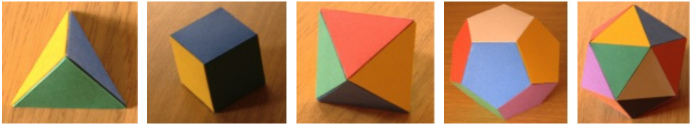
The number of string C-groups of high rank

Dimitri Leemans
Université Libre de Bruxelles



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Introduction

Abstract regular polytopes - problems

Given a family of groups (e.g. S_n), can we determine

1. what is the highest rank of an abstract regular polytope that has one of the groups of the family as full automorphism group?
2. what are the possible ranks?
3. how many pairwise nonisomorphic polytopes are there?
4. ...

Abstract regular polytopes

An **abstract polytope** $(\mathcal{P}, \leq)$ is a poset satisfying four extra conditions:

- the poset has a least face and a greater face;
- each maximal chain of the poset has same length $r + 2$ (r will be called the **rank**);
- a diamond condition;
- a strong connectedness condition.

An abstract polytope is **regular** if its group of automorphisms is transitive on the set of maximal chains (also called **flags**).

Abstract regular polytopes

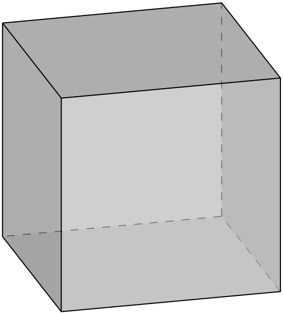


Figure: A Cube

Abstract regular polytopes

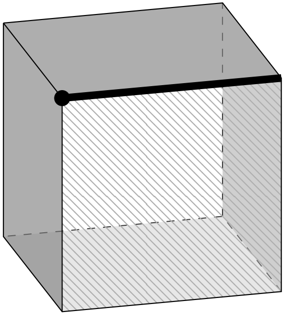
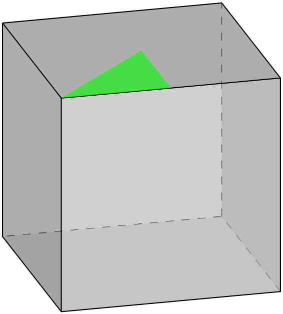


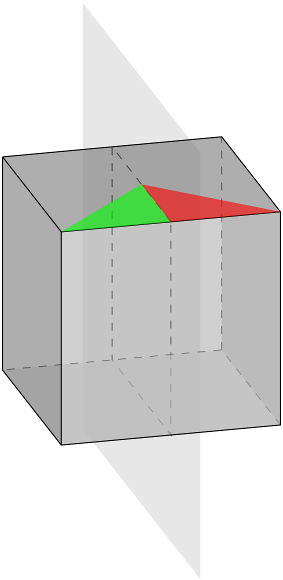
Figure: A chain on the Cube consisting of a vertex, an edge containing that vertex and a face containing the edge

Abstract regular polytopes and String C-groups

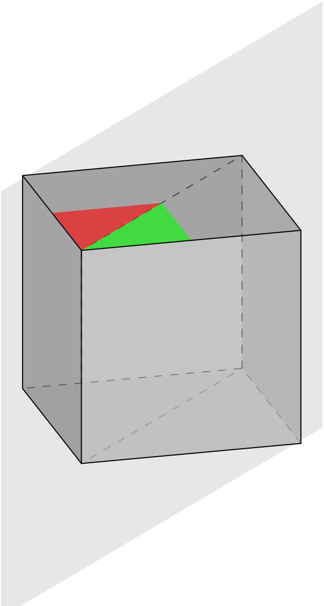
There is a natural one-to-one correspondence between abstract regular polytopes and string C-groups.



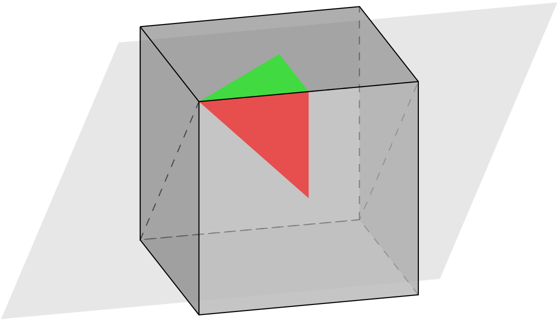
String C-groups



String C-groups



String C-groups



String C-groups

And the other way around ... use Jacques Tits algorithm to construct a coset geometry and order the types.

