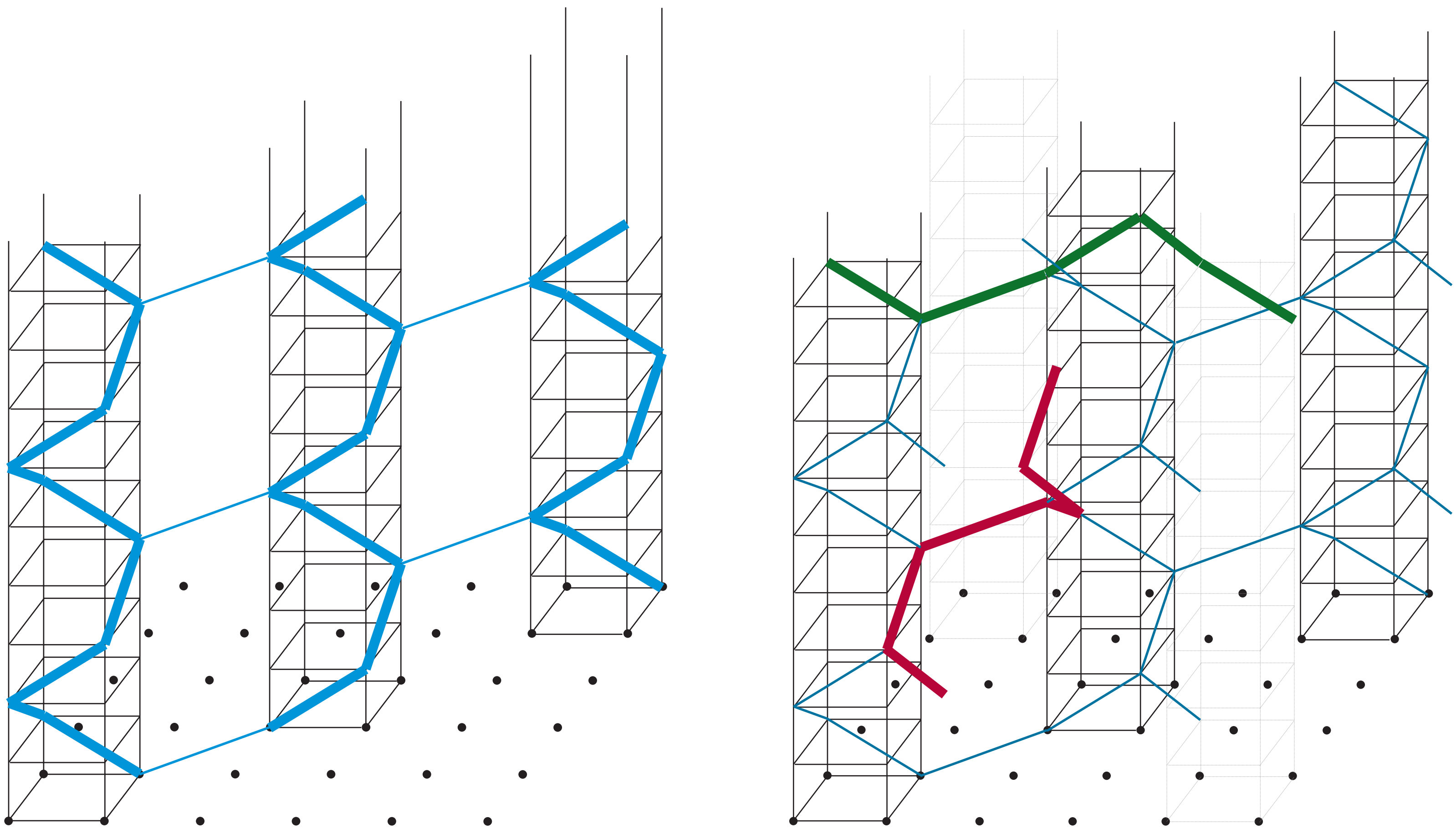
Flag: incident triple of a vertex, an edge, and a face.

• P called chiral if $G(P)$ has two orbits on the flags such that adjacent flags are in distinct orbits.

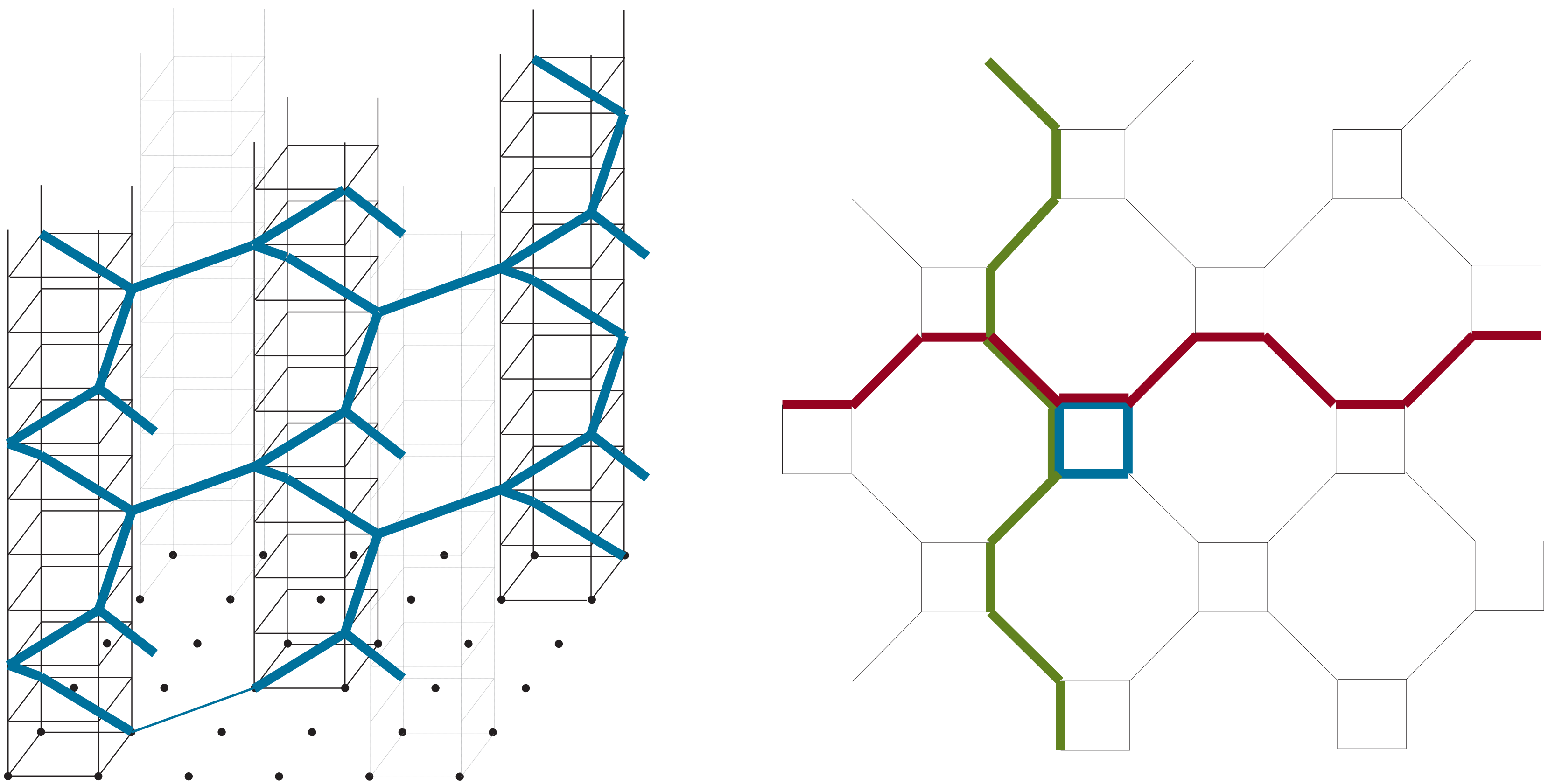
• P called Archimedean if $G(P)$ is vertex-transitive and P has regular polygons as faces.



The Petrie dual of the square tessellation. An infinite regular polyhedron with zig-zag faces. Type $\{\infty, 4\}$.



The *helix-faced regular* *polyhedron* $\{\infty, 3\}^{(b)}$.



The helix-faced polyhedron $\{\infty, 3\}^{(b)}$

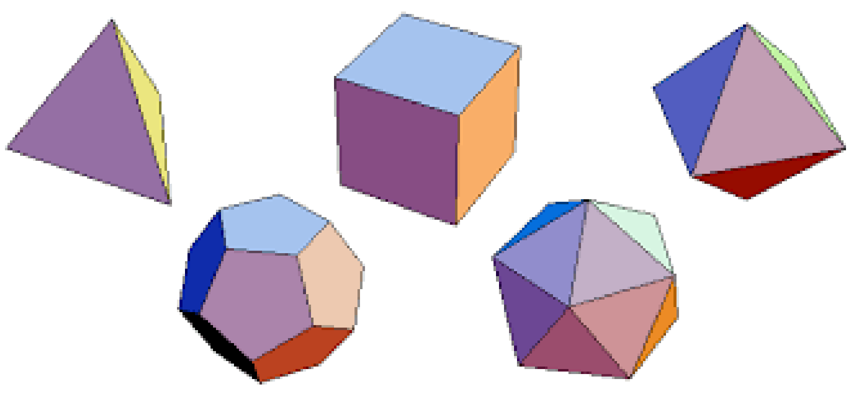
The 48 Regular Polyhedra in $\mathbb{E}^3$

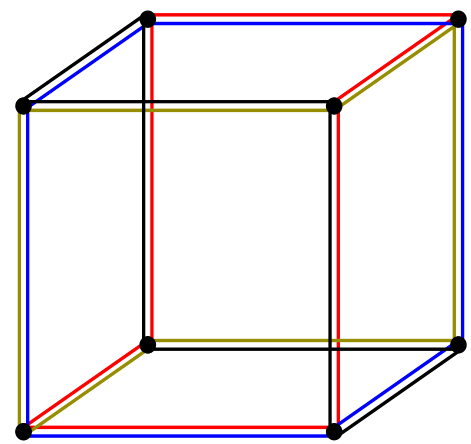
(Grünbaum 1970's, Dress 1981. New approach in McMullen & S. 1997)

- Symmetry group generated by reflections R_0, R_1, R_2 in points, lines, or planes. Classification of such triples of reflections (R_0, R_1, R_2)!

18 finite polyhedra: 5 Platonic, 4 Kepler-Poinsot, 9 Petrials.

