On the algebraic connectivity of token graphs

Cristina Dalfó
Universitat de Lleida, Igualada (Barcelona), Catalonia

joint work with
Miquel Àngel Fiol, Universitat Politècnica de Catalunya
Arnau Messegué, Universitat de Lleida
Mónica A. Reyes, Universitat de Lleida

Algebraic Graph Theory International Webinar
December 27, 2022

Outline

Outline

1. Introduction

2. Known results

3. New results

Introduction: Token graphs

Definition

Let G be a simple graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. For a given integer k such that $1 \leq k \leq n$, the **k-token graph** $F_k(G)$ of G is the graph in which

Introduction: Token graphs

Definition

Let G be a simple graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. For a given integer k such that $1 \leq k \leq n$, the **k-token graph** $F_k(G)$ of G is the graph in which

- the **vertices** of $F_k(G)$ correspond to configurations of k indistinguishable tokens placed at distinct vertices of G,

Introduction: Token graphs

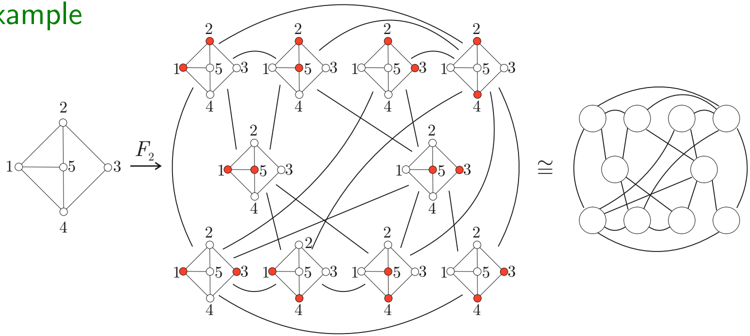
Definition

Let G be a simple graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. For a given integer k such that $1 \leq k \leq n$, the **k-token graph** $F_k(G)$ of G is the graph in which

- the **vertices** of $F_k(G)$ correspond to configurations of k indistinguishable tokens placed at distinct vertices of G,
- two configurations are **adjacent** whenever one configuration can be reached from the other by moving one token along an edge from its current position to an unoccupied vertex.

Token graphs

Example



Token graphs

Observations:

Token graphs

Observations:

- If $k = 1$, then $F_1(G) \simeq G$.

Token graphs

Observations:

- If $k = 1$, then $F_1(G) \simeq G$.
- $\Delta(F_k(G)) \leq k\Delta(G)$.

Token graphs

Observations:

- If $k = 1$, then $F_1(G) \simeq G$.
- $\Delta(F_k(G)) \leq k\Delta(G)$.
- If G is the complete graph K_n, then $F_k(K_n) \simeq J(n, k)$, the **Johnson graphs**.

Token graphs

Observations:

- If $k = 1$, then $F_1(G) \simeq G$.
- $\Delta(F_k(G)) \leq k\Delta(G)$.
- If G is the complete graph K_n, then $F_k(K_n) \simeq J(n, k)$, the **Johnson graphs**.
- In 2002, Rudolph defined the k-token graph, calling it the k-level matrix (in the context of qubits).

Token graphs

Observations:

- If $k = 1$, then $F_1(G) \simeq G$.
- $\Delta(F_k(G)) \leq k\Delta(G)$.
- If G is the complete graph K_n, then $F_k(K_n) \simeq J(n, k)$, the **Johnson graphs**.
- In 2002, Rudolph defined the k-token graph, calling it the k-level matrix (in the context of qubits).
- A **token graph** is also called **symmetric k-th power** (Audenaert, Godsil, Royle, and Rudolph, 2007).