String C-groups

Definition

A **C-group of rank** r is a pair (G, S) such that G is a group and $S := \{\rho_0, \ldots, \rho_{r-1}\}$ is a generating set of involutions of G that satisfy the following property.

$$\forall I, J \subseteq \{0, \ldots, r-1\}, \langle \rho_i \mid i \in I \rangle \cap \langle \rho_j \mid j \in J \rangle = \langle \rho_k \mid k \in I \cap J \rangle$$

This property is called the **intersection property** and denoted by (IP). We call any subgroup of G generated by a subset of S a *parabolic subgroup* of the C-group (G, S).

String C-groups

Definition

A **C-group** (G, S) **of rank** r is a **string C-group** if its set of generating involutions S can be ordered in such a way that $S := \{\rho_0, \ldots, \rho_{r-1}\}$ satisfies

$$\forall i, j \in \{0, \ldots, r-1\}, o(\rho_i \rho_j) = 2 \text{ if } |i - j| > 1$$

This property is called the **string property** and denoted by (SP).

Definition

For a given group G, we will call (G, S) a **string C-group representation** of G provided it satisfies (SP) and (IP).

E. H. Moore (1896) : $(n-1)$-simplex.

Theorem (Moore, 1896)

For every $n \geq 3$, there is a string C-group representation of S_n in its natural permutation representation, of rank $n-1$ whose generating involutions are the transpositions $(i, i+1)$ with $i = 1, \ldots, n-1$.

String C-group representations of symmetric groups

Proposition (Whiston, 2000)

The size of an independent set in S_n is at most $n-1$, with equality only if the set generates the whole group S_n.

String C-group representations of symmetric groups

Sjerve and Cherkassoff (1993) (see also Conder 1980): S_n is a group generated by three involutions, two of which commute, provided that $n \geq 4$.

Theorem ("Moore, Sjerve, Cherkassoff, Conder")

Every group S_n with $n \geq 4$ has a string C-group representation of rank three and one of rank $n-1$.

String C-group representations of symmetric groups

G	Rank 3	Rank 4	Rank 5	Rank 6	Rank 7	Rank 8
S_5	4	1	0	0	0	0
S_6	2	4	1	0	0	0
S_7	35	7	1	1	0	0
S_8	68	36	11	1	1	0
S_9	129	37	7	7	1	1

Source: http://leemans.dimitri.web.ulb.be/∼dleemans/polytopes

String C-group representations of symmetric groups

Theorem (Fernandes, Leemans, 2011)

For $n \geq 5$ or $n = 3$, Moore's generators give, up to isomorphism, the unique string C-group representation of rank $n-1$ for S_n. For $n = 4$, there are, up to isomorphism and duality, two representations, namely the ones corresponding to the hemicube and the tetrahedron.

String C-group representations of symmetric groups

Theorem (Fernandes, Leemans, 2011)

For $n \geq 7$, there exists, up to isomorphism and duality, a unique string C-group representation of rank $(n-2)$ for S_n.

String C-group representations of symmetric groups

Theorem (Fernandes, Leemans, 2011)

Let $n \geq 4$. For every $r \in \{3, \ldots, n-1\}$, there exists at least one string C-group representation of rank r for S_n.

Let $\{\rho_0, \ldots, \rho_{r-1}\}$ be a set of involutions of a permutation group G of degree n. We define the **permutation representation graph** $\mathcal{G}$ as the r-edge-labeled multigraph with n vertices and with a single i-edge $\{a, b\}$ whenever $a\rho_i = b$ with $a \neq b$.