(2 full tetrahedral symmetry, 4 full octahedral, 12 full icosahedral)





$\{4,3\}^{\pi}$

Finite regular polyhedra

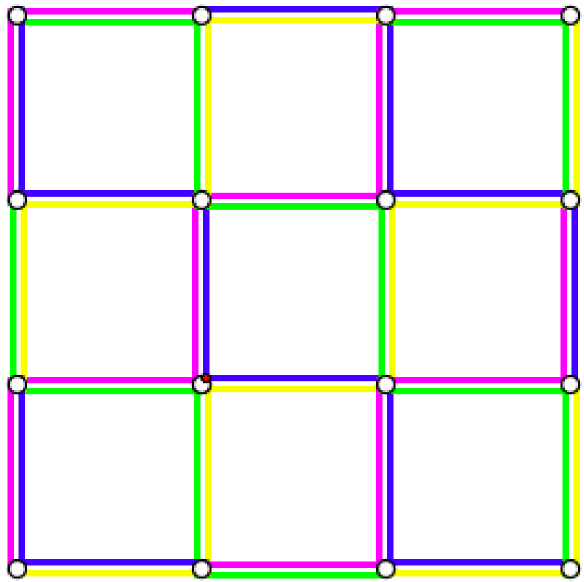
18 finite (5 Platonic, 4 Kepler-Poinsot, 9 Petrials)

tetrahedral			$\{3,3\}$	$\overset{\pi}{\longleftrightarrow}$	$\{4,3\}_3$			
octahedral	$\{6,4\}_3$	$\overset{\pi}{\longleftrightarrow}$	$\{3,4\}$	$\overset{\delta}{\longleftrightarrow}$	$\{4,3\}$	$\overset{\pi}{\longleftrightarrow}$	$\{6,3\}_4$
icosahedral	$\{10,5\}$	$\overset{\pi}{\longleftrightarrow}$	$\{3,5\}$	$\overset{\delta}{\longleftrightarrow}$	$\{5,3\}$	$\overset{\pi}{\longleftrightarrow}$	$\{10,3\}$
	$\updownarrow \varphi_2$		$\updownarrow \varphi_2$					
	$\{6,\frac{5}{2}\}$	$\overset{\pi}{\longleftrightarrow}$	$\{5,\frac{5}{2}\}$	$\overset{\delta}{\longleftrightarrow}$	$\{\frac{5}{2},5\}$	$\overset{\pi}{\longleftrightarrow}$	$\{6,5\}$
					$\updownarrow \varphi_2$		$\updownarrow \varphi_2$
	$\{\frac{10}{3},3\}$	$\overset{\pi}{\longleftrightarrow}$	$\{\frac{5}{2},3\}$	$\overset{\delta}{\longleftrightarrow}$	$\{3,\frac{5}{2}\}$	$\overset{\pi}{\longleftrightarrow}$	$\{\frac{10}{3},\frac{5}{2}\}$

duality δ: R_2, R_1, R_0; Petrie π: R_0R_2, R_1, R_0; facetting φ_2: $R_0, R_1R_2R_1, R_2$

30 apeirohedra (infinite polyhedra)! Crystallographic groups!

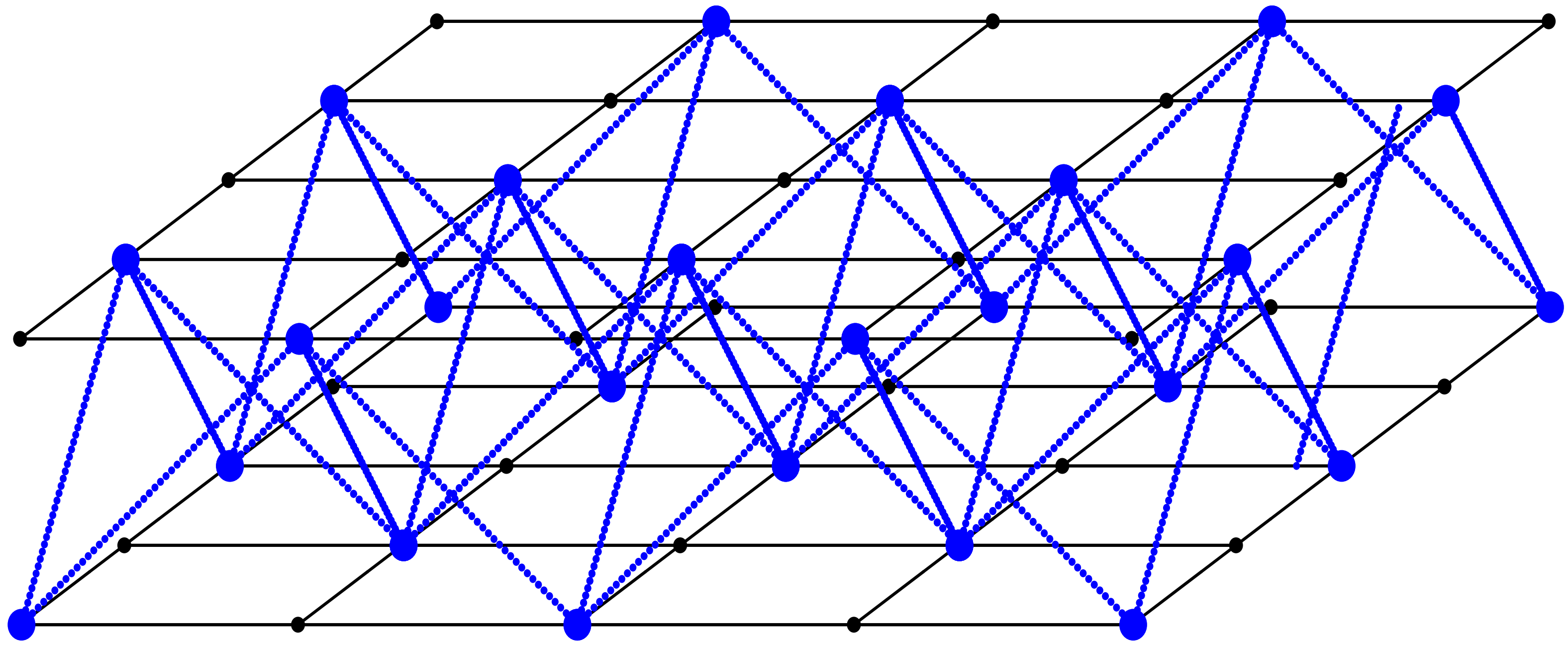
6 planar (three regular tessellations, and their Petrials)



The Petrie dual $\{4,4\}^{\pi}$*, of type* $\{\infty,4\}$*.*

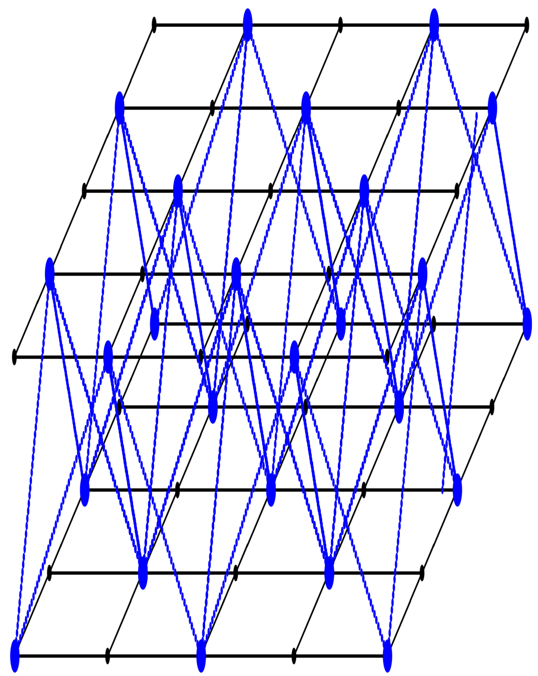
12 reducible apeirohedra. Blends of a planar polyhedron and a linear polygon (line segment or line tessellation).

Blends of a planar polyhedron and a line segment



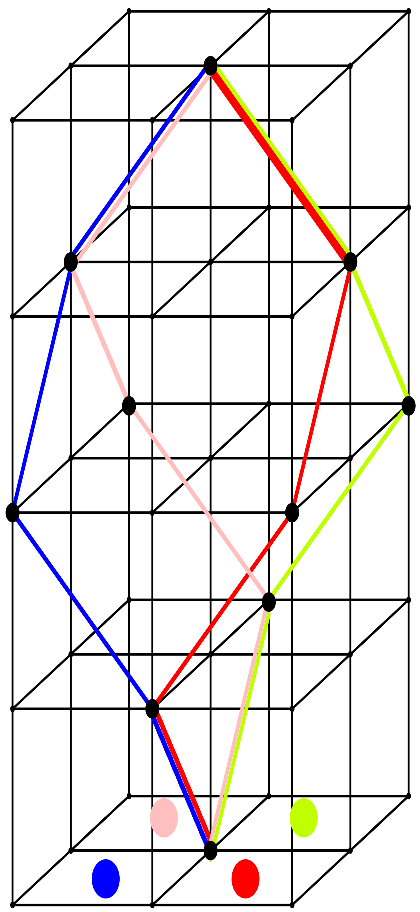
Square tessellation $\{4,4\}$ blended with the line segment $\{\,\}$. Symbol $\{4,4\}\#\{\,\}$.

Same blend, different ratio between components



Square tessellation $\{4,4\}$ blended with the line segment $\{\,\}$. Symbol $\{4,4\}\#\{\,\}$.

Blends of a planar polyhedron and a line tessellation

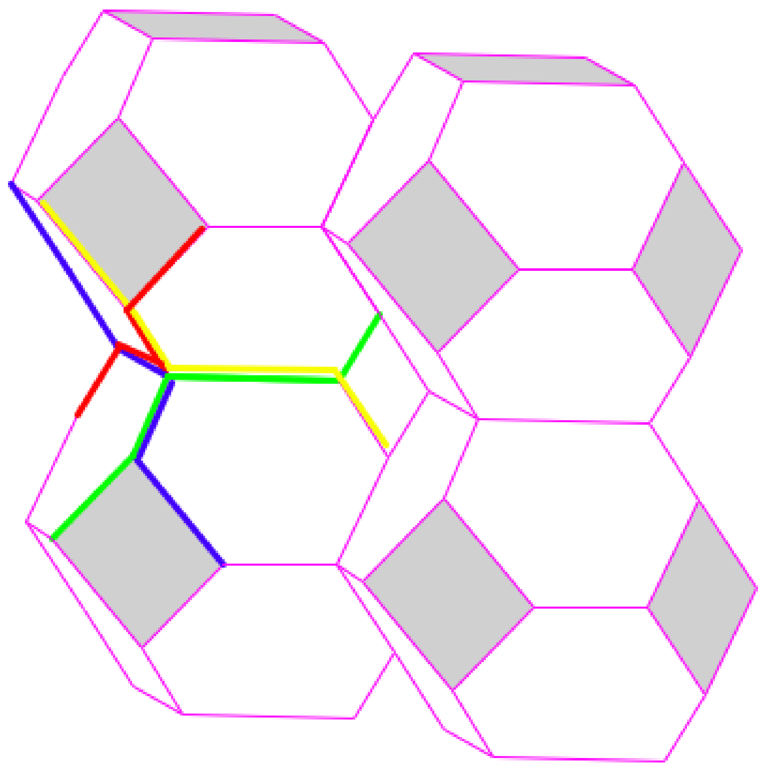


The square tessellation $\{4,4\}$ blended with a line tessellation $\{\infty\}$. Symbol $\{4,4\}\#\{\infty\}$. Each vertical column is occupied by a single helical facet spiraling around the column.

12 irreducible apeirohedra.