Token graphs

Observations:

- If $k = 1$, then $F_1(G) \simeq G$.
- $\Delta(F_k(G)) \leq k\Delta(G)$.
- If G is the complete graph K_n, then $F_k(K_n) \simeq J(n,k)$, the **Johnson graphs**.
- In 2002, Rudolph defined the k-token graph, calling it the k-level matrix (in the context of qubits).
- A **token graph** is also called **symmetric k-th power** (Audenaert, Godsil, Royle, and Rudolph, 2007).
- The name of **token graph** is due to Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia, and Wood, 2012.

Token graphs

Observations:

- If $k = 1$, then $F_1(G) \simeq G$.
- $\Delta(F_k(G)) \leq k\Delta(G)$.
- If G is the complete graph K_n, then $F_k(K_n) \simeq J(n,k)$, the **Johnson graphs**.
- In 2002, Rudolph defined the k-token graph, calling it the k-level matrix (in the context of qubits).
- A **token graph** is also called **symmetric k-th power** (Audenaert, Godsil, Royle, and Rudolph, 2007).
- The name of **token graph** is due to Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia, and Wood, 2012.
- **Applications:** The graph isomorphism problem and quantum mechanics.

Outline

Known results on the Laplacian spectra of token graphs

Notation:

- Let $[n] := \{1, \ldots, n\}$ and $\binom{[n]}{k}$ denote the set of k-subsets of $[n]$.

Known results on the Laplacian spectra of token graphs

Notation:

- Let $[n] := \{1, \ldots, n\}$ and $\binom{[n]}{k}$ denote the set of k-subsets of $[n]$.
- W_n: the set of all column vectors $\boldsymbol{v}$ such that $\boldsymbol{v}^\top \mathbf{1} = 0$.

Known results on the Laplacian spectra of token graphs

Notation:

- Let $[n] := \{1, \ldots, n\}$ and $\binom{[n]}{k}$ denote the set of k-subsets of $[n]$.
- W_n: the set of all column vectors $\boldsymbol{v}$ such that $\boldsymbol{v}^\top \mathbf{1} = 0$.
- Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the Laplacian matrix $\boldsymbol{L}(G)$ of a graph G, with $(0 =)\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. The second smallest eigenvalue λ_2 is known as the **algebraic connectivity** $\alpha(G)$.

Known results on the Laplacian spectra of token graphs

Notation:

- Let $[n] := \{1, \ldots, n\}$ and $\binom{[n]}{k}$ denote the set of k-subsets of $[n]$.
- W_n: the set of all column vectors $\boldsymbol{v}$ such that $\boldsymbol{v}^\top \mathbf{1} = 0$.
- Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the Laplacian matrix $\boldsymbol{L}(G)$ of a graph G, with $(0 =)\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. The second smallest eigenvalue λ_2 is known as the **algebraic connectivity** $\alpha(G)$.
- $1 + \Delta(G) \leq \lambda_{\max}(G) \leq 2\Delta(G)$,
  where $\lambda_{\max} = \lambda_n$ is the Laplacian spectral radius.

Known results on the Laplacian spectra of token graphs

Lemma (DDFFHTZ, 2021)

*Let G be a graph with Laplacian matrix $\boldsymbol{L}_1$. Let $F_k = F_k(G)$ be its token graph with Laplacian $\boldsymbol{L}_k$. Let $\boldsymbol{B}$ be the so-called $(n;k)$-**binomial matrix**, which is an $\binom{n}{k} \times n$ matrix whose rows are the characteristic vectors of the k-subsets of $[n]$ in a given order. Then, the following holds:*

(i) *If $\boldsymbol{v}$ is a λ-eigenvector of $\boldsymbol{L}_1$, then $\boldsymbol{B}\boldsymbol{v}$ is a λ-eigenvector of $\boldsymbol{L}_k$. Thus, the Laplacian spectrum (eigenvalues and their multiplicities) of $\boldsymbol{L}_1$ is contained in the Laplacian spectrum of $\boldsymbol{L}_k$.*