String C-group representations of symmetric groups

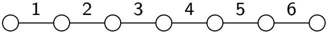
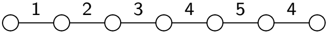
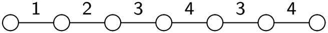
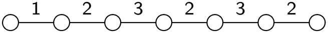
Generators	Permutation representation	Schläfli type
(1,2),(2,3),(3,4),(4,5),(5,6),(6,7)		{3,3,3,3,3}
(1,2),(2,3),(3,4),(4,5)(6,7),(5,6)		{3,3,6,4}
(1,2),(2,3),(3,4)(5,6),(4,5)(6,7)		{3,6,5}
(1,2),(2,3)(4,5)(6,7),(3,4)(5,6)		{6,6}

Table: The induction process used on S_7

Number of representations, up to duality, for S_n ($5 \leq n \leq 14$)

G\r	3	4	5	6	7	8	9	10	11	12	13
S_5	4	1	0	0	0	0	0	0	0	0	0
S_6	2	4	1	0	0	0	0	0	0	0	0
S_7	35	7	1	1	0	0	0	0	0	0	0
S_8	68	36	11	1	1	0	0	0	0	0	0
S_9	129	37	7	7	1	1	0	0	0	0	0
S_{10}	413	203	52	13	7	1	1	0	0	0	0
S_{11}	1221	189	43	25	9	7	1	1	0	0	0
S_{12}	3346	940	183	75	40	9	7	1	1	0	0
S_{13}	7163	863	171	123	41	35	9	7	1	1	0
S_{14}	23126	3945	978	303	163	54	35	9	7	1	1

Number of representations, up to duality, for S_n $(5 \leq n \leq 14)$

G\r	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
S_5	4	**1**	0	0	0	0	0	0	0	0	0
S_6	2	4	**1**	0	0	0	0	0	0	0	0
S_7	35	7	**1**	**1**	0	0	0	0	0	0	0
S_8	68	36	11	**1**	**1**	0	0	0	0	0	0
S_9	129	37	7	7	**1**	**1**	0	0	0	0	0
S_{10}	413	203	52	13	7	**1**	**1**	0	0	0	0
S_{11}	1221	189	43	25	9	7	**1**	**1**	0	0	0
S_{12}	3346	940	183	75	40	9	7	**1**	**1**	0	0
S_{13}	7163	863	171	123	41	35	9	7	**1**	**1**	0
S_{14}	23126	3945	978	303	163	54	35	9	7	**1**	**1**

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S_5	4	**1**	0	0	0	0	0	0	0	0	0
S_6	2	4	**1**	0	0	0	0	0	0	0	0
S_7	35	7	**1**	**1**	0	0	0	0	0	0	0
S_8	68	36	11	**1**	**1**	0	0	0	0	0	0
S_9	129	37	7	**7**	**1**	**1**	0	0	0	0	0
S_{10}	413	203	52	13	**7**	**1**	**1**	0	0	0	0
S_{11}	1221	189	43	25	**9**	**7**	**1**	**1**	0	0	0
S_{12}	3346	940	183	75	40	**9**	**7**	**1**	**1**	0	0
S_{13}	7163	863	171	123	41	35	**9**	**7**	**1**	**1**	0
S_{14}	23126	3945	978	303	163	54	35	**9**	**7**	**1**	**1**

String C-group representations of symmetric groups

Theorem (Fernandes-Leemans-Mixer, 2018)

For $n \geq 9$, there exists, up to isomorphism and duality, seven string C-group representations of rank $(n-3)$ for S_n.
For $n \geq 11$, there exists, up to isomorphism and duality, nine string C-group representations of rank $(n-4)$ for S_n.

Conjecture

Let r be a positive integer and $n \geq 2r+3$. The number of pairwise nonisomorphic string C-group representations of rank $n-r$ is independent on n.

The sequence looks like 1, 1, 7, 9, 35, 48, ...

What about alternating groups ?

String C-group representations of alternating groups

G	Rank 3	Rank 4	Rank 5	Rank 6	Rank 7	Rank 8
A_5	2	0	0	0	0	0
A_6	0	0	0	0	0	0
A_7	0	0	0	0	0	0
A_8	0	0	0	0	0	0
A_9	41	6	0	0	0	0