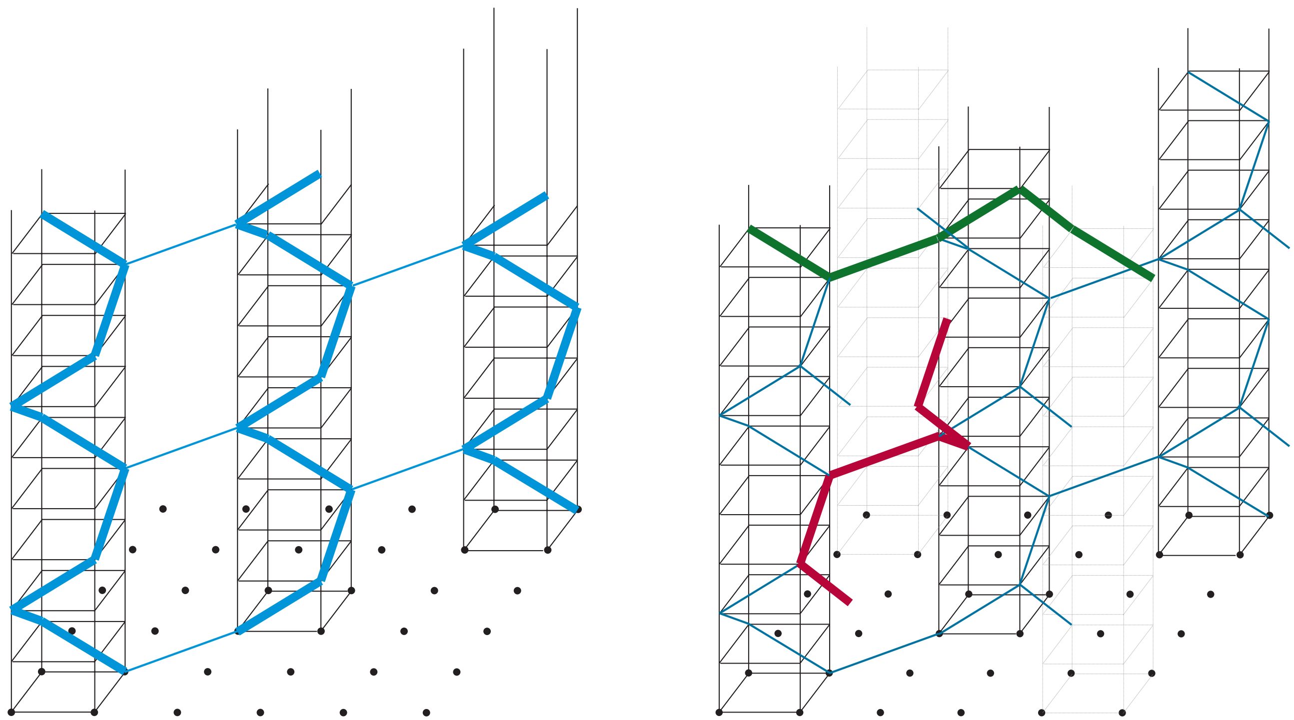
$$
\begin{array}{ccccccc}
\{\infty,4\}_{6,4} & \stackrel{\pi}{\longleftrightarrow} & \{6,4|4\} & \stackrel{\delta}{\longleftrightarrow} & \{4,6|4\} & \stackrel{\pi}{\longleftrightarrow} & \{\infty,6\}_{4,4} \\
 & & \sigma\downarrow & & \downarrow\eta & & \\
 & & \{\infty,4\}_{\cdot,*3} & & \{6,6\}_4 & \stackrel{\varphi_2}{\longrightarrow} & \{\infty,3\}^{(a)} \\
 & & & & \pi\updownarrow & & \updownarrow\pi \\
 & & \{6,4\}_6 & \stackrel{\delta}{\longleftrightarrow} & \{4,6\}_6 & \stackrel{\varphi_2}{\longrightarrow} & \{\infty,3\}^{(b)} \\
 & & \sigma\delta\downarrow & & \downarrow\eta & & \\
 & & \{\infty,6\}_{6,3} & \stackrel{\pi}{\longleftrightarrow} & \{6,6|3\} & &
\end{array}
$$

$\eta\colon R_0R_1R_0, R_2, R_1$; $\sigma = \pi\delta\eta\pi\delta\colon R_1, R_0R_2, (R_1R_2)^2$; $\varphi_2\colon R_0, R_1R_2R_1, R_2$



$\{6,4|4\}^\pi$, *the Petrie dual of the Petrie-Coxeter polyhedron* $\{6,4|4\}$.
Alternative notation: $\{\infty,4\}_{6,4}$.

Not every regular polyhedron has a geometric dual! For example, $\{\infty,4\}_{6,4}$ *does not!*



The helix-faced regular polyhedron $\{\infty, 3\}^{(b)}$. *Its Petrie dual is* $\{\infty, 3\}^{(a)}$.
Neither has a geometric dual!

Symmetry group of $\{\infty, 3\}^{(b)}$ *requires the single extra relation*

$$(R_0R_1)^4(R_0R_1R_2)^3 = (R_0R_1R_2)^3(R_0R_1)^4.$$

Breakdown by mirror vector (for reflection generators R_0, R_1, R_2)

Vector (m_0, m_1, m_2), where m_i is the dimension of the mirror of R_i.

mirror vector	$\{3,3\}$	$\{3,4\}$	$\{4,3\}$	faces	vertex-figures
(2,1,2)	$\{6,6\|3\}$	$\{6,4\|4\}$	$\{4,6\|4\}$	planar	skew
(1,1,2)	$\{\infty,6\}_{4,4}$	$\{\infty,4\}_{6,4}$	$\{\infty,6\}_{6,3}$	helical	skew
(1,2,1)	$\{6,6\}_4$	$\{6,4\}_6$	$\{4,6\}_6$	skew	planar
(1,1,1)	$\{\infty,3\}^{(a)}$	$\{\infty,4\}_{\cdot,*3}$	$\{\infty,3\}^{(b)}$	helical	planar

Last row: polyhedra occur in two enantiomorphic forms, yet geometrically regular!

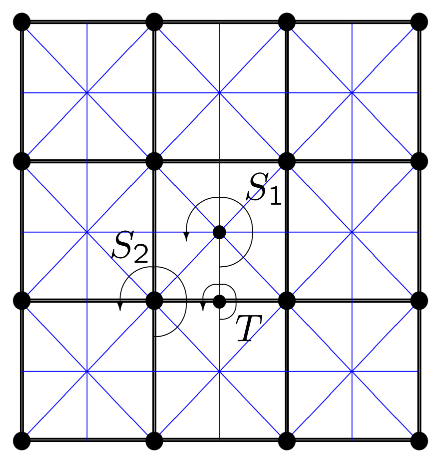
Presentations for the symmetry groups are known. The "fine" Schläfli symbol signifies defining relations. Extra relations specify order of $R_0R_1R_2$, $R_0R_1R_2R_1$, or $R_0(R_1R_2)^2$.

Chiral Polyhedra in $\mathbb{E}^3$

Chirality in the presence of very high symmetry! Combinatorialized notion of chirality!

- Two flag orbits under symmetry group $G(P)$, with adjacent flags in different orbits. Maximal "rotational" symmetry, no "reflexive" symmetry!

- Local picture

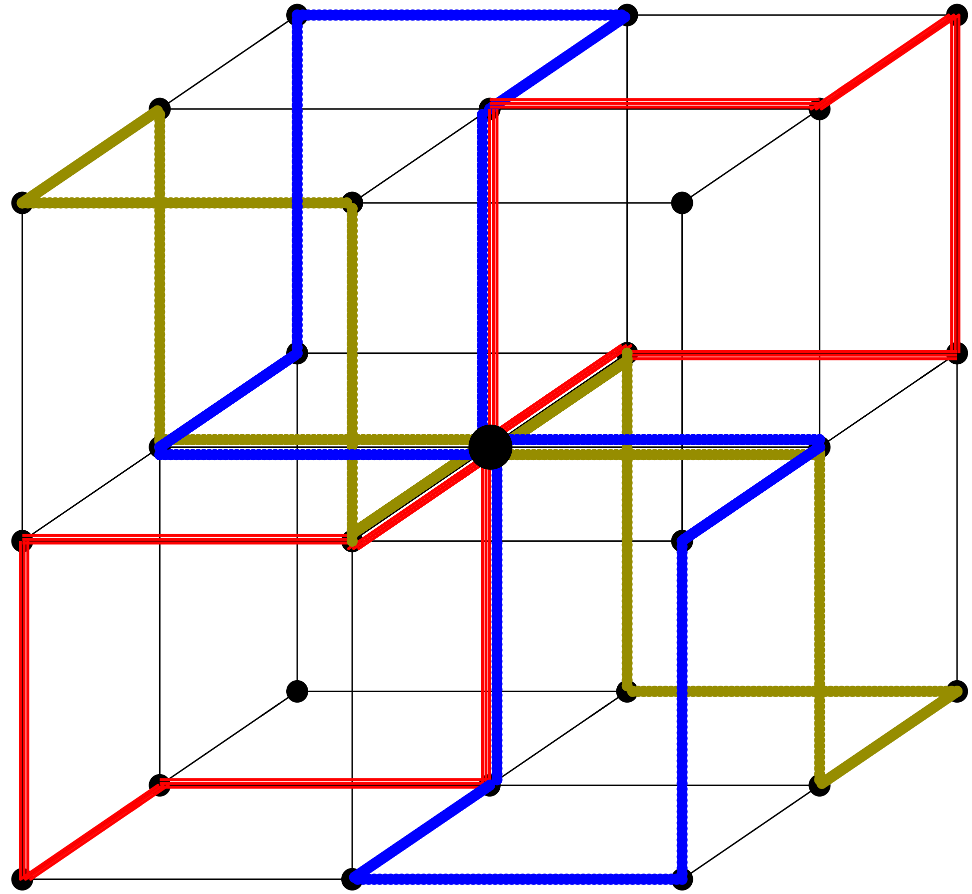


$T = S_1 S_2$

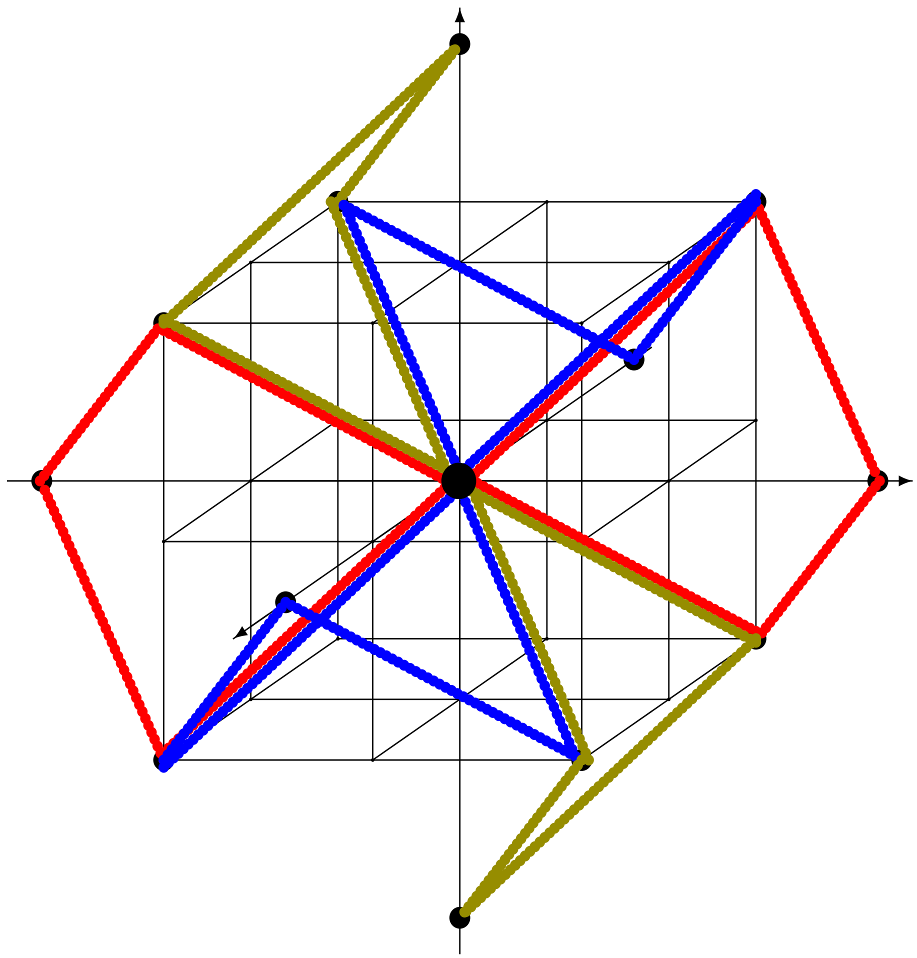
Geometric symmetries S_1 and S_2 must exist! S_1, S_2 are NOT geometric rotations in general, but combinatorially they act like rotations would!