(ii) *If $\boldsymbol{u}$ is a λ-eigenvector of $\boldsymbol{L}_k$ such that $\boldsymbol{B}^\top \boldsymbol{u} \neq \boldsymbol{0}$, then $\boldsymbol{B}^\top \boldsymbol{u}$ is a λ-eigenvector of $\boldsymbol{L}_1$.*

Known results on the Laplacian spectra of token graphs

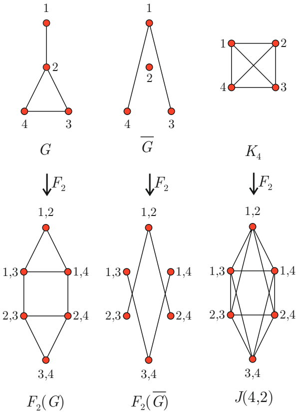
Theorem (DDFFHTZ, 2021)

Let $G = (V, E)$ be a graph on $n = |V|$ vertices, and let $\overline{G}$ be its complement. For a given k, with $1 \leq k \leq n-1$, let us consider the token graphs $F_k(G)$ and $F_k(\overline{G})$. Then, the Laplacian spectrum of $F_k(\overline{G})$ is the complement of the Laplacian spectrum of $F_k(G)$ with respect to the Laplacian spectrum of the **Johnson graph** *$J(n,k) = F_k(K_n)$. That is, every eigenvalue λ_J of $J(n,k)$ is the sum of one eigenvalue $\lambda_{F_k(G)}$ of $F_k(G)$ and one eigenvalue $\lambda_{F_k(\overline{G})}$ of $F_k(\overline{G})$, where each $\lambda_{F_k(G)}$ and each $\lambda_{F_k(\overline{G})}$ is used once:*

$$\lambda_{F_k(G)} + \lambda_{F_k(\overline{G})} = \lambda_J. \quad (1)$$

Known results on the Laplacian spectra of token graphs

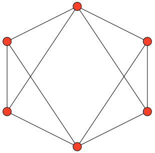
Example (DDFFHTZ, 2021)



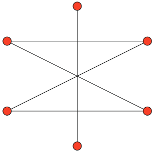
Spectrum	ev G	ev $\overline{G}$	ev Johnson
$\mathrm{sp}(F_1) = \mathrm{sp}(G)$	0	0	0
	1	3	4
	3	1	4
	4	0	4
$\mathrm{sp}(F_2) - \mathrm{sp}(F_1)$	3	3	6
	5	1	6

Known results on the Laplacian spectra of token graphs

Example



G



$\overline{G}$

Spectrum	ev G	ev $\overline{G}$	ev Johnson
$\mathrm{sp}(F_1) = \mathrm{sp}(G)$	0	0	0
	2	4	6
	4	2	6
	4	2	6
	4	2	6
	6	0	6
$\mathrm{sp}(F_2) - \mathrm{sp}(F_1)$	4	6	10
	4	6	10
	6	4	10
	6	4	10
	6	4	10
	8	2	10
	8	2	10
	8	2	10
	10	0	10
$\mathrm{sp}(F_3) - \mathrm{sp}(F_2)$	4	8	12
	8	4	12
	8	4	12
	10	2	12
	10	2	12

Known results on the Laplacian spectra of token graphs

Conjecture (DDFFHTZ, 2021)

Let G be a graph on n vertices. Then, for every $k = 1, \ldots, n-1$, the **algebraic connectivity** *of its token graph $F_k(G)$ equals the one of G, that is,*

$$\alpha(F_k(G)) = \alpha(G) \qquad \text{for every } k = 1, \ldots, |V| - 1.$$

Known results on the Laplacian spectra of token graphs

Conjecture (DDFFHTZ, 2021)

Let G be a graph on n vertices. Then, for every $k = 1, \ldots, n-1$, the **algebraic connectivity** *of its token graph $F_k(G)$ equals the one of G, that is,*

$$\alpha(F_k(G)) = \alpha(G) \qquad \text{for every } k = 1, \ldots, |V| - 1.$$

- Since $F_k(G) = F_{n-k}(G)$, the conjecture only needs to be proved for the case $k = \lfloor n/2 \rfloor$.