Source: http://leemans.dimitri.web.ulb.be/∼dleemans/polytopes

String C-group representations of alternating groups

G	Rank 3	Rank 4	Rank 5	Rank 6	Rank 7	Rank 8
A_5	2	0	0	0	0	0
A_6	0	0	0	0	0	0
A_7	0	0	0	0	0	0
A_8	0	0	0	0	0	0
A_9	41	6	0	0	0	0
A_{10}	94	2	4	0	0	0
A_{11}	64	0	0	3	0	0
A_{12}	194	90	22	0	0	0
A_{13}	1558	102	25	10	0	0
A_{14}	4347	128	45	9	0	0
A_{15}	5820	158	20	42	6	0

Source: http://leemans.dimitri.web.ulb.be/∼dleemans/polytopes

String C-group representations of alternating groups

Theorem (Fernandes, Leemans, Mixer, 2012)

For each $n \notin \{3,4,5,6,7,8,11\}$, there is a rank $\lfloor \frac{n-1}{2} \rfloor$ string C-group representation of the alternating group A_n.

We found a striking example! A_{11} has string C-group representations of rank 3 and 6, but not 4 nor 5!

String C-group representations of alternating groups

A conjecture arose thanks to the collected data and the struggle to construct the above mentioned examples.

Conjecture

The highest rank of a string C-group representation of A_n is $\lfloor \frac{n-1}{2} \rfloor$ when $n \geq 12$.

String C-group representations of alternating groups

Strategy of the proof:
Take A_n the group of even permutations of n points.
First show that a string C-group representations of A_n of rank $r > \lfloor \frac{n-1}{2} \rfloor$, if it exists, must have all its maximal parabolic subgroups (that is the subgroups generated by all but one generator) intransitive.
Then use this fact and permutation representation graphs to show that it is impossible.

String C-group representations of alternating groups

Theorem (Cameron, Fernandes, Leemans, Mixer, 2016)

Let Γ be a string C-group of rank r which is isomorphic to a transitive subgroup of S_n other than S_n or A_n. Then one of the following holds:

1. $r \leq n/2$;
2. $n \equiv 2 \mod 4$, $r = n/2+1$ *and* Γ *is* $C_2 \wr S_{n/2}$. *The generators are*

$$\rho_0 = (1, n/2+1)(2, n/2+2)\ldots(n/2, n);$$
$$\rho_1 = (2, n/2+2)\ldots(n/2, n);$$
$$\rho_i = (i-1, i)(n/2+i-1, n/2+i) \text{ for } 2 \leq i \leq n/2.$$

   Moreover the Schläfli type is $[2, 3, \ldots, 3, 4]$.

3. Γ *is transitive imprimitive and is one of the examples appearing in the next Table.*
4. Γ *is primitive. In this case,* Γ *is obtained from the permutation representation of degree 6 of* $S_5 \cong PGL_2(5)$ *and it is the 4-simplex of Schäfli type* $[3, 3, 3]$.

String C-group representations of alternating groups

Degree	*Number*	Structure	Order	Schäfli type
6	9	$S_3 \times S_3$	36	$[2,3,3]$
6	11	$2^3 : S_3$	48	$[2,3,3]$
6	11	$2^3 : S_3$	48	$[2,3,4]$
8	45	$2^4 : S_3 : S_3$	576	$[3,4,4,3]$

Table: Examples of transitive imprimitive string C-groups of degree n and rank $n/2+1$ for $n \leq 9$.

Corollary

Suppose $G = A_n$ of degree n. Let (G,S) be a string C-group with $S = \{\rho_0, \ldots, \rho_{r-1}\}$. If $r \geq n/2+2$, all subgroups G_i must be intransitive.

String C-group representations of alternating groups

The "Aveiro" theorem:

Theorem (Cameron, Fernandes, Leemans, Mixer, 2017)

The rank of A_n is 3 if $n = 5$; 4 if $n = 9$; 5 if $n = 10$; 6 if $n = 11$ and $\lfloor \frac{n-1}{2} \rfloor$ if $n \geq 12$. Moreover, if $n = 3, 4, 6, 7$ or 8, the group A_n is not a string C-group.

The proof of this result takes 39 pages and uses induction in some parts.

- 8 pages to refine our result on transitive groups and get to prove that all the maximal parabolic subgroups must be intransitive if the conjecture is false.
- 10 pages to handle the case where we assume there exists a 2-fracture graph.
- 21 pages to handle the case where we assume no 2-fracture graph exists.