- Classification breaks down into
  - polyhedra with finite faces and
  - polyhedra with infinite faces!
- Three very large 2-parameter families of chiral polyhedra of each kind!
- Chiral polyhedra must be infinite (apeirohedra)! No finite or blended examples! Finite faces must be *skew*, and infinite faces must be *helical*.

S. (2004, 2005)



Chiral polyhedron $P(1,0)$ of type $\{6,6\}$. Neighborhood of a single vertex.



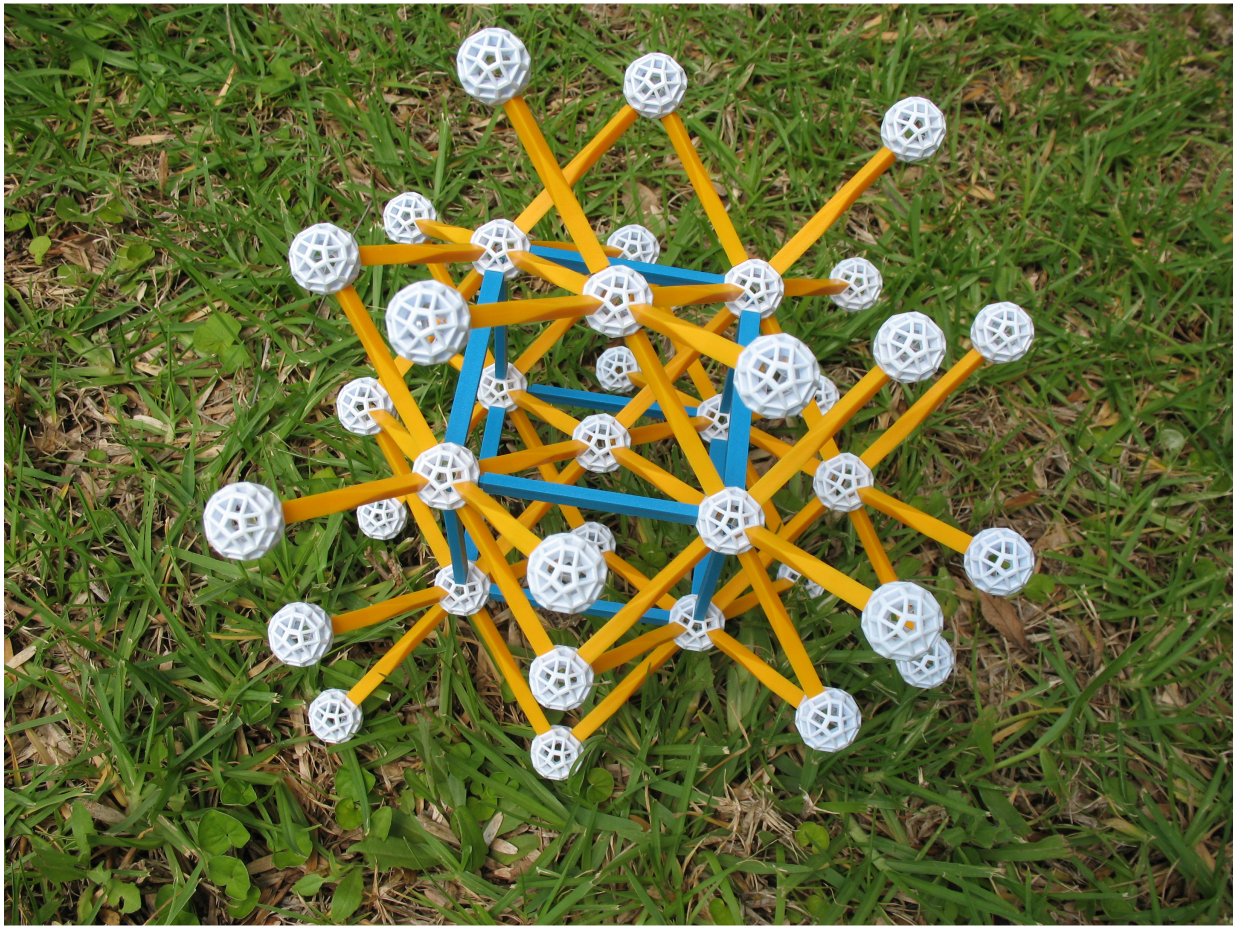
Chiral polyhedron $Q(1,1)$, type $\{4,6\}$. Neighborhood of a single vertex.

Three Classes of Finite-Faced Chiral Polyhedra

(S_1, S_2 rotatory reflections, hence skew faces and skew vertex-figures.)

Schläfli	$\{6, 6\}$	$\{4, 6\}$	$\{6, 4\}$
Notation	$P(a, b)$	$Q(c, d)$	$Q(c, d)^*$
Param.	$a, b \in \mathbb{Z}$, $(a, b) = 1$	$c, d \in \mathbb{Z}$, $(c, d) = 1$	$c, d \in \mathbb{Z}$, $(c, d) = 1$
	geom. self-dual $P(a, b)^* \cong P(a, b)$		
Regular cases	$P(a,-a)=\{6,6\}_4$ $P(a,a)=\{6,6\|3\}$	$Q(c,0)=\{4,6\}_6$ $Q(0,d)=\{4,6\|4\}$	$Q(c,0)^*=\{6,4\}_6$ $Q(0,d)^*=\{6,4\|4\}$

Each extended family contains two regular polyhedra (for these parameter values the faces or vertex-figures become flat).



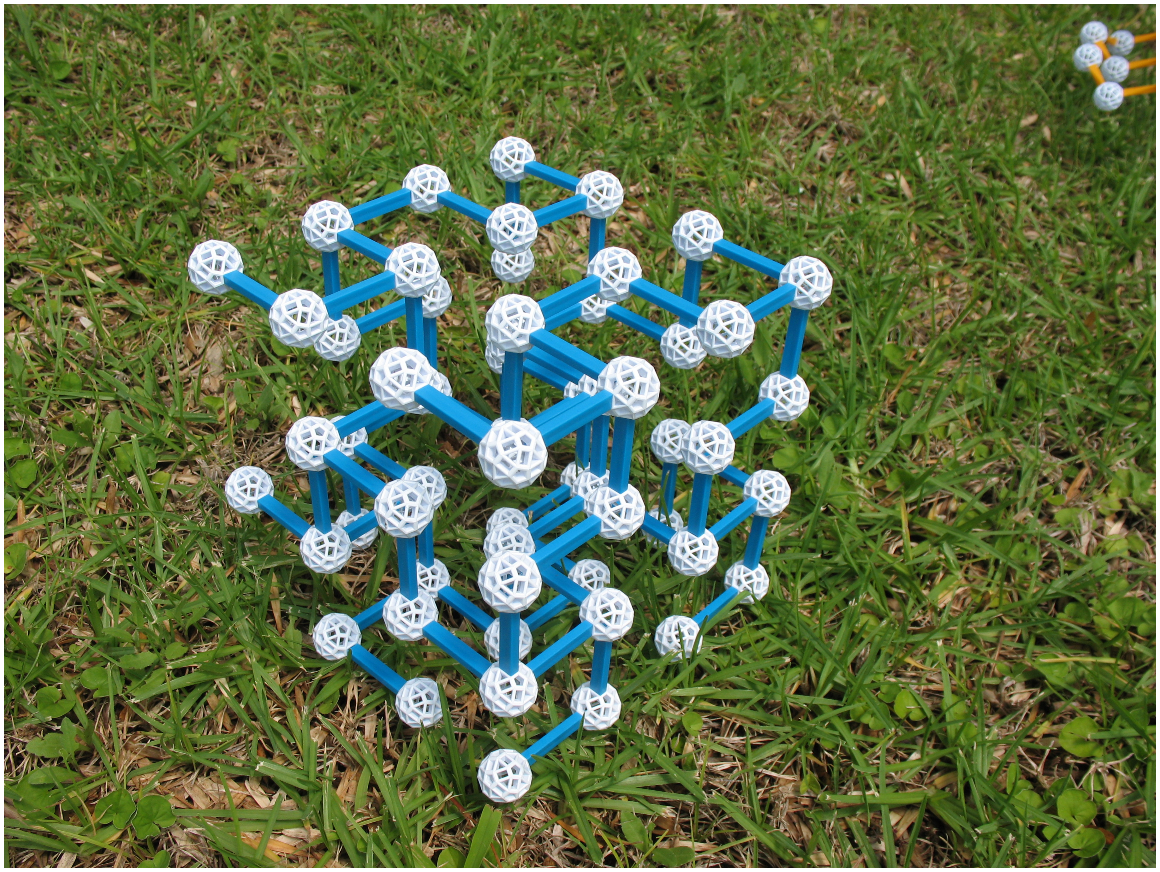
Chiral polyhedron $Q(1,1)$, type $\{4,6\}$. Skew squares, six at each vertex. Vertex set is Λ_3. (All models built and photographed by Daniel Pellicer.)

Three Classes of Helix-Faced Chiral Polyhedra

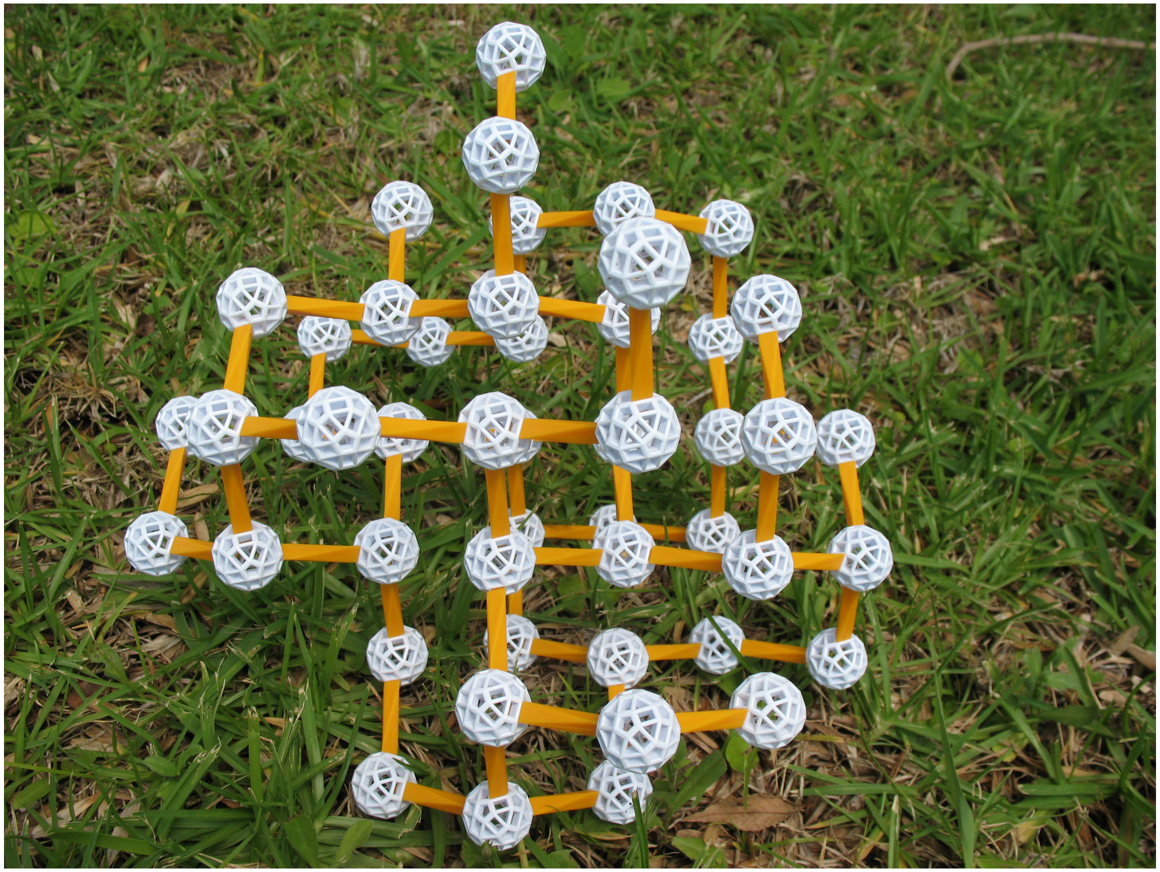
(S_1 screw motion, S_2 rotation; helical faces and planar vertex-figures.)

