Which simple groups are automorphism groups of chiral maps (or not).

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Motivation: Study of finite simples groups by means of actions properties

Theorem

Every finite simple group is isomorphic to one of the following groups:

- *A **cyclic group of prime order** (abelian).*
- *An **alternating group of degree at least 5**.*
- *A **group of Lie type**: Chevalley, Steinberg, Suzuki and Ree groups.*
- *One of the 26 **sporadic groups**: Mathieu (5 groups), Janko (4 groups), Conway (3 groups), Fisher (3 groups), Higman-Sims, McLaughlin (McL), Held, Rudvalis, Suzuki, O'Nan, Harada-Norton, Lyons, Thompson, Baby Monster, Monster group.*
- *The **Tits group**.*

Which non-abelian finite simple groups are generated by three involutions, two of which commute? (Mazurov)

Answer

Nuhzin and others (1990,1997,2003): All except

$PSL(3,q)$, $PSU(3,q)$, $PSL(4,2^n)$, $PSU(4,2^n)$, A_6, A_7, M_{11}, M_{22}, M_{23} and McL.

M. Mačaj and G. Jones correct the above list by adding $PSU(4,3)$ and $PSU(5,2)$ as exceptions, too.

In other words:

With the above exceptions, all non-abelian finite simple groups are automorphism groups of regular maps.

Note: In the following, with a simple group we will always mean a non-abelian finite simple group.

Kourovka Notebook, Problem 7.30 (1980)

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What comes next?

Obviously, the following question:

Which simple groups are automorphism groups of regular oriented maps (orientably-regular maps)?

Terminology: With a regular oriented map we mean a regular object in the **category of oriented maps**, that is, an oriented map (an orientable map with a given orientation), whose monodromy (or connection) group acts regularly on darts (oriented edges or arcs).

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General context

Study of regular oriented maps, described by a triple $(G; L, R)$, where G is a group generated by an involution L and by R.

More generally,

Definition

Given positive integers i, j, we say that G is (i, j)-generated if there is a pair of generators of G having order i and j.

Remarks

If G is $(1, j)$-generated, then G is cyclic.
If G is $(2, 2)$-generated, then G is dihedral.
$\Rightarrow$ First interesting case: $(2, 3)$-generated groups.
This case includes Hurwitz groups, that is, non-trivial finite quotients of the triangle group $\Delta(2, 3, 7)$.

$(2,k)$-generated groups. Some examples.

The question of which simple groups are $(2,3)$-generated has been studied extensively by many authors. Some examples:

- Alternating groups A_n except for $n = 3, 6, 7, 8$.
  (A_n is Hurwitz for $n \geqslant 168$: Conder, 1980).
- $PSL(n,q)$ for $(n,q) \neq (2,9), (3,4), (4,2)$ (Pellegrini 2017).

Moreover, $SL(3,q)$ and $PSL(3,q)$ are $(2,4)$-generated (Breda and –,3 days ago).

Theorem (King 2017)

Every (non-abelian finite) simple group is generated by an involution and an element of prime order.

Corollary

Every (non-abelian finite) simple group is the automorphism group of a regular oriented map.

Regular oriented maps

Triples $(G; L, R)$, where $\{L, R\}$ is a generating set of the group G and L is an involution.

$$\boxed{(G_1; L_1, R_1) \text{ is isomorphic to } (G_2; L_2, R_2)}$$

if there is a group isomorphism $G_1 \to G_2$ sending (L_1, R_1) to (L_2, R_2).

Remark:

Inner automorphisms of G produce isomorphic maps, i.e.

$\boxed{(G; L, R) \text{ is isomorphic to } (G; L^g, R^g)}$ for any $g \in G$.

$\Rightarrow$ One can choose (and fix) a representative L for each conjugacy class of involutions.

Reflexible and chiral maps

Remark: Given a regular oriented map $(G; L, R)$,

the triple $(G; L, R^{-1})$ is also a regular oriented map, called the mirror image of $(G; L, R)$.