Back to symmetric groups

Compiling the previous results, we get the following.

Corollary

If G is a transitive group of degree n having a string C-group of rank $r \geq (n+3)/2$, then G is necessarily the symmetric group S_n.

String C-groups of high rank

S_n	Rk $n-1$	Rk $n-2$	Rk $n-3$	Rk $n-4$	Rk $n-5$	Rk $n-6$
S_5	1	4				
S_6	1	4	2			
S_7	1	1	7	35		
S_8	1	1	11	36	68	
S_9	1	1	7	7	37	129
S_{10}	1	1	7	13	52	203
S_{11}	1	1	7	9	25	43
S_{12}	1	1	7	9	40	75
S_{13}	1	1	7	9	35	41
S_{14}	1	1	7	9	35	54
S_{15}	1	1	7	9	35	48
S_{16}	1	1	7	9	35	48

Table: The number of pairwise nonisomorphic string C-groups of rank $n-k$ for S_n with $1 \leq k \leq 6$ and $5 \leq n \leq 16$.

Let $\mathcal{S}(n,r)$ be the set of all string C-group representations of rank r for S_n. Define a relation $\sim$ on $\mathcal{S}(n,r) \times \mathcal{S}(n,r)$ by saying that for any elements $P, Q \in \mathcal{S}(n,r)$, $P \sim Q$ if and only if P is isomorphic to Q or to the dual of Q. The relation $\sim$ is an equivalence relation.

String C-groups of high rank

Let $\Sigma^\kappa(n) = \mathcal{S}(n, n-\kappa)/\sim$. The results of Fernandes-Leemans 2011 and Fernandes-Leemans-Mixer 2018 give the following sequence.

$$|\Sigma^1(n)| = 1 \text{ for } n \geq 5$$
$$|\Sigma^2(n)| = 1 \text{ for } n \geq 7$$
$$|\Sigma^3(n)| = 7 \text{ for } n \geq 9$$
$$|\Sigma^4(n)| = 9 \text{ for } n \geq 11$$

In addition, relying on computational results, it was conjectured that

$$|\Sigma^5(n)| = 35 \text{ for } n \geq 13 \text{ and}$$
$$|\Sigma^6(n)| = 48 \text{ for } n \geq 15.$$

String C-groups of high rank

The Brussels Theorem:

Theorem (Cameron-Fernandes-Leemans 2022)

For each fixed integer $\kappa \geq 1$, there exists an integer c_κ such that, for all $n \geq 2\kappa + 3$, $|\Sigma^\kappa(n)| = c_\kappa$.

This theorem and the tools used in its proof, in particular the rank and degree extension, imply that if one knows the string C-groups of rank $(n+3)/2$ for S_n with n odd, one can construct from them all string C-groups of rank $(n+3)/2+k$ for S_{n+k} for any positive integer k.
The classification of the string C-groups of rank $r \geq (n+3)/2$ for S_n is thus reduced to classifying string C-groups of rank r for S_{2r-3}.

String C-groups of high rank

The study of string C-groups of high rank, specially for the alternating and symmetric groups, was accomplished using permutation representation graphs of these groups on n points. For a better understanding of these representations we used a subgraph called **fracture graph** introducted by Fernandes, Leemans and Mixer. The concept of fracture graph turns out to be essential to the classification of the string C-groups for the symmetric group S_n with ranks $n-3$ and $n-4$.

String C-groups of high rank

Let $\Gamma := (G, \{\rho_0, \ldots, \rho_{r-1}\})$ be a sggi such that G acts faithfully on a set $\{1, \ldots, n\}$.
Suppose that for every $i \in \{0, \ldots, r-1\}$, the subgroup G_i has at least one more orbit than G.
Then, for each i, the involution ρ_i permutes a pair of points lying in different G_i-orbits.
Choosing one such transposition of ρ_i for each i, and regarding them as the edges of a graph on the vertex set $\{1, \ldots, n\}$, we obtain a graph with r edges that we call a **fracture graph** for Γ.

This concept was developed further by Cameron, Fernandes, Leemans and Mixer where a new subgraph appeared in the proof of the “Aveiro Theorem”, the **2-fracture graph**.