Schläfli	$\{\infty,3\}$	$\{\infty,3\}$	$\{\infty,4\}$
Notat.	$P_1(a,b)$	$P_2(c,d)$	$P_3(c,d)^*$
Param.	$a,b\in\mathbb{R}$, $(a,b)\neq(0,0)$	$c,d\in\mathbb{R}$, $(c,d)\neq(0,0)$	$c,d\in\mathbb{R}$, $(c,d)\neq(0,0)$
Helices over	triangles	squares	triangles
	$P(a,b)^{\varphi_2}$	$Q(c,d)^{\varphi_2}$	$Q^*(c,d)^{\kappa}$
Regular	$P_1(a,-a)=\{\infty,3\}^{(a)}$ $P_1(a,a)=\{3,3\}$	$P_2(c,0)=\{\infty,3\}^{(b)}$ $P_2(0,d)=\{4,3\}$	$P_3(0,d)=\{\infty,4\}_{\cdot,*3}$ (self-Petrie) $P_3(c,0)=\{3,4\}$

Each extended family contains two regular polyhedra, one finite and one infinite. Helices collapse or vertex-stars become planar.



Chiral polyhedron $P_1(0,1)$ of type $\{\infty,3\}$. Helical faces over triangles, three at each vertex. Photo taken in the direction of a helix; triangular projection of a helical face visible.



Chiral polyhedron $P_2(1,1)$ of type $\{\infty, 3\}$. Helical faces over squares, three at each vertex. Photo taken in the direction of a helix.

Remarkable facts about chiral polyhedra

- Essentially: any two finite-faced polyhedra are combinatorially non-isomorphic.

$$P(a,b) \cong P(a',b') \text{ iff } (a',b') = \pm(a,b), \pm(b,a).$$

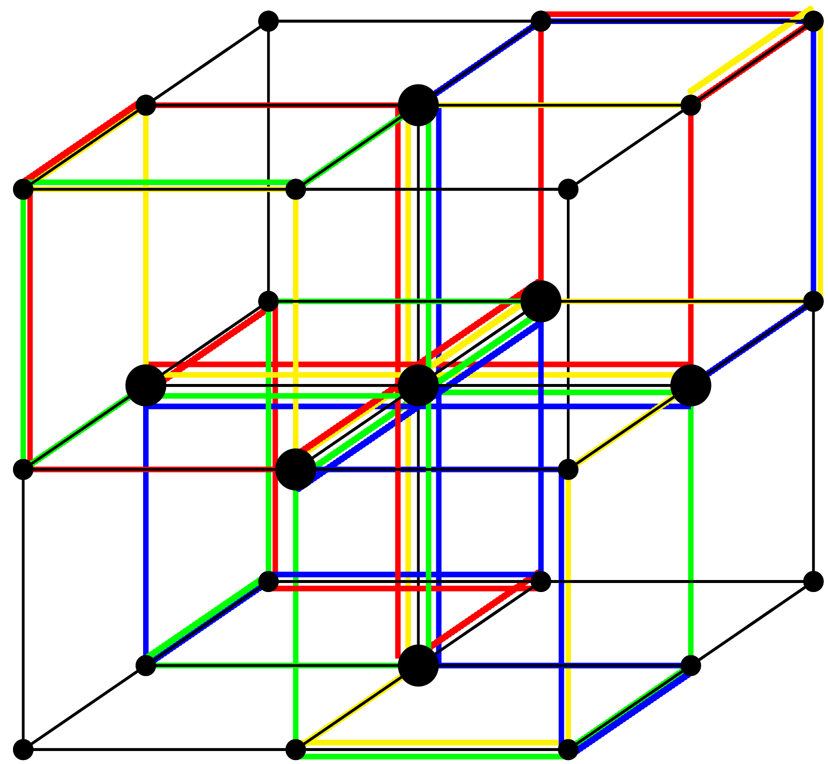
$$Q(c,d) \cong Q(c',d') \text{ iff } (c',d') = \pm(c,d), \pm(-c,d).$$

- Finite-faced polyhedra are combinatorially chiral! Helix-faced polyhedra combinatorially regular! Chiral helix-faced polyhedra are deformations of regular helix-faced polyhedra! [Pellicer & Weiss 2009].

- Chiral helix-faced polyhedra unravel Platonic solids! Coverings

$$\{\infty,3\} \mapsto \{3,3\}, \ \{\infty,3\} \mapsto \{4,3\}, \ \ \{\infty,4\} \mapsto \{3,4\}.$$

More polygons on an edge ....



Vertex neighborhood in $\mathcal{K}_4(1,2)$: 4 faces at an edge; 12 at a vertex (octahedral vertex-figure). All Petrie polygons of every other cube. Net **pcu**.

Regular Polygonal Complexes in $\mathbb{E}^3$

(joint with D.Pellicer, 2010, 2013)

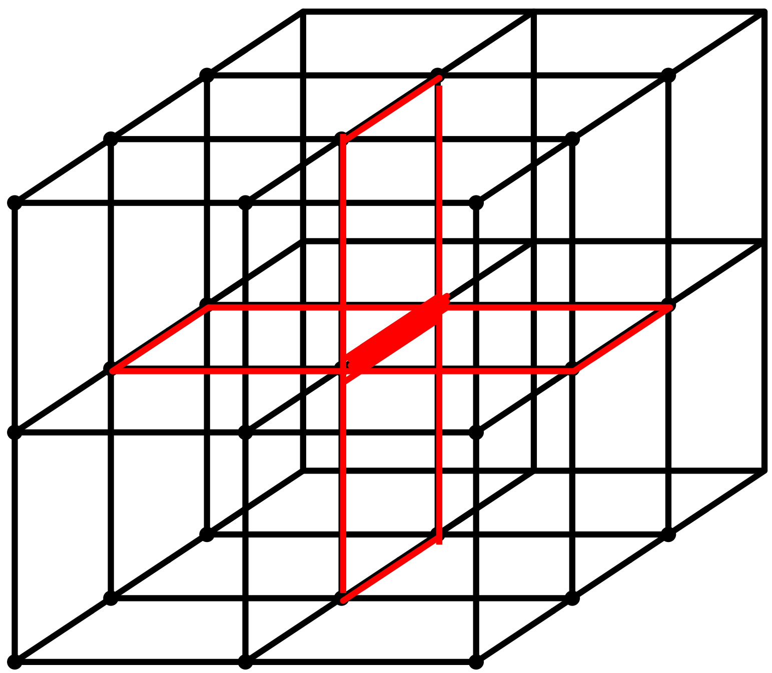
A polygonal complex K in $\mathbb{E}^3$ is a family of simple polygons, called *faces*, such that

- each edge of a face is an edge of exactly r faces ($r \geq 2$);
- the vertex-figure at each vertex is a connected graph, possibly with multiple edges;
- the edge graph of K is connected;
- each compact set meets only finitely many faces (discreteness).

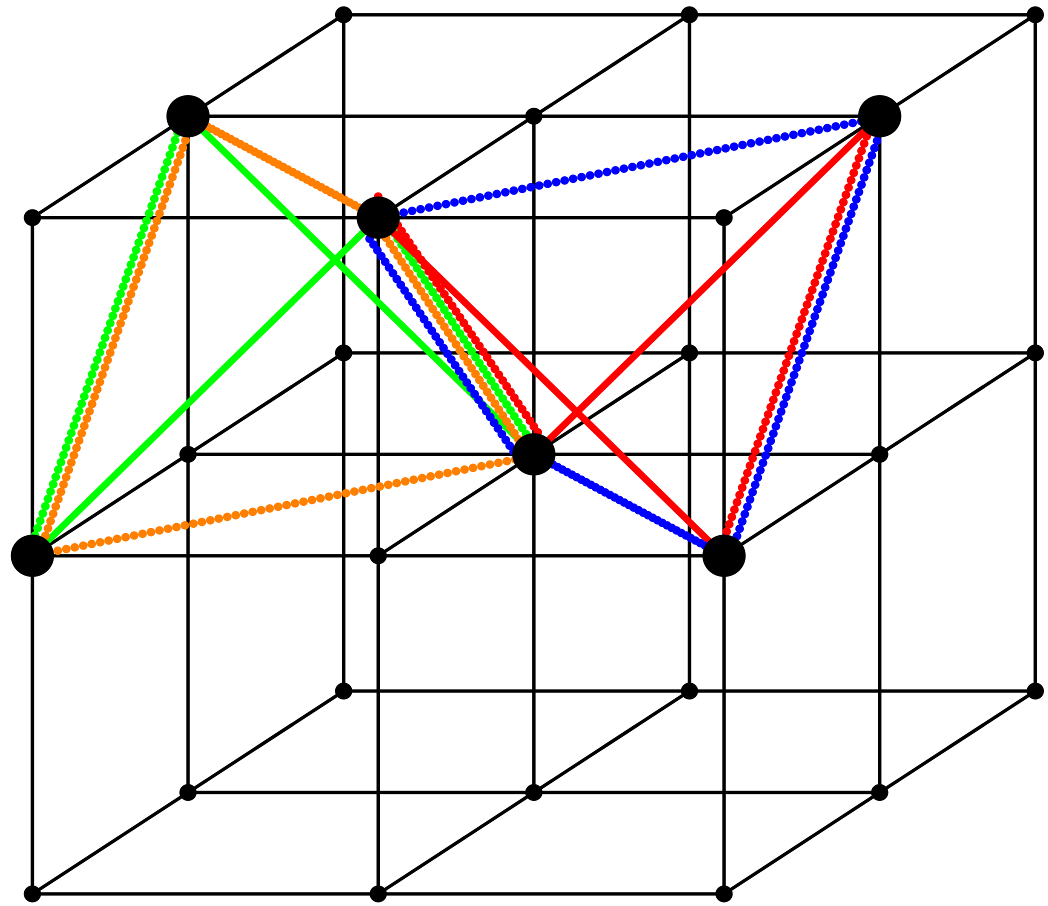
K is regular if its geometric symmetry group $G(K)$ is transitive on the flags of K.
(flag: incident vertex-edge-face triple)

Examples

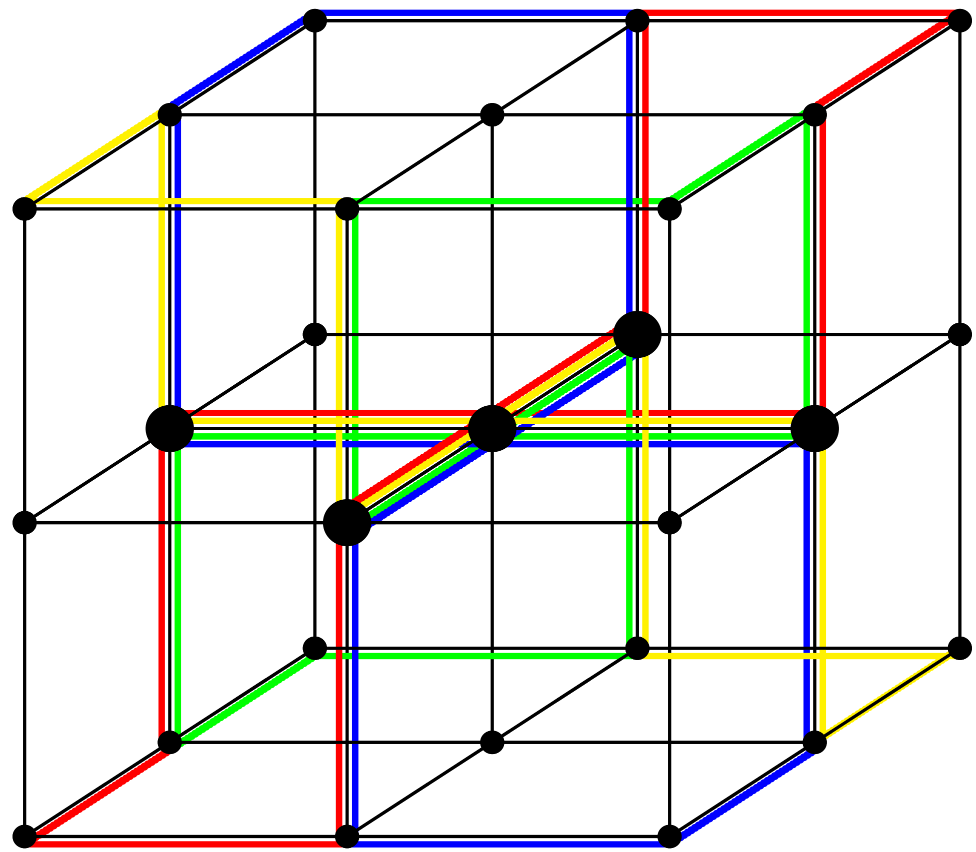
- All regular polyhedra ($r = 2$). There are 48.
- All squares of the cubical tessellation ($r = 4$).