Known results on the Laplacian spectra of token graphs

Conjecture (DDFFHTZ, 2021)

Let G be a graph on n vertices. Then, for every $k = 1, \ldots, n-1$, the **algebraic connectivity** *of its token graph $F_k(G)$ equals the one of G, that is,*

$$\alpha(F_k(G)) = \alpha(G) \qquad \text{for every } k = 1, \ldots, |V| - 1.$$

- Since $F_k(G) = F_{n-k}(G)$, the conjecture only needs to be proved for the case $k = \lfloor n/2 \rfloor$.
- Computer exploration showed that $\alpha(F_2(G)) = \alpha(G)$ for all graphs with at most 8 vertices.

Known results on the Laplacian spectra of token graphs

Theorem (DDFFHTZ (2021))

For each of the following classes of graphs, the algebraic connectivity of a token graph $F_k(G)$ equals the algebraic connectivity of G. For $k = 1, \ldots, n-1$ and every n, we have the following:

(i) *Let $G = K_n$ be the* **complete graph** *on n vertices. Then,* $\alpha(F_k(G)) = \alpha(G) = n$.

(ii) *Let $G = K_{n_1,n_2}$ be the* **complete bipartite graph** *on $n = n_1 + n_2$ vertices, with $n_1 \leq n_2$. Then,* $\alpha(F_k(G)) = \alpha(G) = n_1$.

(iii) *Let $G = S_n$ be the* **star graph** *on n vertices. Then,* $\alpha(F_k(G)) = \alpha(G) = 1$.

(iv) *Let $G = P_n$ be the* **path graph** *on n vertices. Then,* $\alpha(F_k(G)) = \alpha(G) = 2(1 - \cos(\pi/n))$.

Outline

New results on the Laplacian spectra of token graphs: Notation

- W_n: the set of all column vectors $\boldsymbol{v}$ such that $\boldsymbol{v}^\top \mathbf{1} = 0$.

New results on the Laplacian spectra of token graphs: Notation

- W_n: the set of all column vectors $\boldsymbol{v}$ such that $\boldsymbol{v}^\top \mathbf{1} = 0$.
- Given a graph $G = (V, E)$ of order n, a vector $\boldsymbol{v} \in \mathbb{R}^n$ is an **embedding** of G if $\boldsymbol{v} \in W_n$.

New results on the Laplacian spectra of token graphs: Notation

- W_n: the set of all column vectors $\boldsymbol{v}$ such that $\boldsymbol{v}^\top \mathbf{1} = 0$.
- Given a graph $G = (V, E)$ of order n, a vector $\boldsymbol{v} \in \mathbb{R}^n$ is an **embedding** of G if $\boldsymbol{v} \in W_n$.
- **Rayleigh quotient:**

$$\lambda_G(\boldsymbol{v}) := \frac{\boldsymbol{v}^\top \boldsymbol{L}(G)\boldsymbol{v}}{\boldsymbol{v}^\top \boldsymbol{v}} = \frac{\sum_{(i,j)\in E} [\boldsymbol{v}(i) - \boldsymbol{v}(j)]^2}{\sum_{i\in V} \boldsymbol{v}^2(i)}$$

where $\boldsymbol{v}(i)$ denotes the entry of $\boldsymbol{v}$ corresponding to the vertex $i \in V(G)$.