$(G; L, R)$ is reflexible if it is isomorphic to its mirror image, otherwise $(G; L, R)$ is chiral.

The group G is strong map symmetric if any regular oriented map $(G; L, R)$ is reflexible.

In other words

G is strong map symmetric if for any pair (L, R) such that $(G; L, R)$ is a regular oriented map, there is an automorphism of G inverting both L and R $(L^{-1} = L)$.

Simple groups which are automorphism groups of chiral maps

A result of Leemans and Liebeck (2017): Every simple group is the automorphism group of a chiral map with the following exceptions: A_7, $PSL(2, q)$, $PSL(3, q)$ and $PSU(3, q)$, for every prime power q.

Remarks

- An exhaustive computer search show that A_7 is strong map symmetric. $\surd$ok
- By a result of MacBeath, $PSL(2, q)$ is strong map symmetric, as noted by Singerman. $\surd$ok
- No proof in the above paper for strong map symmetry of $PSL(3, q)$ and $PSU(3, q)$ (remind to a forthcoming paper).

Next story

Complete aware of King results and Leemans-Liebeck results Antonio Breda and me started to study maps in an other category: The category of **pseudo-oriented maps** (so named by S. Wilson). The regular objects of this category are also groups generated by a pair of elements one of which is an involution, but with another geometrical meaning. For instance, the underlying surface can be non-orientable.

Reflexibility of a regular oriented map $(G; L, R)$, correspond to reflexibility of the regular pseudo-oriented map $(G; L', R')$ (applying Petrie-dual, which is involutory).

For some reason, we needed to know if such maps are reflexible for $G = PSL(3, q)$, for some values of q.
(I confess that I don't remember the reason).

The groups $SL(n, K)$ and $PSL(n, K)$

$SL(n, K)$ is the group of $n \times n$ matrices with entries in the Galois field K and determinant 1. Its center is

$$\mathcal{Z} = \{\lambda I_n : \lambda \in K,\ \lambda^n = 1\}. \qquad (I_n \in SL(n, K) \text{ identity matrix})$$

$PSL(n, K) = SL(n, K)/\mathcal{Z}$.

Some results

- $SL(3, K)$ is $(3, 3)$-generated if $|K|$ is an odd prime (Conder, Robertson, Williams, 1992).
- For any pair of generators of $PSL(2, K)$ there is an automorphism of $PSL(2, K)$ inverting both generators (Breda, Jones, Nedela, Škoviera, 2009). Hence, in particular, $PSL(2, K)$ is strong map-symmetric.
- Pairs of generators of $PSL(2, K)$ are known (Conder, Potočnik, Širan, 2008).

Involutions in $SL(3, K)$ and $PSL(3, K)$

Let $I \in SL(3, K)$ be the identity matrix and let $\mathcal{Z}$ be the center of $SL(3, K)$. Then $\boxed{\mathcal{Z} = \{\lambda I : \lambda \in K,\ \lambda^3 = 1\}}$

Lemma

If $T \in SL(3, K)$ is such that $T^2 \in \mathcal{Z}$ and $T \notin \mathcal{Z}$, then T is conjugate in $SL(3, K)$ to λL for some $\lambda \in K$ such that $\lambda^3 = 1$,

where $\boxed{\begin{array}{c} L = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \text{if } char(K) \neq 2 \end{array}}$ *and* $\boxed{\begin{array}{c} L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \text{if } char(K) = 2 \end{array}}$

Corollary

There is a only one conjugacy class of involutions in $SL(3, K)$ and in $PSL(3, K) = SL(3, K)/\mathcal{Z}$.

Automorphisms of $SL(3, K)$

Some automorphisms of $SL(3, K)$:

- Apply a field automorphism to the entries of the matrix $M \in SL(3, K)$.
- The involution ϕ sending M to $(M^\top)^{-1}$ (the inverse of the transpose).
- Conjugation by a matrix $g \in GL(3, K)$: $M \mapsto M^g$.

Lemma

The above set of automorphisms of $SL(3, K)$ generates the automorphism group of $SL(3, K)$.