Vertex-figure: octahedral graph!



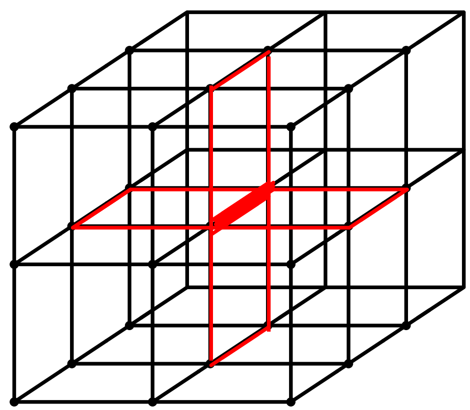
$\mathcal{K}_1(1,2)$: four tetragons on an edge. Petrie polygons of tetrahedra inscribed in cubes, in an alternating fashion. The net is **fcu**.



$\mathcal{K}_5(1,2)$: 4 faces at an edge, 8 at a vertex (double square as vertex-figure). One Petrie-polygon for each cube. Net is **nbo** (Niobium Monoxide, NbO).

<u>Case:</u> Symmetry group $G(K)$ not simply flag-transitive

• K is the 2-skeleton of a certain rank 4 structure in $\mathbb{E}^3$, called a regular 4-apeirotope. There are *eight* such rank 4 structures contributing four regular polygonal complexes!



Eight regular 4-apeirotopes in $\mathbb{E}^3$ (in pairs of Petrie duals).

Infinite! Two have square faces, the others zigzag faces. Face mirrors!

The eight regular 4-apeirotopes in $\mathbb{E}^3$

$\{4,3,4\}$	$\{\{4,6\,\|\,4\},\{6,4\}_3\}$
$\text{apeir}\{3,3\} = \{\{\infty,3\}_6\#\{\,\},\{3,3\}\}$	$\{\{\infty,4\}_4\#\{\infty\},\{4,3\}_3\} = \text{apeir}\{4,3\}_3$
$\text{apeir}\{3,4\} = \{\{\infty,3\}_6\#\{\,\},\{3,4\}\}$	$\{\{\infty,6\}_3\#\{\infty\},\{6,4\}_3\} = \text{apeir}\{6,4\}_3$
$\text{apeir}\{4,3\} = \{\{\infty,4\}_4\#\{\,\},\{4,3\}\}$	$\{\{\infty,6\}_3\#\{\infty\},\{6,3\}_4\} = \text{apeir}\{6,3\}_4$

<u>Case:</u> Symmetry group $G(K)$ simply flag-transitive

- Includes all regular polyhedra.

- Finite complexes must be polyhedra (18 examples).

- 21 simply flag-transitive regular polygonal complexes in $\mathbb{E}^3$ which are not polyhedra and are infinite.

The 21 simply flag-transitive regular polygonal complexes in $\mathbb{E}^3$ which are not polyhedra, and their nets (edge graphs).

Complex	G_2	r	Face	Vertex-Figure	Vertex Set	G^*	Net
$\mathcal{K}_1(1,2)$	D_2	4	4_s	cuboctahedron	Λ_2	$[3,4]$	**fcu**
$\mathcal{K}_2(1,2)$	C_3	3	4_s	cube	Λ_3	$[3,4]$	**bcu**
$\mathcal{K}_3(1,2)$	D_3	6	4_s	double cube	Λ_3	$[3,4]$	**bcu**
$\mathcal{K}_4(1,2)$	D_2	4	6_s	octahedron	Λ_1	$[3,4]$	**pcu**
$\mathcal{K}_5(1,2)$	D_2	4	6_s	double square	V	$[3,4]$	**nbo**
$\mathcal{K}_6(1,2)$	D_4	8	6_s	double octah.	Λ_1	$[3,4]$	**pcu**
$\mathcal{K}_7(1,2)$	D_3	6	6_s	double tetrah.	W	$[3,4]$	**dia**
$\mathcal{K}_8(1,2)$	D_2	4	6_s	cuboctahedron	Λ_2	$[3,4]$	**fcu**
$\mathcal{K}_1(1,1)$	D_3	6	∞_3	double cube	Λ_3	$[3,4]$	**bcu**
$\mathcal{K}_2(1,1)$	D_2	4	∞_3	double square	V	$[3,4]$	**nbo**

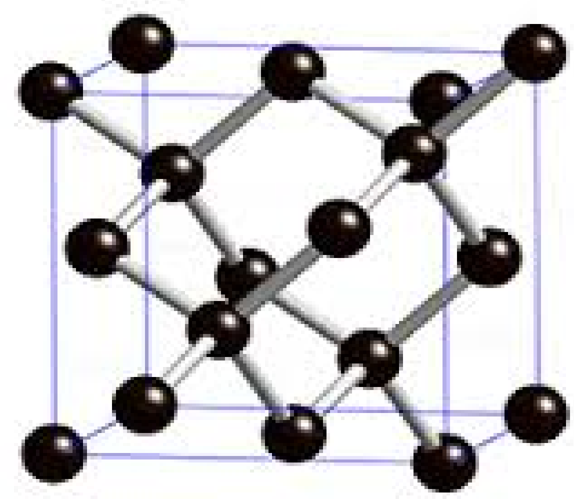
nbo = net of Niobium Monoxide, NbO

The 21 complexes and their nets (cont.).

Complex	G_2	r	Face	Vertex-Figure	Vertex Set	G^*	Net
$\mathcal{K}_3(1,1)$	D_4	8	∞_3	double octah.	Λ_1	$[3,4]$	**pcu**
$\mathcal{K}_4(1,1)$	D_3	6	∞_4	double tetrah.	W	$[3,4]$	**dia**
$\mathcal{K}_5(1,1)$	D_2	4	∞_4	ns-cuboctah.	Λ_2	$[3,4]$	**fcu**
$\mathcal{K}_6(1,1)$	C_3	3	∞_4	tetrahedron	W	$[3,4]^+$	**dia**
$\mathcal{K}_7(1,1)$	C_4	4	∞_3	octahedron	Λ_1	$[3,4]^+$	**pcu**
$\mathcal{K}_8(1,1)$	D_2	4	∞_3	ns-cuboctah.	Λ_2	$[3,4]$	**fcu**
$\mathcal{K}_9(1,1)$	C_3	3	∞_3	cube	Λ_3	$[3,4]^+$	**bcu**
$\mathcal{K}(0,1)$	D_2	4	∞_2	ns-cuboctah.	Λ_2	$[3,4]$	**fcu**
$\mathcal{K}(0,2)$	D_2	4	∞_2	cuboctah.	Λ_2	$[3,4]$	**fcu**
$\mathcal{K}(2,1)$	D_2	4	6_c	ns-cuboctah.	Λ_2	$[3,4]$	**fcu**
$\mathcal{K}(2,2)$	D_2	4	3_c	cuboctahedron	Λ_2	$[3,4]$	**fcu**

$V := \mathbb{Z}^3 \setminus ((0,0,1)+\Lambda_{(1,1,1)}),\;\; W := 2\Lambda_{(1,1,0)} \cup ((1,-1,1)+2\Lambda_{(1,1,0)})$

Edge-graph (net) of $K_7(1,2)$: diamond net, modeling the diamond crystal. (Carbon atoms sit at the vertices, and bonds between adjacent atoms are represented by edges. The "hexagonal rings" are the faces of $K_7(1,2)$.)



Edges of $K_7(1,2)$ run along main diagonals of cubes in $\{4,3,4\}$. Six skew hexagonal faces around an edge (r = 6). Vertex-figure is the double edge-graph of the tetrahedron (so 12 faces meet at a vertex).

Archimedean (Uniform) Skeletal Polyhedra in $\mathbb{E}^3$

- Faces are *regular* polygons (flat, skew, helical, zigzag).
- $G(P)$ transitive on vertices of P.

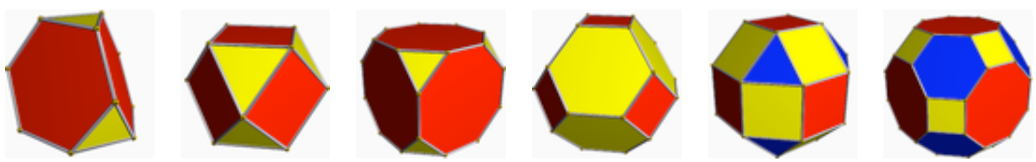
What is known?

- *Convex* polyhedra: Archimedean solids
  Skeletal analogues of the Archimedean solids!
- *Finite* Archimedean polyhedra with *planar* faces
  — Classical paper by Coxeter, Longuet-Higgins and Miller (1954)
  — Completeness proof by Skilling (1974), Har'El (1993).
- Arbitrary Archimedean skeletal polyhedra wide open!
  — Finite polyhedra with skew faces not classified.

Tractable class: Wythoffians ("truncations")

(E.S. & Abigail Williams 2016)

• Archimedean solids from Platonic solids via Wythoff's construction (exploits reflection group structure)!
Archimedean solids: Wythoffians of Platonics!



....

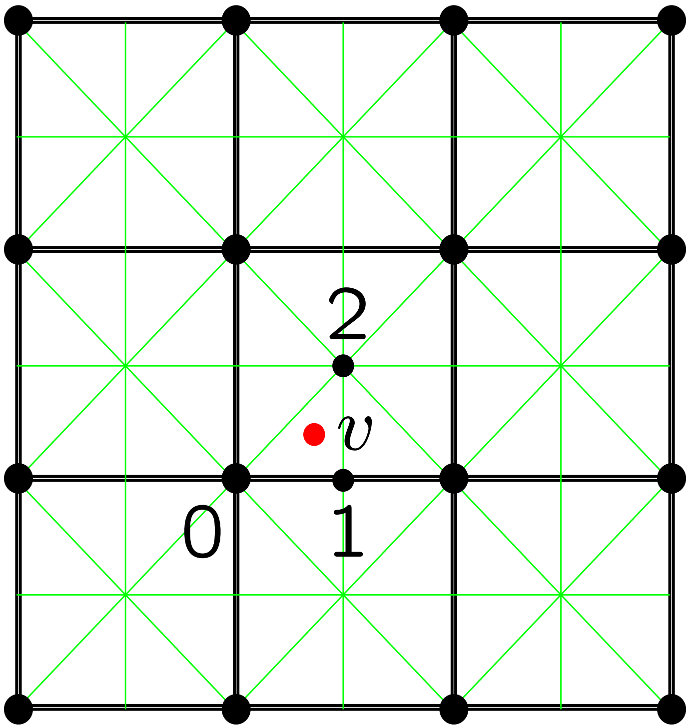
• The 48 regular skeletal polyhedra in $\mathbb{E}^3$ have symmetry groups generated by reflections (in points, lines or planes)!