If it happens that for each i we can find two transpositions of ρ_i such that, for each transposition, its points are in different G_i-orbits, then taking an i-edge between each of these pairs of points we obtain a graph on n vertices with $2r$ edges that we call a 2-**fracture graph**.

String C-groups of high rank

A first step in the proof of the Brussels Theorem is to classify all possible permutation representation graphs of string C-groups, which have rank at least $(n-1)/2$ and admit a 2-fracture graph. This classification includes string C-groups of the highest rank for alternating groups of odd degree. So it gives an important contribution to the classification of the string C-groups of the highest rank for alternating groups. This classification has not yet been accomplished.

We introduce the notion of a **split** in a permutation representation graph of a string group generated by involutions (sggi). To be a little bit more precise, a permutation representation graph has a split if it has a fracture graph but no 2-fracture graph.

String C-groups of high rank

We analyse the string C-groups that have a permutation representation graph with a split. For that we also define the notion of a **perfect split**.

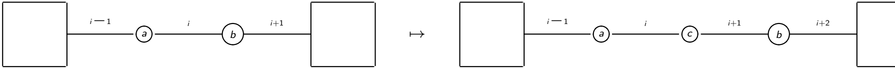
Roughly speaking, a **perfect split** is an edge with label i that splits the permutation representation graph in two parts such that on one part all edges have labels $\leq i$ and on the other part, all edges have labels $\geq i$.
The main result here, is the classification of the sggi's for S_n of rank $r \geq n/2$ that have a permutation representation graph with splits but no perfect splits.

String C-groups of high rank

We then introduce a rank and degree operation (which we call the r&d-extension) that, given a sggi of degree n and rank r, constructs a sggi of degree $n+1$ and rank $r+1$ provided the sggi given has a perfect split.

Assume that $\Gamma = (G, \{\rho_0, \ldots, \rho_{r-1}\})$ is a sggi whose permutation representation graph, with vertex-set $\{1, \ldots, n\}$, and that Γ has a perfect i-split $\{a, b\}$.
In addition, let $\{O_1, O_2\}$ be a partition of $\{1, \ldots, n\}$ with $a \in O_1$, $b \in O_2$ such that $G_{<i}$ and $G_{>i}$ are, respectively, the actions of G_i on O_1 and O_2.
Let $\rho_i = \alpha_i\beta_i(a, b)$ with α_i and β_i being the actions of ρ_i in O_1 and O_2 respectively.

Now we extend the sggi Γ to another sggi $\Gamma^{i\uparrow} = (G^{i\uparrow}, S^{i\uparrow})$ where G is acting on a set $\{1, \ldots, n\} \cup \{c\}$ of size $n+1$, $\Gamma^{i\uparrow}$ has two adjacent perfect splits $\{a, c\}$ and $\{c, b\}$, and such that $G^{i\uparrow}_{>i+1} \cong G_{>i}$ and $G^{i\uparrow}_{<i} \cong G_{<i}$.



More precisely, $S^{i\uparrow} = \{\delta_0, \ldots, \delta_r\}$ and $G^{i\uparrow} = \langle S \rangle$ where

$$\delta_j = \begin{cases} \rho_j, & \text{if } j \in \{0, \ldots, i-1\}, \\ \alpha_i(a, c), & \text{if } j = i, \\ \beta_i(c, b), & \text{if } j = i+1, \\ \rho_{j-1}, & \text{if } j \in \{i+2, \ldots, r\}. \end{cases} .$$

String C-groups of high rank

We also describe this operation on string C-groups, having as main concern to give sufficient conditions to ensure that the r&d extension of a string C-group is a string C-group.

We then consider transitive string C-groups having a perfect split and show that when the rank is sufficiently large the r&d-extension is possible.

Finally, we use the r&d-extension to determine a one-to-one correspondence between $\Sigma^\kappa(n)$ and $\Sigma^\kappa(n+1)$ when $n \geq 2\kappa + 3$ and therefore prove the Brussels Theorem.

Future work

Can we find the whole sequence ? Or at least some of the next numbers?

1, 1, 7, 9, 35, 48, ...

This sequence will be available in the OEIS as sequence number A359367.

Can we classify all string C-group representations of rank $\lfloor (n-1)/2 \rfloor$ for A_n?

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