By King Theorem we know that $PSL(3, K)$ is $(2, p)$-generated for some prime p.
$\Rightarrow$ the set of regular oriented maps with automorphism group $PSL(3, K)$ is not empty.

$SL(3, K) = PSL(3, K)$ if $|K| \not\equiv 1 \mod 3$.
$|SL(3, K)| = 3|PSL(3, K)|$ if $|K| \equiv 1 \mod 3$ and $SL(3, K)$ has no normal subgroup isomorphic to $PSL(3, K)$.
$\Rightarrow$ any pair of generators of $PSL(3, K)$ is the projection of a pair of generators of $SL(3, K)$ (by the canonical epimorphism $SL(3, K) \to SL(3, K)/\mathcal{Z}$).

Conclusion

The set of regular oriented maps with automorphism group $SL(3, K)$ is not empty and

$$\boxed{\begin{array}{lll} & SL(3, K) & \text{strong map symmetric} \\ \Rightarrow & PSL(3, K) & \text{strong map symmetric} \end{array}}$$

Strong map-symmetry of $SL(3, K)$

Theorem

$SL(3, K)$ is strong map symmetric.

Some steps of the proof:

- For $\langle L, R\rangle = SL(3, K)$ assume (up to conjugation) that
  $L = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ if $char(K) \neq 2$ and
  $L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ if $char(K) = 2$.
- Show the existence of $g \in GL(3, K)$ such that

  $$(\phi(L))^g = L \quad \text{and} \quad (\phi(R))^g = R^{-1},$$

  where $\phi(M) = (M^\top)^{-1}$. Remark that $\phi(L) = L$, so g centralizes L.

Some steps of the proof

Remark

It is now enough to prove that there is g in the centralizer of L in $GL(3,K)$ such that $(\phi(R))^g = R^{-1} \Leftrightarrow \boxed{(R^\top)^g = R}$

This reduce the proof to show that the system of linear equations $(R^\top)^g = R$ for the entries of g has a solution. Taking care that the following sets of matrices in $SL(3,K)$ are (maximal) subgroups of $SL(3,K)$

$$\begin{bmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}; \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}; \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{bmatrix}; \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}; \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}; \begin{bmatrix} * & 0 & * \\ * & * & * \\ * & 0 & * \end{bmatrix}$$

as $\langle L, R\rangle = SL(3,K)$, if L belongs to some of such subgroups H, then $R \notin H$. This guarantees the existence of the desired solution g for $(R^\top)^g = R$.

Next story: What about $PSU(3, q)$?

After my talk on strong map symmetry of $SL(3, q)$ and $PSL(3, q)$ at the

9th Slovenian International Conference on Graph Theory,
23-29 of June 2019, Bled (Slovenia)

D. Leemans gave us to know his results obtained in 2017 with M. Liebeck and gave a suggestion for proving strong map symmetry of $PSU(3, q)$.

Note

At that time, António and me, did not yet submit our results to a journal. Moreover, from a talk with G. Jones, we realize the importance of completing the list of strong map symmetric simple groups for his work on edge-transitive maps.

Strong map symmetry of $PSU(3, q)$

Corollary

If $G = \langle L, R \rangle$ is an irreducible subgroup of $SL(3, K)$, then G is strong map symmetric.

A subgroup of $GL(n, K)$ is irreducible if it does not fix any proper subspace of K^n; a property which is invariant under conjugation.

Lemma

$SU(3, q) < SL(3, q^2)$ is irreducible and hence strong map symmetric.

Corollary

$PSU(3, q) < PSL(3, q^2)$ is strong map symmetric.

Thank you
and
Happy New Year

All details can be found in

Antonio Breda d'Azevedo and Domenico A. Catalano,
Strong map symmetry of $SL(3, K)$ *and* $PSL(3, K)$ *for every finite field* K,
Journal of Algebra and its Applications, Vol. 20, No. 04, 2150048 (2021).

or in its references or in the references of the references of it.