• Run a variant of Wythoff to produce skeletal Wythoffians of the 48 regular skeletal polyhedra.

• Initial vertex in $\mathbb{E}^3$ (not on a surface, less confined).

• Wythoffians always vertex-transitive, not always Archimedean!

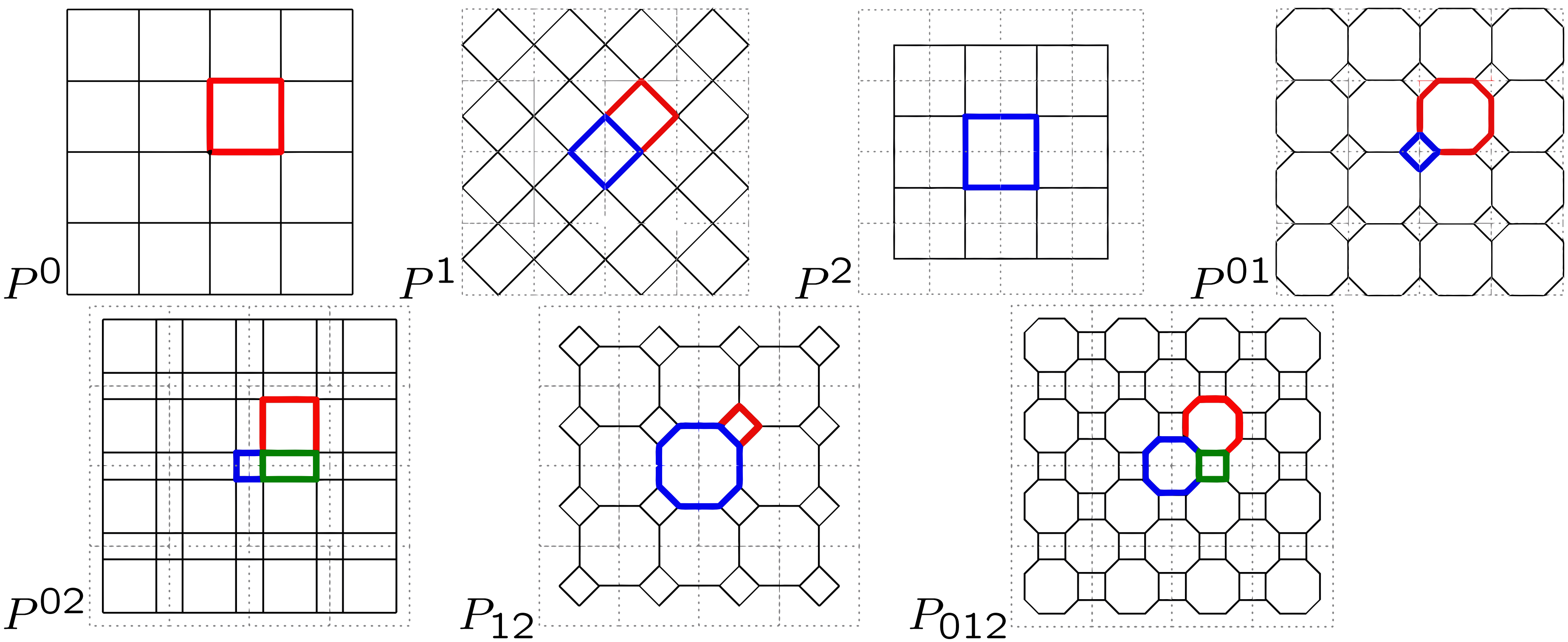
Wythoff's construction in the classical case



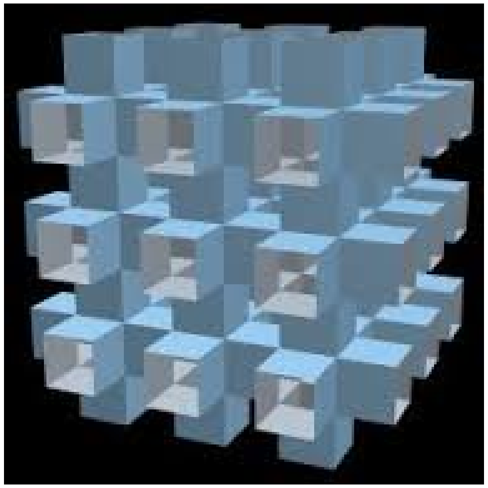
initial vertex v of type $\{0, 1, 2\}$

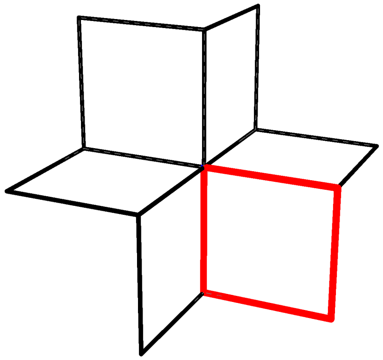
Labeled simplicial 2-complex (barycentric subdivision, order complex)! Seven possible types I for initial vertices v (non-empty subsets of $\{0, 1, 2\}$)

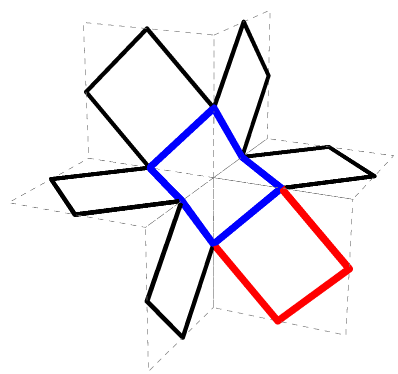
Wythoffian's of the square tessellation $\{4,4\}$

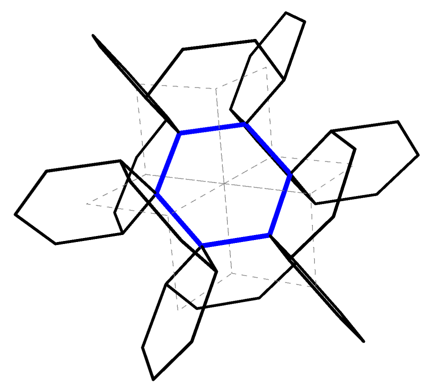


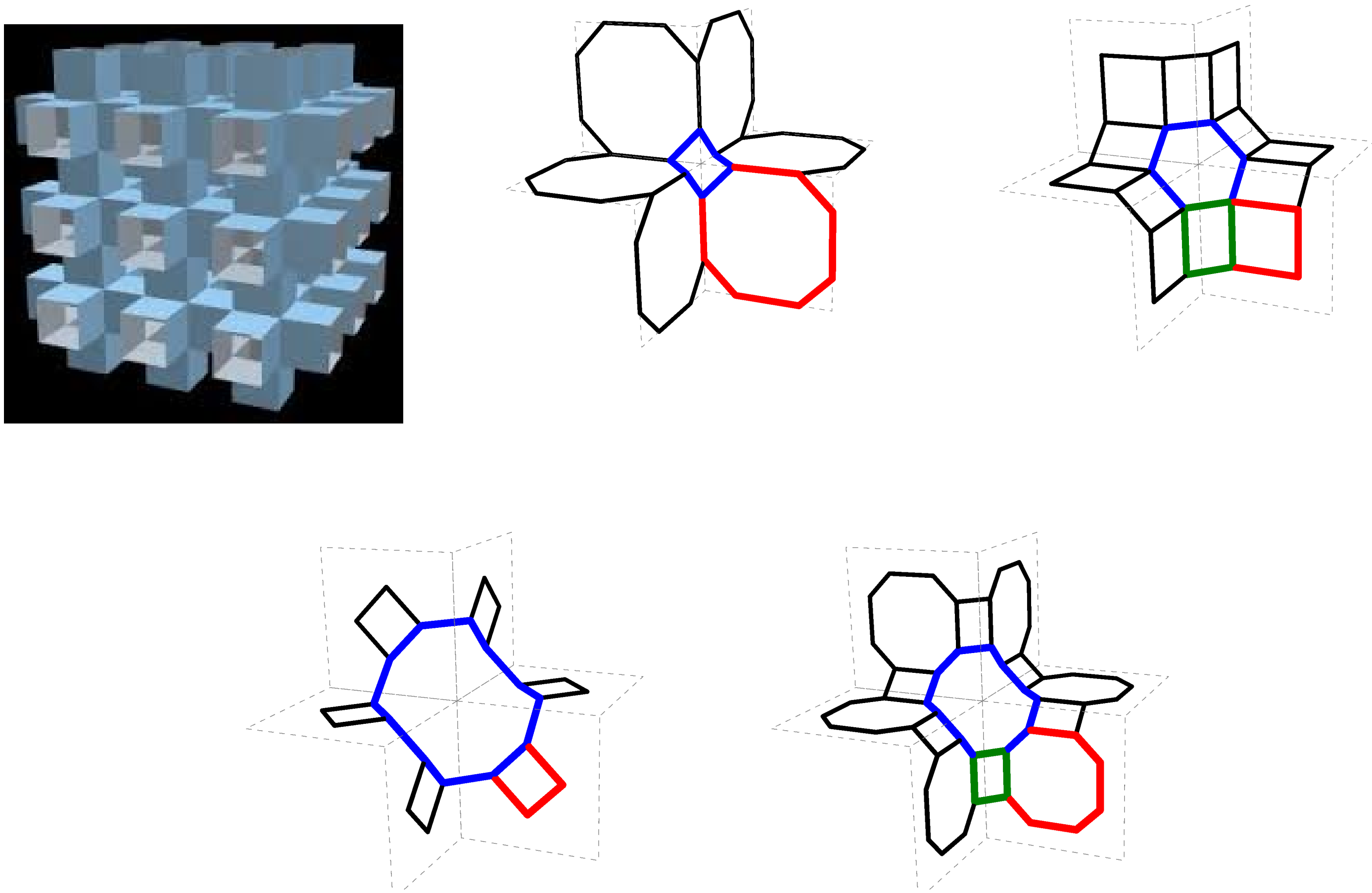
The seven Wythoffians of the Petrie-Coxeter polyhedron $\{4, 6|4\}$





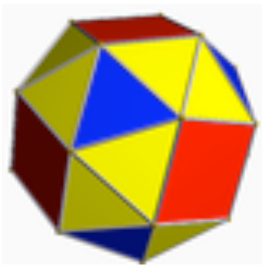
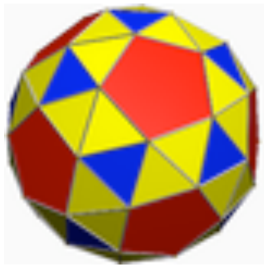