New results on the Laplacian spectra of token graphs: Notation

- W_n: the set of all column vectors $\boldsymbol{v}$ such that $\boldsymbol{v}^\top \mathbf{1} = 0$.
- Given a graph $G = (V, E)$ of order n, a vector $\boldsymbol{v} \in \mathbb{R}^n$ is an **embedding** of G if $\boldsymbol{v} \in W_n$.
- **Rayleigh quotient:**

$$\lambda_G(\boldsymbol{v}) := \frac{\boldsymbol{v}^\top \boldsymbol{L}(G)\boldsymbol{v}}{\boldsymbol{v}^\top \boldsymbol{v}} = \frac{\sum_{(i,j)\in E} [\boldsymbol{v}(i) - \boldsymbol{v}(j)]^2}{\sum_{i \in V} \boldsymbol{v}^2(i)}$$

  where $\boldsymbol{v}(i)$ denotes the entry of $\boldsymbol{v}$ corresponding to the vertex $i \in V(G)$.
- If $\boldsymbol{v}$ is an eigenvector of G, then its corresponding eigenvalue is $\lambda(\boldsymbol{v})$. For an embedding $\boldsymbol{v}$ of G, we have

$$\alpha(G) \leq \lambda_G(\boldsymbol{v}),$$

  and there is equality when $\boldsymbol{v}$ is an $\alpha(G)$-eigenvector of G.

New results on the Laplacian spectra of token graphs

Lemma

Let $G^+ = (V^+, E^+)$ be a graph on the vertex set $V = \{1, 2, \ldots, n+1\}$, having a vertex of degree 1, say the vertex $n+1$ that is adjacent to n. Let $G = (V, E)$ be the graph obtained from G^+ by deleting the vertex $n+1$. Then,

$$\alpha(G) \geq \alpha(G^+),$$

with equality if and only if the $\alpha(G)$-eigenvector $\boldsymbol{v}$ of G has entry $\boldsymbol{v}(n) = 0$.

New results on the Laplacian spectra of token graphs

- Let G be a graph with k-token graph $F_k(G)$.

New results on the Laplacian spectra of token graphs

- Let G be a graph with k-token graph $F_k(G)$.
- For a vertex $a \in V(G)$, let $S_a := \{A \in V(F_k(G)) : a \in A\}$ and $S'_a := \{B \in V(F_k(G)) : a \notin B\}$.

New results on the Laplacian spectra of token graphs

- Let G be a graph with k-token graph $F_k(G)$.
- For a vertex $a \in V(G)$, let $S_a := \{A \in V(F_k(G)) : a \in A\}$ and $S'_a := \{B \in V(F_k(G)) : a \notin B\}$.
- Let H_a and H'_a be the subgraphs of $F_k(G)$ induced by S_a and S'_a, respectively.

New results on the Laplacian spectra of token graphs

- Let G be a graph with k-token graph $F_k(G)$.
- For a vertex $a \in V(G)$, let $S_a := \{A \in V(F_k(G)) : a \in A\}$ and $S'_a := \{B \in V(F_k(G)) : a \notin B\}$.
- Let H_a and H'_a be the subgraphs of $F_k(G)$ induced by S_a and S'_a, respectively.
- Note that $H_a \cong F_{k-1}(G \setminus \{a\})$ and $H'_a \cong F_k(G \setminus \{a\})$.

New results on the Laplacian spectra of token graphs

- Let G be a graph with k-token graph $F_k(G)$.
- For a vertex $a \in V(G)$, let $S_a := \{A \in V(F_k(G)) : a \in A\}$ and $S'_a := \{B \in V(F_k(G)) : a \notin B\}$.
- Let H_a and H'_a be the subgraphs of $F_k(G)$ induced by S_a and S'_a, respectively.
- Note that $H_a \cong F_{k-1}(G \setminus \{a\})$ and $H'_a \cong F_k(G \setminus \{a\})$.

Lemma

Given a vertex $a \in G$ and an eigenvector $\boldsymbol{v}$ of $F_k(G)$ such that $\boldsymbol{B}^\top \boldsymbol{v} = \boldsymbol{0}$, let

$$\boldsymbol{w}_a := \boldsymbol{v}|_{S_a} \quad \text{and} \quad \boldsymbol{w}'_a := \boldsymbol{v}|_{S'_a}.$$

Then, $\boldsymbol{w}_a$ and $\boldsymbol{w}'_a$ are embeddings of H_a and H'_a, respectively.