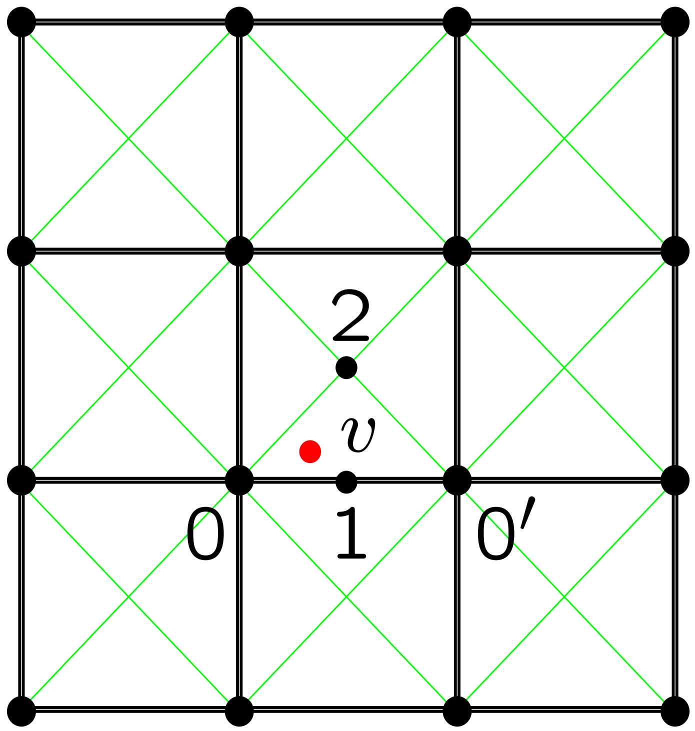
Another tractable class: Snub Wythoffians (“truncations”) (in progress, joint with Tomas Skacel)

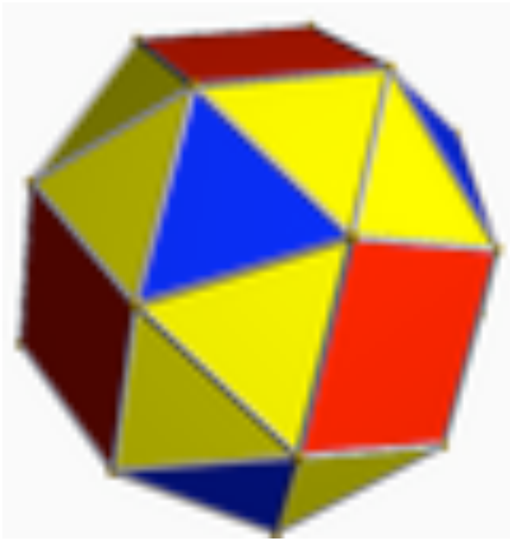
Snub cube and snub dodecahedron

- Use a “rotational variant” of Wythoff’s construction to produce skeletal snub Wythoffians of the regular skeletal polyhedra. .
- Exploit structure of the (combinatorial) “rotation subgroup” of $G(P)$!
- Snub Wythoffians always vertex-transitive, not always Archimedean!

Wythoff's construction in the classical case

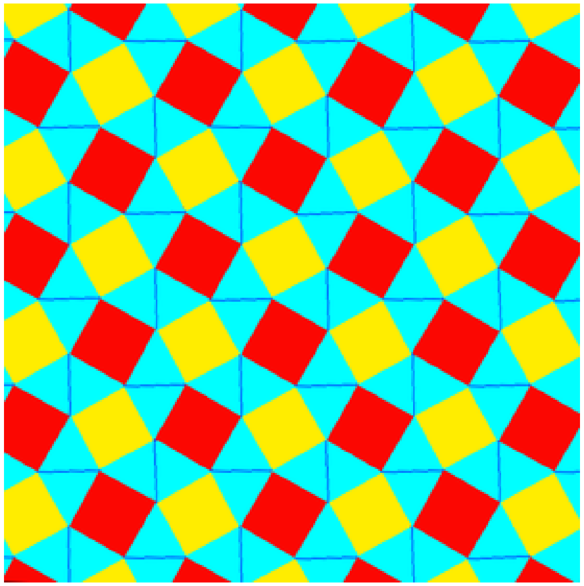


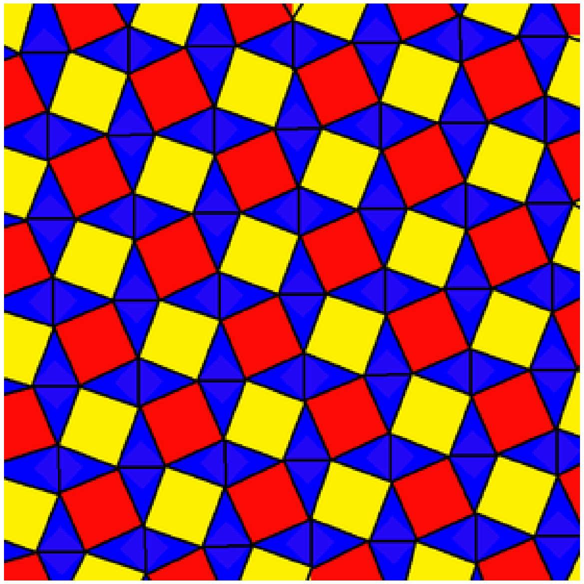


Allow only "rotational" symmetries.

Skeletal variant of Wythoff (surface free)!

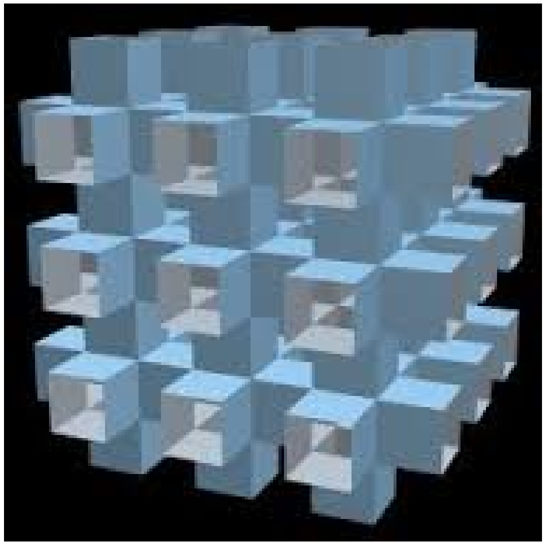
Snub Wythoffian of the square tessellation $\{4, 4\}$

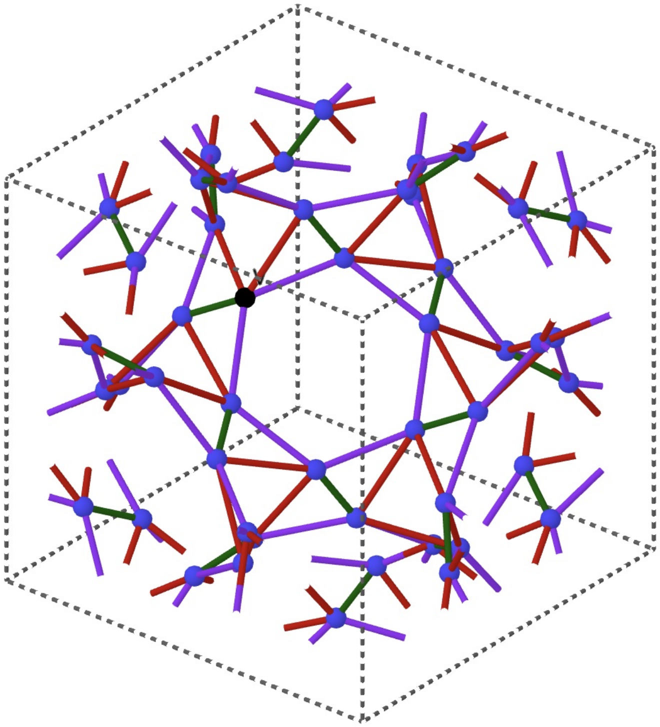




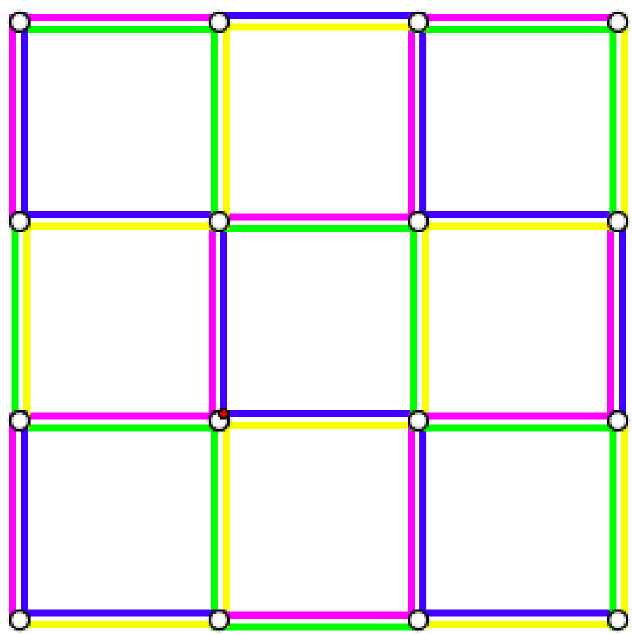
uniform vs. nonuniform

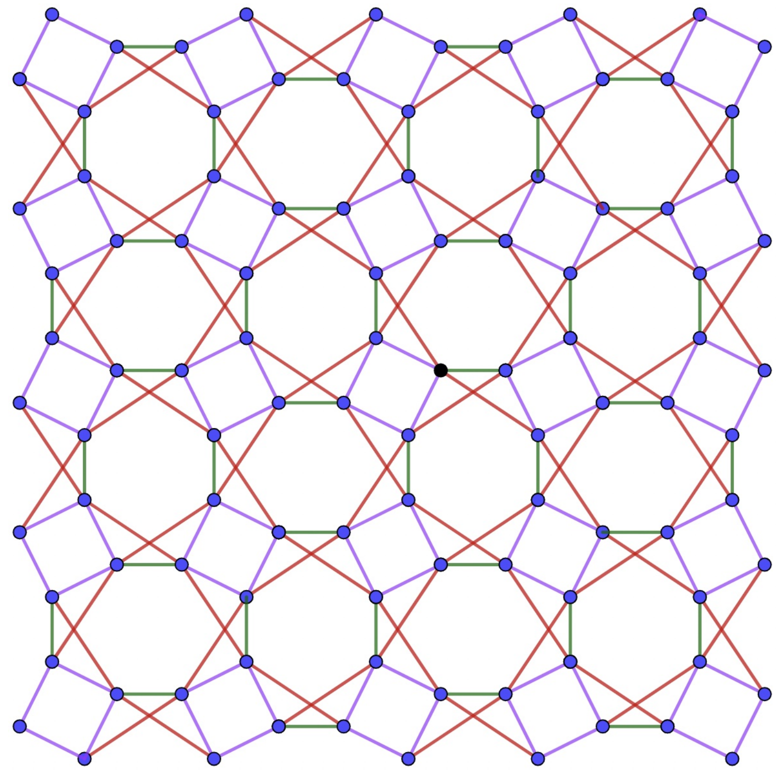
Snub Wythoffian of Petrie-Coxeter $\{4, 6|4\}$





Snub Wythoffian of $\{4,4\}^{\pi}$





Other interesting classes of skeletal polyhedra

- 2-orbit polyhedra

Pellicer & Williams (2023): classes 2_0 and 2_2 classified

- 3-orbit polyhedra

Cunningham & Pellicer (2021): finite polyhedra classified

- regular polyhedra of index 2 (combinatorially regular, symmetry group of index 2 in combinatorial automorphism group)

Cutler & S. (2011/2012): finite polyhedra classified

..... The End .....

Thank you

Abstract The study of highly symmetric structures in Euclidean 3-space has a long and fascinating history tracing back to the early days of geometry. With the passage of time, various notions of polyhedral structures have attracted attention and have brought to light new exciting figures intimately related to finite or infinite groups of isometries. A radically different, skeletal approach to polyhedra was pioneered by Grunbaum in the 1970's building on Coxeter's work. A polyhedron is viewed not as a solid but rather as a finite or infinite periodic geometric edge graph in space equipped with additional polyhedral super-structure imposed by the faces. Since the mid 1970's there has been a lot of activity in this area. Much work has focused on classifying skeletal polyhedra and complexes by symmetry, with the degree of symmetry defined via distinguished transitivity

properties of the geometric symmetry groups. These skeletal figures exhibit fascinating geometric, combinatorial, and algebraic properties and include many new finite and infinite structures.