New results on the Laplacian spectra of token graphs

Theorem

For each of the following classes of graphs, the algebraic connectivity of a token graph $F_k(G)$ satisfies the following.

(i) *Let T_n be a* **tree** *on n vertices. Then, $\alpha(F_k(T_n)) = \alpha(T_n)$ for every n and $k = 1, \ldots, n-1$.*

(ii) *Let G be a graph such that $\alpha(F_k(G)) = \alpha(G)$. Let T_G be a graph where each vertex of G is the root vertex of some (possibly empty) tree. Then $\alpha(F_k(T_G)) = \alpha(T_G)$.*

(iii) *Let $G = C_n$ be a* **cycle graph** *on $n \geq 3$ vertices. Then, $\alpha(F_k(G)) = \alpha(G) = 2(1 - \cos(2\pi/n))$.*

New results on the Laplacian spectra of token graphs

Theorem

Let G be a graph on n vertices satisfying $\alpha(F_{k-1}(G)) = \alpha(G)$ and minimum degree

$$\delta(G) \geq \frac{k(n+k-3)}{2k-1}$$

for some integer $k = 1, \ldots, \lfloor n/2 \rfloor$. Then, the algebraic connectivity of its k-token graph equals the algebraic connectivity of G,

$$\alpha(F_k(G)) = \alpha(G).$$

New results on the Laplacian spectra of token graphs

Corollary

Let G be a graph on n vertices and minimum degree $\delta(G)$.

(i) *If $\delta(G) \geq \frac{2}{3}(n-1)$, then $\alpha(F_2(G)) = \alpha(G)$.*

(ii) *If $\delta(G) \geq \frac{3}{4}n$, then G satisfies $\alpha(F_k(G)) = \alpha(G)$ for every $k = 1, \ldots, n-1$.*

New results on the Laplacian spectra of token graphs

Some examples of known graphs satisfying Conjecture are:

- With (regular) minimum degree $n-1$, the **complete** graph.
- With (regular) degree $n-2$, the **cocktail party graph** (obtained from the complete graph with even number of vertices minus a matching).
- With degree $n-3$, the complement (regular) $\overline{C_n}$ of the cycle with $n \geq 12$ vertices.
- The complete r-**partite graph** $G = K_{n_1,n_2,\ldots,n_r} \neq K_r$ for $r \geq 2$, with number of vertices $n = n_1 + n_2 + \cdots + n_r$, for $n_1 \leq n_2 \leq \cdots \leq n_r$, with minimum degree $\delta(G) = n_1 + \cdots + n_{r-1}$, and $n \geq 3n_r - 2$.

References

N. Abreu, Old and new results on algebraic connectivity of graphs, *Linear Algebra Appl.* **423** (2007) 53—73.

K. Audenaert, C. Godsil, G. Royle, and T. Rudolph, Symmetric squares of graphs, *J. Combin. Theory B* **97** (2007) 74–90.

W. Carballosa, R. Fabila-Monroy, J. Leaños, and L. M. Rivera, Regularity and planarity of token graphs, *Discuss. Math. Graph Theory* **37** (2017), no. 3, 573–586.

C. Dalfó, F. Duque, R. Fabila-Monroy, M. A. Fiol, C. Huemer, A. L. Trujillo-Negrete, and F. J. Zaragoza Martínez, On the Laplacian spectra of token graphs, *Linear Algebra Appl.* **625** (2021) 322–348.

R. Fabila-Monroy, D. Flores-Peñaloza, C. Huemer, F. Hurtado, J. Urrutia, and D. R. Wood, Token graphs, *Graphs Combin.* **28** (2012), no. 3, 365–380.

M. Fiedler, Algebraic connectivity of graphs, *Czech. Math. Journal* **23** (1973), no. 2, 298–305.

To the memory of Susana-Clara López, from Universitat de Lleida, who died yesterday.